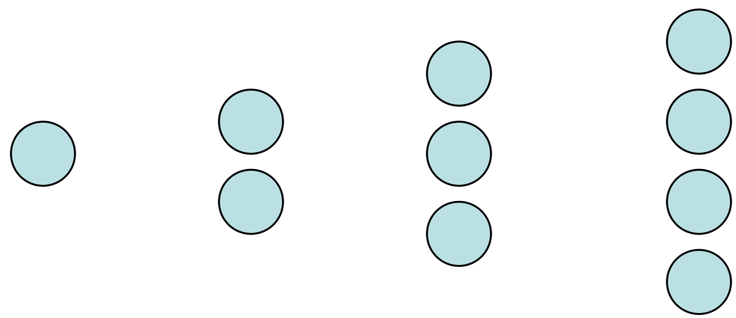
Stacking Cannonballs

Dr. Bob

Cannonballs in a layer

Let L_n be the number of cannonballs on layer n.

Each row in a layer has 1 more than the previous row:

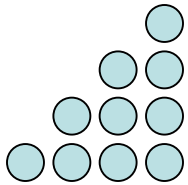


$$L_n = 1 + 2 + 3 + 4 + \ldots + n$$

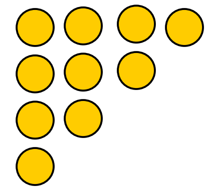
$$L_n = \sum_{i=1}^{n} i$$

Cannonballs in a Layer

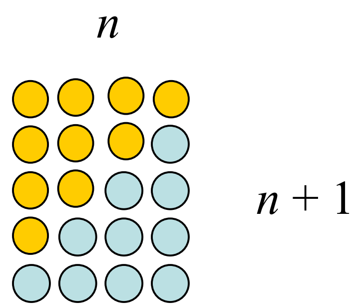
To find the sum of cannonballs in a layer graphically, rearrange the rows as like this:



Copy and rotate



Stack the two layers on top of each other

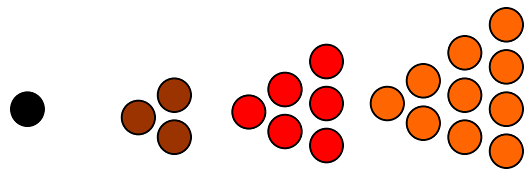


The double layer has $2 * L_n = n*(n + 1)$, *or*

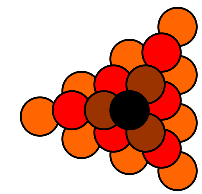
$$L_n = n * (n + 1) / 2 = \sum_{i=1}^{n} i$$

Stack the Cannonballs

Let T_N be the number of cannonballs in a stack of N layers.



Stack the layers.



$$T_N = 1 + 3 + 6 + 10 + \ldots + L_N$$

$$T_N = \sum_{n=1}^{N} L_n$$

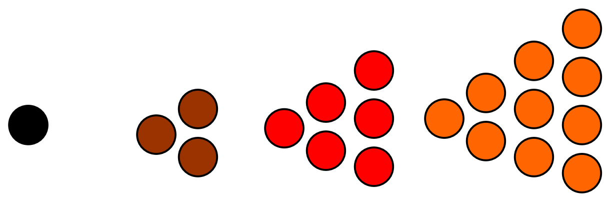
The answer to the original question:
When is the number above a layer = the number in the layer?

$$T_4 = (1+3+6)+10 = 10+10$$

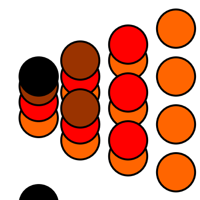
At layer 4.

Another way to count.

Suppose each layer was moved so that that the point ball of each layer is right above the other.

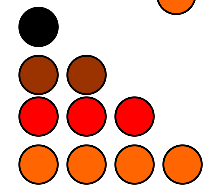


Stack the layers.



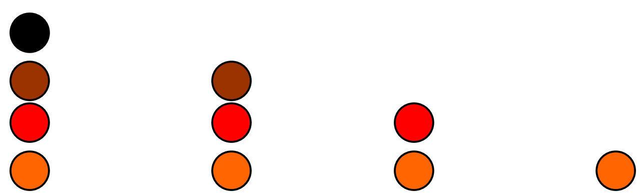
w=width

From the side it looks like this.



Add up the cannonballs:

h=height



$$T_4 = 1*4 + 2*3 + 3*2 + 4*1 = \sum w*h$$

$$T_N = \sum_{n=1}^{N} n*(N+1-n) = (N+1)\sum_{n=1}^{N} n - \sum_{n=1}^{N} n^2$$

Combine both ways.

For N layers, the last page gives:

$$T_N = \sum_{n=1}^{N} n*(N+1-n) = (N+1)\sum_{n=1}^{N} n - \sum_{n=1}^{N} n^2 \qquad \text{(eq. 1)}$$

The first page gives:

$$T_N = \sum_{n=1}^{N} L_n = \sum_{n=1}^{N}\sum_{i=1}^{n} i = \sum_{n=1}^{N} n*(n+1)/2 = \sum_{n=1}^{N} n^2/2 + \sum_{n=1}^{N} n/2$$

or $$2*T_N = \sum_{n=1}^{N} n + \sum_{n=1}^{N} n^2 \qquad \text{(eq. 2)}$$

Add eq. 2 + eq 1.

$$2*T_N + T_N = 3*T_N = (N+2)\sum_{n=1}^{N} n = (N+2)*N*(N+1)/2$$

or

$$T_N = N*\frac{(N+1)}{2}*\frac{(N+2)}{3}$$

Question:

When is the number above a layer = the number in the layer?

$$T_{N-1} = L_N$$

or

$$(N-1) * \frac{N}{2} * \frac{(N+1)}{3} = N * \frac{(N+1)}{2}$$

Dividing both sides by N*((N+1)/6 gives:

$$N - 1 = 3$$

$$N = 4$$

Checking we have:

T_3=3*(4/2)*(5/3) = 10

L_4 =4*(5/2) = 10

So the equations work!