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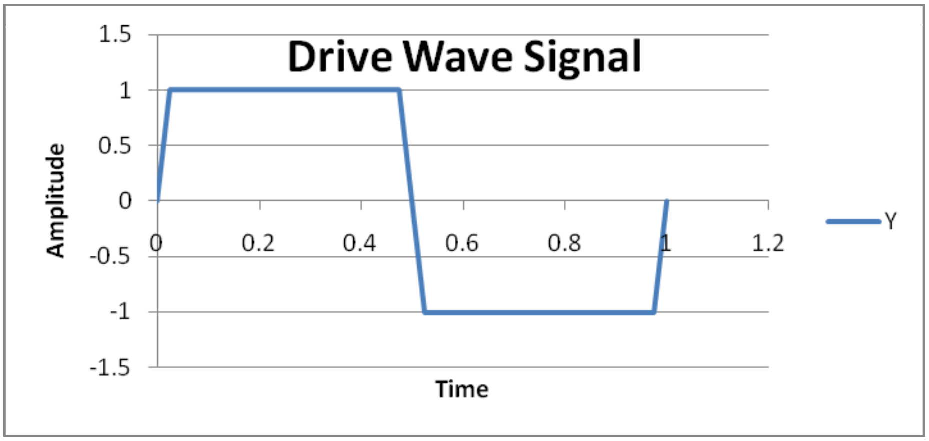
Using Excel's FFT Function, Goldwave (a .wav-file Editor) and a PC Sound Card to Build a Poor Man's Function Generator

Piezo actuators and motors can be driven with a variety of wave shapes, including sine waves, square waves and quasi-square waves. Quasi-square waves, similar to the one shown below, have some advantages from an energy efficiency point of view (see reference 2 below.) This shape is characterized by fast transitions (relative to the fundamental frequency) between constant voltage levels. The fundamental frequency is matched to the piezo device's resonant frequency for many motors. The fast transitions are used to facilitate energy recovery from the constant charging and discharging of the piezo element (ref. 2).



This particular signal consists of piecewise continuous line segments, each segment being defined by a single equation, so six individual equations are required to represent one cycle of the wave. *Goldwave* is a wav.-file editor with the capability to generate periodic signals of arbitrary shape from equations. However, (to my knowledge - I could be wrong, since I don't know many of *Goldwave's* capabilites), it requires a single equation, and cannot piece together multiple segments. A single equation for any periodic function of arbitrary shape can be found by expansion of an infinite Fourier series. Said another way, any periodic function can be defined by an infinite number of sine and cosine terms of various amplitudes and frequencies. The Fourier Series is given by

$$(1)\quad F(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right)$$

Where

$$(2)\quad a_n = \frac{1}{L}\int_{-L}^{L} F(t) cos\frac{n\pi t}{L}dt \quad and \quad (3)\quad b_n = \frac{1}{L}\int_{-L}^{L} F(t) sin\frac{n\pi t}{L}dt$$

$$n = 0,1,2, \dots \dots.., \qquad L \;\; is \;\; \frac{1}{2} \;\; of\ the\ period,$$

$and\ a_0\ turns\ out\ to\ be\ equal\ to\ the\ mean\ value\ of\ the\ function, F(t)\ from -L\ to\ L.$

It should be noted that for even functions $[g(-x) = g(x)]$, such as the cosine function, only cosine terms can be present, and for odd functions $[g(-x) = -g(x)]$, such as the sine function and our quasi-square wave, only sine terms can be present. Also, $a_0 = 0$ for odd functions. (Functions that are neither odd nor even can have both sine and cosine terms, and a value of a_0 other than zero.)

A Fast Fourier Transform (FFT) mathematically approximates the infinite series, with a finite number of terms, retaining the most significant terms. So, an FFT generated series can come close to generating the desired quasi-square wave above as a sum of sine terms with various amplitudes and frequencies. Excel has a fairly easy to use FFT function, which will provide these amplitudes and frequencies. (See reference 1, below, for instructions on the use of Excel's FFT function.)

The quasi-square wave above has been made non-dimensional to have a period of 1 and an amplitude of 1, as illustrated for a sine wave below.

Non-Dimensional Form

Let $\sin(\omega t) = \sin(2\pi f t) = \sin\left(2\pi \frac{t}{t_p}\right)$

Where $f\ is\ the\ frequency\ and\ t_p\ is\ the\ period\ and\ \omega\ is\ the\ circular\ frequency$

Let $t' = \frac{t}{t_p}\ be\ the\ non-dimensional\ time.$

Then $\sin(\omega t) = \sin(2\pi f t) = \sin\left(2\pi \frac{t}{t_p}\right) = \sin(2\pi t') = \sin(\omega' t')$

In this scheme,

$the\ non-dimensional\ circular\ frequency\ is,\quad \omega' = 2\pi$

$the\ non-dimensional\ frequency, f' = 1$

$and\ the\ non-dimensional\ period, t_p' = 1$

In a signal made up of multiple sine waves with different frequencies, if we designate the fundamental frequency as

$$f_0, \quad with\ period\ \ t_{p_0}$$

With non-dimensional frequency counterparts

$f_0' = 1, \quad with\ period\ \ t_{p_0}' = 1$

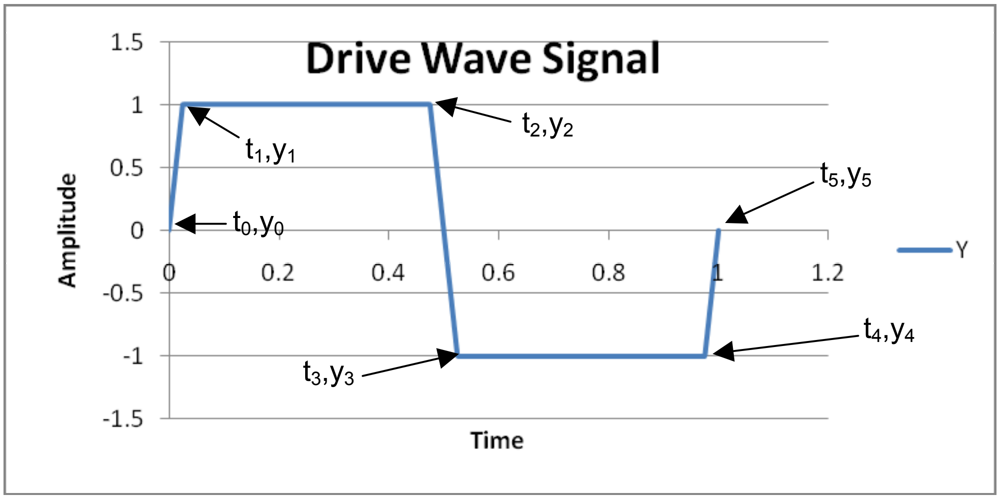
Then other frequencies, such as $f_n,\ with\ period\ t_{p_n},$ can be related to the fundamental frequency by ratios.

$Let \quad f_n = c_n f_0 \quad then \quad c_n = \frac{f_n}{f_0} = \frac{t_{p_0}}{t_{p_n}}$

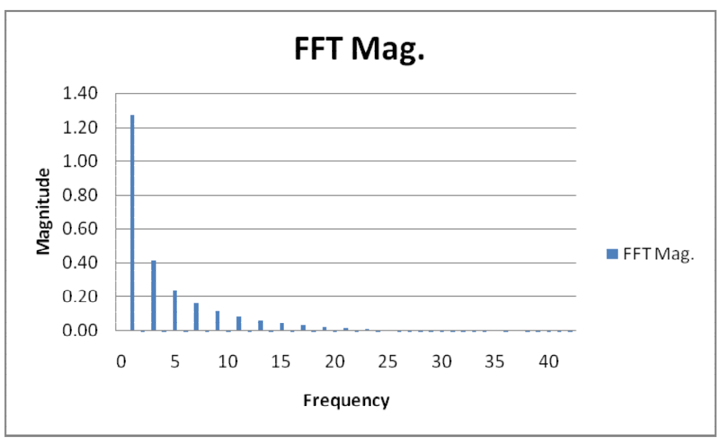
The n^{th} non-dimensional frequency then becomes $f_n' = c_n f_0' = c_n$

Excel's FFT

To set up the Excel spreadsheet, each line segment was defined by its two end points as shown below. The entire wave was arbitrarily divided into 1024 equal time slices, instead of say 1000, because Excel requires exactly 2^n $(1024 = 2^{10})$ points to perform an FFT. Amplitudes for each point in time were generated from the five line segment equations.



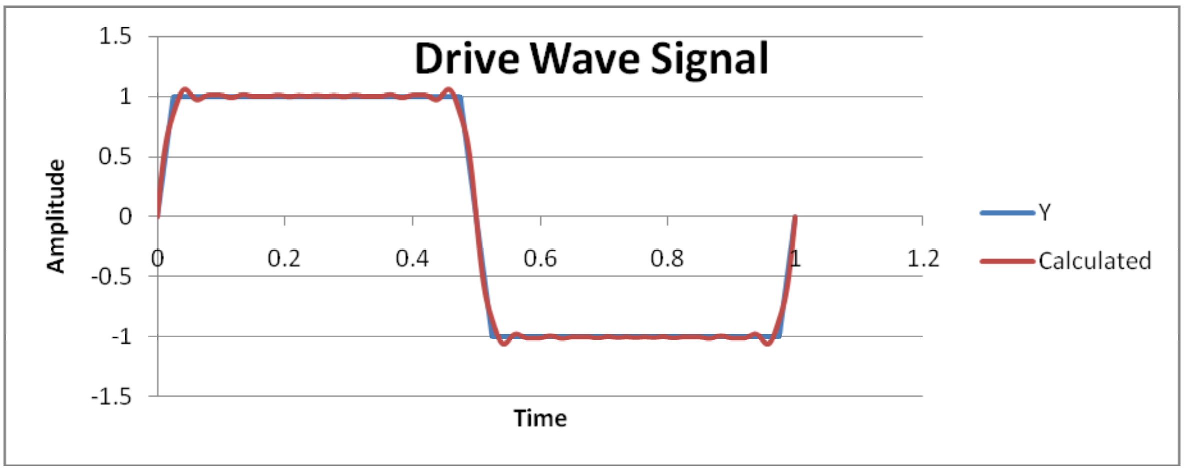
The FFT function was ran in Excel yielding 1024 frequencies and amplitudes (although only the first 512 are useful). Graphical results are shown in the plot of magnitude vs frequency, below, for frequencies up to 42.



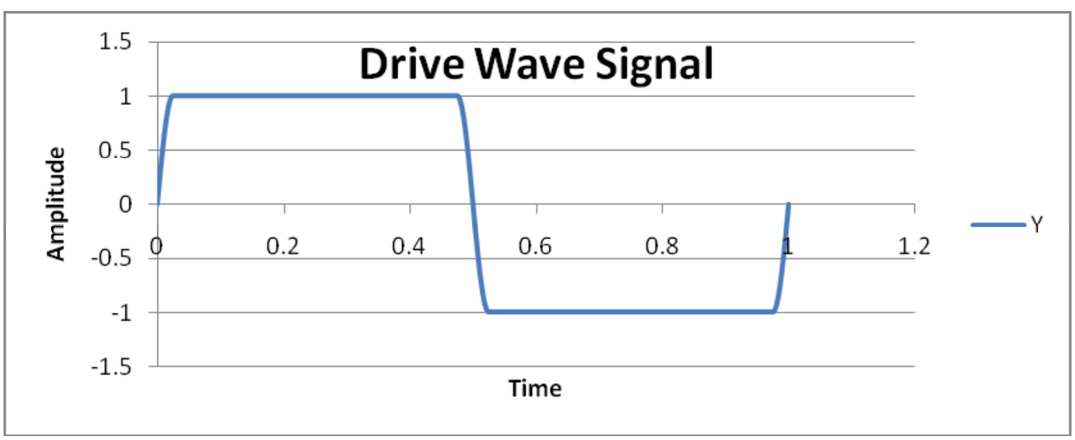
Fifteen frequencies were arbitrarily selected based on the fifteen largest amplitudes, and the following equation was formed:

$$y = 1.2679\sin(2\pi ft) + 0.4084\sin(6\pi ft) + 0.2286\sin(10\pi ft) + 0.1465\sin(14\pi ft) + 0.097731\sin(18\pi ft) + 0.0649\sin(22\pi ft) + 0.0414\sin(26\pi ft) + 0.02413\sin(30\pi ft) + 0.01142\sin(34\pi ft) + 0.00793\sin(46\pi ft) + 0.0010\sin(50\pi ft) + 0.0106\sin(54\pi ft) + 0.0100\sin(58\pi ft) + 0.0087\sin(62\pi ft) + 0.00685\sin(66\pi ft)$$

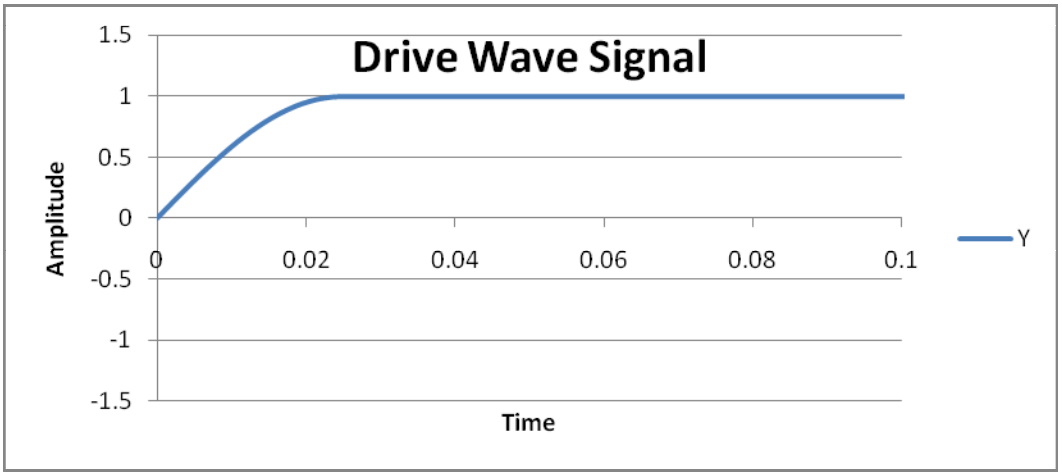
This equation was plotted, shown below overlaid on our original input wave. The equation was also copied into Goldwave to generate the periodic function. As can be seen from the plot below, the largest differences occur near where the different line segments meet, where the slopes change abruptly (i.e., the first derivative of the piecewise function is discontinuous.)



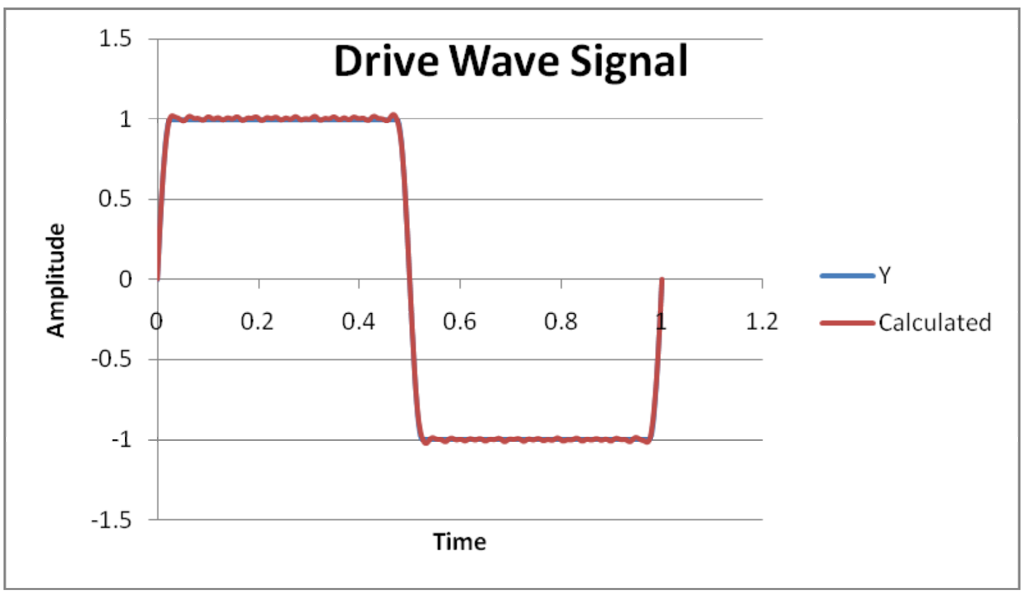
A second wave, shown below, was formed in which the fast transition segments between the constant amplitude sections were made from sine and cosine segments, in order to provide continuous slopes or first derivatives at the transition points.



Since this doesn't look much different from our line-segment plot, the time axis has been expanded to show the first transition more clearly, below.



Results of the FFT for this case is shown below, where it is obvious that a closer approximation has resulted from the smoother transitions. The desired original signal (colored blue) is barely visible under the FFT generated wave (in red).



Calculated Series

Another, way more laborious, approach to obtaining the signal equation is to calculate it directly from the Fourier equations (1) thru (3) for a finite number of terms. Since the desired periodic signal is an odd function, $a_n = 0,\ for\ n = 0,1,2 \ldots.$ The coefficients, b_n, can be obtained by piece-wise integration of Equation (3). That is, since the equations for each signal segment (line, sine or cosine) are known, the integral (3) can be broken into pieces corresponding to each segment and evaluated separately. This was done, for both the line segmented case and the case with sine & cosine transitions, to see how the two methods compare.

Line-Sement Case

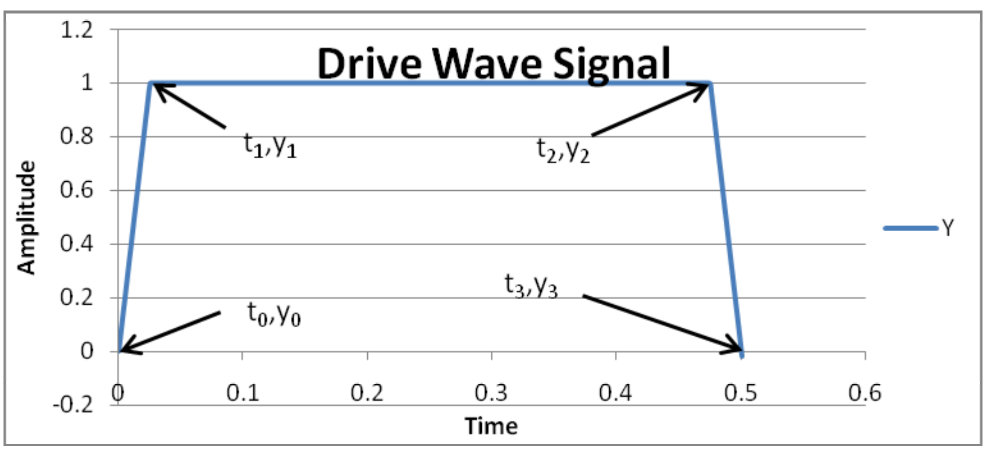
$$(3)\ b_n = \frac{1}{L}\int_{-L}^{L} F(t) sin \frac{n\pi t}{L} dt$$

We can integrate 2x the half-period, so (3) becomes

$$(4)\ b_n = \frac{2}{L}\int_{0}^{L} F(t) sin \frac{n\pi t}{L} dt$$

We will change point nomenclature to accommodate this integration, as shown in the plot below. Each line segment can be expressed as

$$(5)\ F(t)_m = q_m t + p_m \quad where\ m = 1, 2\ or\ 3 \quad for\ our\ three\ line\ segments$$



The slope, $q_m = \frac{y_m - y_{m-1}}{t_m - t_{m-1}}$ and the y-intercept, $p_m = \frac{y_{m-1}t_m - y_m t_{m-1}}{t_m - t_{m-1}}$

Substituting (5) into (4)

$$(6)\ \ b_n = \frac{2}{L}\int_0^L (q_m t + p_m) sin\frac{n\pi t}{L}dt$$

Integrating this, we get

$$(7)\ \ b_n = \frac{2Lq_m}{(n\pi)^2} Sin(\frac{n\pi t}{L})\Big|_0^L - \left(\frac{2q_m t}{n\pi} + \frac{2p_m}{n\pi}\right) Cos(\frac{n\pi t}{L})\Big|_0^L$$

Breaking this into two parts to help keep track of all the terms, define

$$(8)\ \ b_{ns} = \frac{2Lq_m}{(n\pi)^2} Sin(\frac{n\pi t}{L})\Big|_0^L \quad and \quad (9)\ b_{nc} = -\left(\frac{2q_m t}{n\pi} + \frac{2p_m}{n\pi}\right) Cos(\frac{n\pi t}{L})\Big|_0^L$$

Evaluating (8) for the three line segments yields

$$(10)\ \ b_{ns} = \frac{2Lq_1}{(n\pi)^2} Sin(\frac{n\pi t}{L})\Big|_{t_0}^{t_1} + \frac{2Lq_2}{(n\pi)^2} Sin(\frac{n\pi t}{L})\Big|_{t_1}^{t_2} + \frac{2Lq_3}{(n\pi)^2} Sin(\frac{n\pi t}{L})\Big|_{t_2}^{t_3}$$

$$\mathbf{(11)\ \ b_{ns} = \frac{2Lq_1}{(n\pi)^2}\left[Sin\left(\frac{n\pi t_1}{L}\right) - Sin(\frac{n\pi t_0}{L})\right] + \frac{2Lq_2}{(n\pi)^2}\left[Sin\left(\frac{n\pi t_2}{L}\right) - Sin(\frac{n\pi t_1}{L})\right] + \frac{2Lq_3}{(n\pi)^2}\left[Sin\left(\frac{n\pi t_3}{L}\right) - Sin(\frac{n\pi t_2}{L})\right]}$$

Evaluating (9) for the three line segments yields

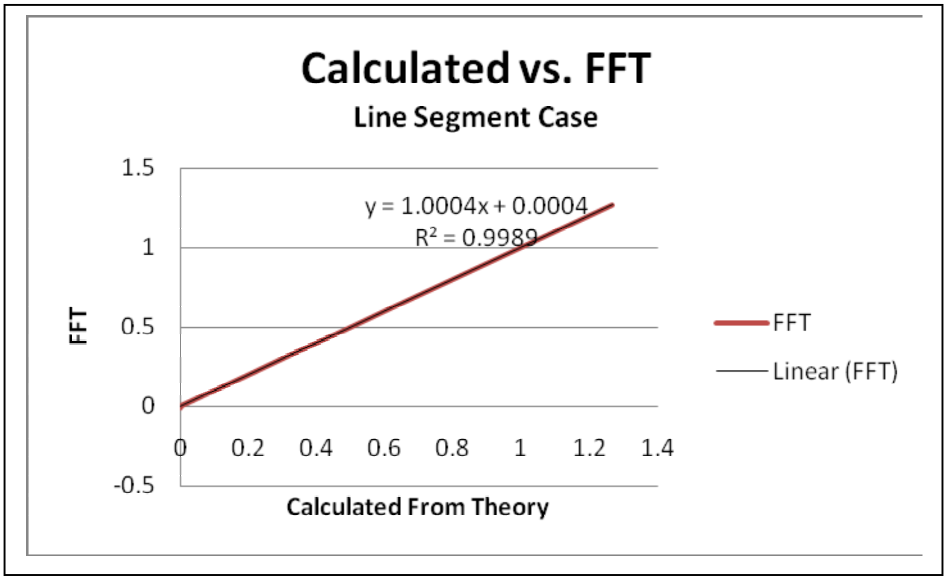
$$(9)\ b_{nc} = -\left(\frac{2q_m t}{n\pi} + \frac{2p_m}{n\pi}\right) Cos(\frac{n\pi t}{L})\Big|_0^L$$

$$(12)\ b_{nc} = -\left(\frac{2q_1 t}{n\pi} + \frac{2p_1}{n\pi}\right) Cos(\frac{n\pi t}{L})\Big|_{t_0}^{t_1} - \left(\frac{2q_2 t}{n\pi} + \frac{2p_2}{n\pi}\right) Cos(\frac{n\pi t}{L})\Big|_{t_1}^{t_2} - \left(\frac{2q_3 t}{n\pi} + \frac{2p_3}{n\pi}\right) Cos(\frac{n\pi t}{L})\Big|_{t_2}^{t_3}$$

$$\mathbf{(13)\ b_{nc} = -\left(\frac{2q_1 t}{n\pi} + \frac{2p_1}{n\pi}\right)\left[Cos\left(\frac{n\pi t_1}{L}\right) - Cos(\frac{n\pi t_0}{L})\right] - \left(\frac{2q_2 t}{n\pi} + \frac{2p_2}{n\pi}\right)\left[Cos\left(\frac{n\pi t_2}{L}\right) - Cos(\frac{n\pi t_1}{L})\right] - \left(\frac{2q_3 t}{n\pi} + \frac{2p_3}{n\pi}\right)\left[Cos\left(\frac{n\pi t_3}{L}\right) - Cos(\frac{n\pi t_2}{L})\right]}$$

$$\mathbf{(14)\ b_n = b_{ns} + b_{nc}}$$

Equations (11), (13) and (14) (in bold type) were calculated in the spreadsheet for n=1,2….1024. Results were almost identical to the Excel's FFT. Amplitudes of the two were plotted against each other and are shown in the plot below. A linear trend line (least squares) fit is also shown overlaid on the plot.



Sinusoidal Transition Case

In the linear segmented case we simply picked the points (t_1,y_1, t_2,y_2) where line segments intersected. In the sinusoidal transition case, we can first pick a non-dimensional frequency for the fast transitions, which will be a multiple of our non-dimensional fundamental frequency, f_0=1. For the spreadsheet analysis, the fast frequency picked was f_s=10. We can now calculate the points as follows:

$$t_0 = 0, \quad t_1 = \frac{1}{2\pi f_s}\sin^{-1} 1, \quad t_2 = 0.5 - t_1, \quad and\ t_3 = 0.5$$

Starting again with Equation (4)

$$(4)\ \ b_n = \frac{2}{L}\int_0^L F(t) sin\frac{n\pi t}{L} dt$$

The functional equations for the three segments of the half-period are:

$$F_1 = \text{Sin}\,(2\pi f_s t)$$
$$F_2 = 1 \qquad\qquad and,$$
$$F_3 = Cos2\pi f_s(t - t_{2)}$$

$Breaking\ (4) into\ three\ parts$

$Let \quad b_n = b_1 + b_2 + b_3, \qquad corresponding\ to\ our\ three\ sections.\ Then,$

For segment 1

$$(15) \quad b_1 = \frac{2}{L}\int_{t_0}^{t_1} Sin(2\pi f_s t) Sin\left(\frac{n\pi t}{L}\right) dt$$

$$b_1 = \frac{2}{L}\left[\frac{Sin\,\pi\left(2f_s - \frac{n}{L}\right)t}{2\pi\left(2f_s - \frac{n}{L}\right)} - \frac{Sin\,\pi\left(2f_s + \frac{n}{L}\right)t}{2\pi\left(2f_s + \frac{n}{L}\right)}\right]_{t_0}^{t_1}$$

$$\mathbf{(16)}$$

$$\boldsymbol{b_1 = \frac{1}{L\pi}\left[\frac{Sin\,\pi\left(2f_s - \frac{n}{L}\right)t_1 - Sin\,\pi\left(2f_s - \frac{n}{L}\right)t_0}{2\pi\left(2f_s - \frac{n}{L}\right)} - \frac{Sin\,\pi\left(2f_s + \frac{n}{L}\right)t_1 - Sin\,\pi\left(2f_s + \frac{n}{L}\right)t_0}{2\pi\left(2f_s + \frac{n}{L}\right)}\right]}$$

$$\boldsymbol{for\ 2f_s \neq \frac{n}{L}}$$

$When\ 2f_s = \frac{n}{L} \qquad (15)\ becomes$

$$(15') \quad b_1 = \frac{2}{L}\int_{t_0}^{t_1} Sin^2\left(\frac{n\pi t}{L}\right) dt$$

$$b_1 = \frac{2}{n\pi}\left[\frac{1}{2}\frac{n\pi t}{L} - \frac{1}{4}Sin\frac{2n\pi t}{L}\right]_{t_0}^{t_1}$$

$$b_1 = \frac{1}{n\pi}\left[\left(\frac{n\pi t_1}{L} - \frac{1}{2}Sin\frac{2n\pi t_1}{L}\right) - \left(\frac{n\pi t_0}{L} - \frac{1}{2}Sin\frac{2n\pi t_0}{L}\right)\right]$$

$$\boldsymbol{(16')\ b_1 = \frac{1}{L}(t_1 - t_0) - \frac{1}{2n\pi}\left(Sin\frac{2n\pi t_1}{L} - Sin\frac{2n\pi t_0}{L}\right)}$$

$$\boldsymbol{when \quad 2f_s = \frac{n}{L}}$$

For segment 2

$$b_2 = \frac{2}{L}\int_{t_1}^{t_2} Sin\frac{n\pi t}{L} dt = -\frac{2}{n\pi}\left[Cos\frac{n\pi t}{L}\right]_{t_1}^{t_2}$$

$$\boldsymbol{(17)\ b_2 = -\frac{2}{n\pi}\left[Cos\frac{n\pi t_2}{L} - Cos\frac{n\pi t_1}{L}\right]}$$

For segment 3

$$(18)\ \ b_3 = \frac{2}{L}\int_{t_2}^{t_3} Cos2\pi f_s(t-t_2)Sin\frac{n\pi t}{L}\ dt$$

Substituting the identity.

$Cos2\pi f_s(t-t_2) = Cos(2\pi f_s t)Cos(2\pi f_s t_2) + Sin(2\pi f_s t)Sin(2\pi f_s t_2)$ into (18)

$$(19)\ b_3 = \frac{2}{L}\int_{t_2}^{t_3} Cos(2\pi f_s t)\ Cos(2\pi f_s t_2)\ Sin\frac{n\pi t}{L}\ dt + \frac{2}{L}\int_{t_2}^{t_3} Sin(2\pi f_s t)\ Sin(2\pi f_s t_2)\ Sin\frac{n\pi t}{L}\ dt$$

Breaking this into 2 parts and taking the constant terms outside of the integral, let

$$b_{3a} = \frac{2}{L}Cos(2\pi f_s t_2)\int_{t_2}^{t_3} Cos(2\pi f_s t)\ Sin\frac{n\pi t}{L}\ dt$$

And

$$b_{3b} = \frac{2}{L}Sin(2\pi f_s t_2)\int_{t_2}^{t_3} Sin(2\pi f_s t)\ \ Sin\frac{n\pi t}{L}\ dt$$

Integration of b_{3a} *gives*

$$b_{3a} = -\frac{1}{L}\ Cos(2\pi f_s t_2)\left[\frac{Cos\left(\frac{n\pi}{L}-2\pi f_s\right)t}{\left(\frac{n\pi}{L}-2\pi f_s\right)} + \frac{Cos\left(\frac{n\pi}{L}+2\pi f_s\right)t}{\left(\frac{n\pi}{L}+2\pi f_s\right)}\right]_{t_2}^{t_3}$$

$$\mathbf{(20)}\ \boldsymbol{b_{3a} = -\frac{1}{\pi L}\ Cos(2\pi f_s t_2)\left[\frac{Cos\left(\frac{n\pi}{L}-2\pi f_s\right)t_3 - Cos\left(\frac{n\pi}{L}-2\pi f_s\right)t_2}{\left(\frac{n\pi}{L}-2\pi f_s\right)} + \frac{Cos\left(\frac{n\pi}{L}+2\pi f_s\right)t_3 - Cos\left(\frac{n\pi}{L}+2\pi f_s\right)t_2}{\left(\frac{n\pi}{L}+2\pi f_s\right)}\right]}$$

$$\boldsymbol{for\ \ 2f_s \neq \frac{n}{L}}$$

When $2f_s = \frac{n}{L}$ $\quad b_{3a}$ *becomes*

$$b_{3a} = \frac{2}{L}Cos\left(\frac{n\pi t_2}{L}\right)\int_{t_2}^{t_3} Cos\frac{n\pi t}{L}\ Sin\frac{n\pi t}{L}\ dt$$

$$b_{3a} = \frac{2}{n\pi}Cos\left(\frac{n\pi t_2}{L}\right)\left[\frac{1}{2}Sin^2(\frac{n\pi t}{L})\right]_{t_2}^{t_3}$$

$$\mathbf{(21)}\ \ \ \boldsymbol{b_{3a} = \frac{1}{n\pi}Cos\left(\frac{n\pi t_2}{L}\right)\left[Sin^2\left(\frac{n\pi t_3}{L}\right) - Sin^2(\frac{n\pi t_2}{L})\right]}$$

$$\boldsymbol{when\ \ \ 2f_s = \frac{n}{L}}$$

$$b_{3b} = \frac{2}{L} Sin(2\pi f_s t_2) \int_{t_2}^{t_3} Sin(2\pi f_s t) \; Sin\frac{n\pi t}{L} \, dt$$

Intergrating

$$\textbf{(21b)} \;\; \boldsymbol{b_{3b}} = \frac{1}{\pi L} \boldsymbol{Sin(2\pi f_s t_2)} \left[\frac{Sin\,\pi\left(2f_s - \frac{n}{L}\right)t_3 - Sin\,\pi\left(2f_s - \frac{n}{L}\right)t_2}{\left(2f_s - \frac{n}{L}\right)} - \frac{Sin\,\pi\left(2f_s + \frac{n}{L}\right)t_3 - Sin\,\pi\left(2f_s + \frac{n}{L}\right)t_2}{\left(2f_s + \frac{n}{L}\right)} \right]$$

$$\boldsymbol{for\ 2f_s \neq \frac{n}{L}}$$

$$When\ 2f_s = \frac{n}{L} \qquad b_{3b}\ becomes$$

$$b_{3b} = \frac{2}{L} Sin\left(\frac{n\pi t_2}{L}\right) \int_{t_2}^{t_3} Sin^2 \frac{n\pi t}{L} \, dt$$

Intergrating this,

$$b_{3b} = \frac{2}{n\pi} Sin\left(\frac{n\pi t_2}{L}\right) \left[\frac{1}{2}\frac{n\pi t}{L} - \frac{1}{4} Sin\frac{2n\pi t}{L}\right]_{t_2}^{t_3}$$

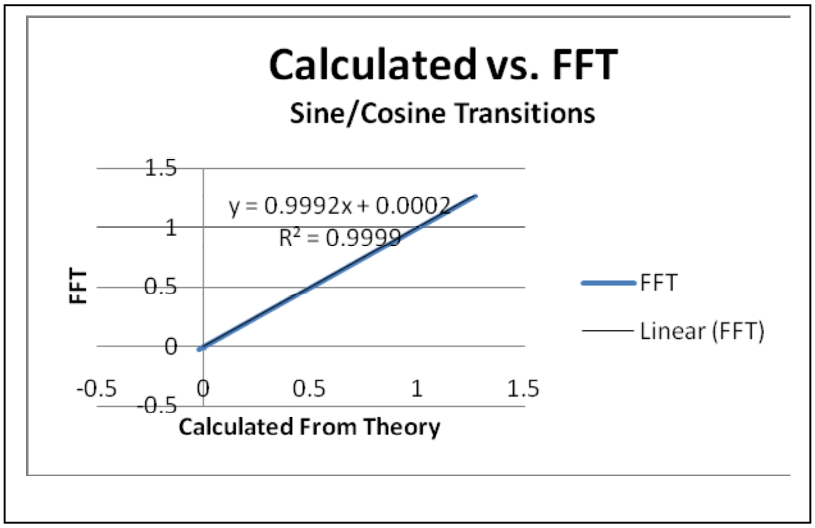
$$b_{3b} = \frac{1}{n\pi} Sin\left(\frac{n\pi t_2}{L}\right) \left[\left(\frac{n\pi t_3}{L} - \frac{1}{2} Sin\frac{2n\pi t_3}{L}\right) - \left(\frac{n\pi t_2}{L} - \frac{1}{2} Sin\frac{2n\pi t_2}{L}\right)\right]$$

$$\textbf{(22)} \quad \boldsymbol{b_{3b}} = \boldsymbol{Sin\left(\frac{n\pi t_2}{L}\right) \left[\frac{1}{L}(t_3 - t_2) - \frac{1}{2n\pi}\left(Sin\frac{2n\pi t_3}{L} - Sin\frac{2n\pi t_2}{L}\right)\right]}$$

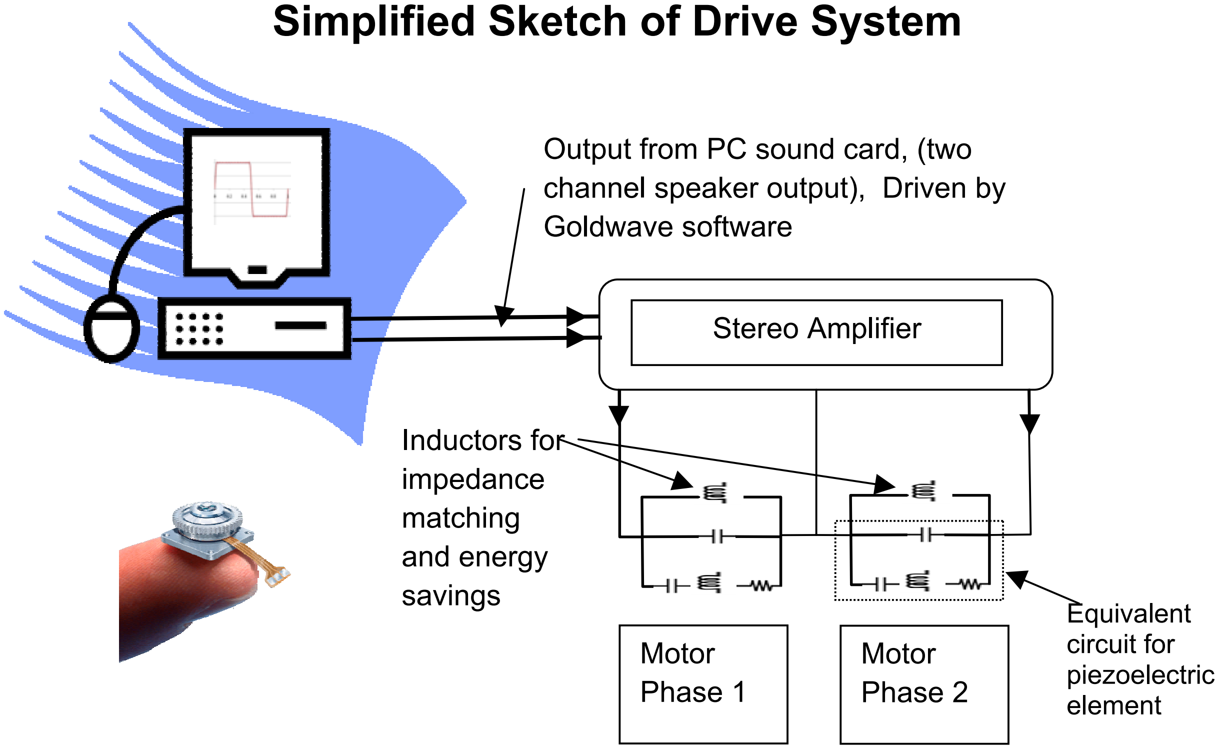
$$\boldsymbol{When\ 2f_s = \frac{n}{L}}$$

$$\textbf{(23)} \quad \boldsymbol{b_n = b_1 + b_2 + b_{3a} + b_{3b}}$$

As in the line-segment case, Equations 16, 16’, 17, 20, 21, 21b, 22 and 23 (in bold type) were calculated in the spreadsheet for n=1,2….1024 for the sinusoidal case. Results were again almost identical to the Excel’s FFT. Amplitudes of the two were plotted against each other are shown in the plot below. A linear trend line (least squares) fit is also shown overlaid on the plot.



Simplified Sketch of Drive System



References

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by Larry Klingenberg
April 2005
Instructions on using Excel's FFT function.
http://online.sfsu.edu/~larryk/Common%20Files/Excel.FFT.pdf

2) **Effcient Charge Recovery Method for Driving Piezoelectric Actuators with Quasi-Square Waves**
Domenico Campolo, *Member, IEEE*, Metin Sitti, *Member, IEEE*, and Ronald S. Fearing, *Member, IEEE*
http://nanolab.me.cmu.edu/publications/papers/Sitti-IEEETransUltrason2003.pdf

3) **Goldwave. To download the latest version:**
http://www.goldwave.com/release.php