Astronomy and Cosmology

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1 Definitions

The earth rotates about the Sun in an elliptical orbit, with the Sun at a focus of the ellipse, and where the period of the orbit is about 365.25 days. The plane of this elliptical orbit is called the ecliptic. At the same time, the Earth rotates around its own axis every 24 hours. This rotational axis of the Earth is not perpendicular to the ecliptic plane. It is tilted with its axis inclined to the ecliptic normal by 23.44 degrees. This rotation of the Earth is relatively constant over time with respect to the "fixed" stars, which are at such enormous distances from the earth that they may be considered fixed. These stars, with respect to a coordinate system on the earth, appear to be located on a celestial sphere of huge radius. Each day this sphere of fixed stars appears to rotate around the earth. The equatorial plane of this celestial sphere coincides with the equator of the earth. To locate a star on this celestial sphere we can use spherical coordinates. The angle between a star and the celestial equator, is called its declination. This is like the geographical latitude on the Earth. We specify the declination with the symbol ϕ. So ϕ takes values between $-\pi/2$ and $\pi/2$. We can locate the star exactly on the celestial sphere with a second angle, an angle measured around the equator. This angle is called the Right Ascension of the star. But we must have a reference point for this angle. We measure the Right Ascension angle in the Equatorial plane from a line that lies in the plane. This line is the line of intersection of the ecliptic plane with the Equatorial plane. When the Sun is in the direction of this line, the Sun appears to lie in both the ecliptic plane and the Equatorial plane. This happens twice a year, at the Vernal Equinox, and at the Autumnal equinox. At these two points, the length of day and night is 12 hours. Thus the term equinox. The Vernal

Equinox, (spring Equinox) occurs at about March 21. This direction is the reference point for the Right Ascension angle. We write the Right Ascension with the symbol Λ. So ϕ and Λ are the fixed coordinates of a star on the Celestial sphere.

The above remarks must be slightly qualified. The Earth is a spinning top, and like a top, its axis does wobble. This wobbling is called precession. This is quite slow. But as the spinning Earth precesses, the equatorial plane changes with respect to the fixed stars. And of course the line of intersection of the Celestial equator and the Ecliptic changes. This phenomenon is known as the precession of the Equinoxes. The precession of the Equinoxes is about 1 degree every 72 years. This means that the Declination and Right Ascension of a star will change very slowly over time.

The Right Ascension is measured in hours, varying from 0 to 24. The word hour is used in several ways in astronomy. First, it is a measure of time. A day is 24 hours. A day is the time it takes for the Earth to spin once on its axis. However this is ambiguous, because this period of revolution is measured in two ways, first with respect to the fixed stars, and second with respect to the Sun. Now the period with respect to the fixed stars is relatively constant, but the period with respect to the Sun varies throughout the year because the earth is revolving around the Sun in a nonuniform way. Also the rotation of the Earth is not completely constant. This measuring of time by the angular rotation of the earth is confusing in astronomy. This is no longer the way in which time is defined. Under the International System of Units, the second is currently defined as the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom. This definition refers to a caesium atom at rest at a temperature of 0 K. This defines the second, and so the hour: an hour is 3600 seconds. So the unit hour refers to time. But hour is also used in astronomy as a measure of rotation angle. Thus an angle of 24 hours is an angle of 360 degrees. This has nothing to do with time, although the earth does rotate approximately an hour of angle in an hour of time. Right Ascension is measured in these angular hours from the Vernal Equinox direction.

Suppose an astronomer wishes to locate a star. He is at a position on the Earth at some time, at some longitude, and at some latitude. The zenith is the point directly above him. His meridian is the plane passing through the center of the earth, the zenith, and the Earth pole. He can adjust the angles of his telescope to point to a star. He can adjust the angle above his horizon, and the angle around the horizon. These angles are known as the altitude and azimuth respectively. But this is not a very convenient way to locate a star because the star positions are recorded in the spherical coordinates consisting of the declination ϕ and the Right Ascension Λ. So he will have a telescope which has a rotational axis in the direction of the pole, and a rotational axis at right angles to the pole. Thus he can directly set the declination, and the Right Ascension angles. However, he needs to know the location of the vernal Equinox, so that he can set the zero of his telescope to this direction. The hour angle of a star is the angle from the meridian to the star measured in a clockwise direction viewed looking downward at the north pole. The hour angle is written as Ψ. If the astronomer knows the hour angle of the star, he can first set his telescope to point to his meridian, then rotate his telescope around the pole axis by the hour angle to locate the star. Suppose he knows the Right Ascension, and wants to calculate the hour angle. He must know the rotational position of the celestial sphere with respect to the earth, which is dependent on the time. He needs the location of the Vernal Equinox. That is he needs the hour angle of the Vernal Equinox. The hour angle of the Vernal equinox is called the Sidereal Time. The Sidereal Time is written with the symbol Σ. These angles, Σ, Ψ, and Λ are never negative. They are measured in hours. The hour angle of an object Ψ (H.A), its right ascension Λ (R.A), and the sidereal time Σ (S.T.) are related by the equation

$$\Sigma = \Psi + \Lambda.$$

or as is sometimes written,

$$S.T. = H.A. + R.A.$$

Again, these angles are all positive and have values between 0 and 24. To better understand the relationship between these angles, see Figure 1. We state these definitions again. The hour angle, Ψ, of an observed celestial body X, is the angle that X makes with the meridian. This angle is measured in the west direction from the meridian to X. It is given in hours, and has a value between 0 and 24 hours. More exactly the hour angle is the angle between the meridian and the orthogonal projection of the object to the celestial plane. The right ascension (R.A.), which we call Λ, is the angle in the celestial plane from the vernal equinox to the orthogonal projection of the object X to the celestial plane. This angle is measured opposite to the hour angle. It is measured in an east direction from the vernal equinox to the object X. It is measured in hours from 0 to 24. The vernal equinox is the intersection of the celestial equator with the ecliptic. The ecliptic is the plane of the earth as it travels around the sun. From the point of view of the celestial sphere, centered at the earth, the ecliptic is the plane of the orbit of the sun around the earth. The Sun passes through the vernal equinox on about March 21st. The sidereal time, which we call Σ, is a measure of the rotation of the earth with respect to the fixed stars. The vernal equinox, the intersection of the celestial plane with the ecliptic. It is a fixed direction, and is located by measuring fixed stars. The sidereal time is the hour angle of the vernal equinox, so should perhaps be called the sidereal time at the meridian.

So from the equation above, we can find the S.T. by observing the H.A. of a body whose R.A. is known. The book by Smart, on page 37, gives an example of the calculation. Smart has a good discussion of, sidereal time, solar time, and mean solar time. However, this is a very old book. It was written before the modern definition of the second. The ascension of a celestial body is when it crosses the meridian and reaches its highest point in the sky. So when a body reaches its ascension point moving to the right in the sky, the vernal equinox is R.A. hours from the meridian. And since the hour angle is zero, the S.T. equals the R.A. That is the right ascension of a body X is the sidereal time when it reaches its ascension. One hour of sidereal time corresponds directly to 15 degrees of the earth's rotation. Twenty four hours of sidereal time is equivalent to 360 degrees of earth rotation relative to the fixed stars. This is no longer exact because the time standard is no longer measured astronomically by earth rotation, but by the frequency of a spectral line of caesium-133. This is necessary because the Earth's rotation is not absolutely constant. Mean solar time is the average time of the rotation of the earth relative to the sun, whereas the sidereal day is the time of rotation of the earth relative to the fixed stars. The sidereal day is equal to about 23 hours and 56 minutes of mean solar time. The sidereal time at an observation point on the earth is the angle of rotation from that point to the vernal equinox. We have seen above the equation relating the observed hour angle, the sidereal time, and the right ascension of an observed celestial object. The mean sun is a fictitious body that travels in a circle on the celestial equator. It has the same period as the Sun. The mean sun sometimes lags the real Sun and sometimes leads it. The mean Sun and the real Sun in general cross the local meridian at different times. This difference in times on a given day, say at the Greenwich

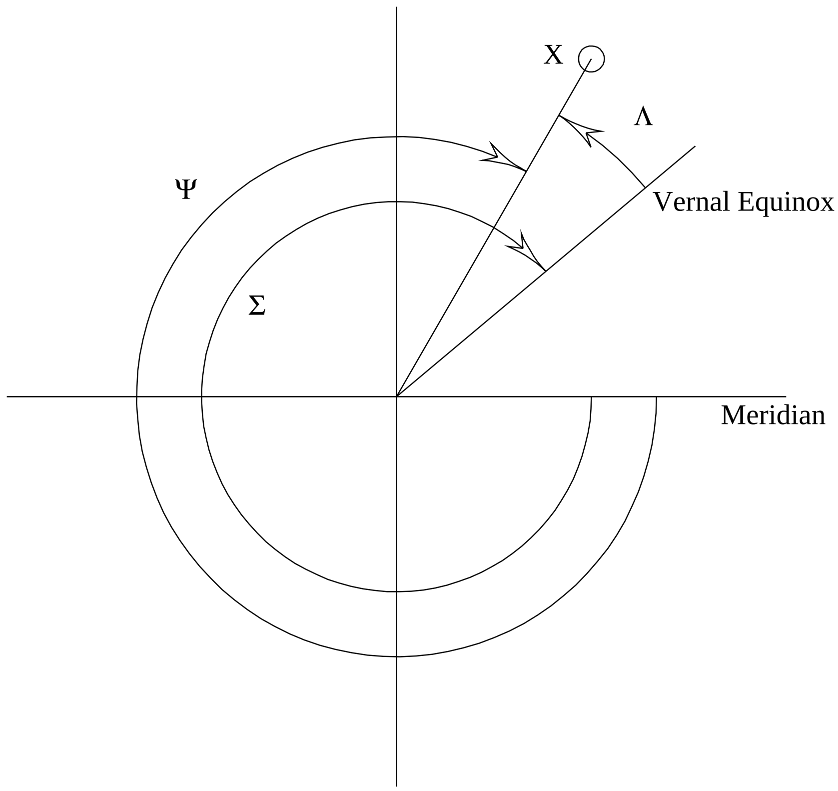

Figure 1: **The relation between Sidereal Time, Σ, the Hour Angle Ψ, and the Right Ascension Λ.** Ojects rotate clockwise. The hour angle of an object X is the time since the object last crossed the meridian. The hour angle of X in this figure is about $\Psi = 24(3/4) = 18$ hours, or about 270 degrees. The Right Ascension Λ is the time for the object to ascend to the zeneth after the Vernal Equinox crosses the meridian, that is, it is the angle from the Vernal Equinox to the object, measured counterclockwise. In this figure the right Ascension of X is about 25 degrees. The sidereal time is the hour angle of the Vernal Equinox. Here it is about $\Sigma = 270 + 25 = 295$ degrees, or about $24(295/360) = 59/3 = 19.666666666666$ hours. We have in general $\Sigma = \Psi + \Lambda$, or $ST = HA + RA$.

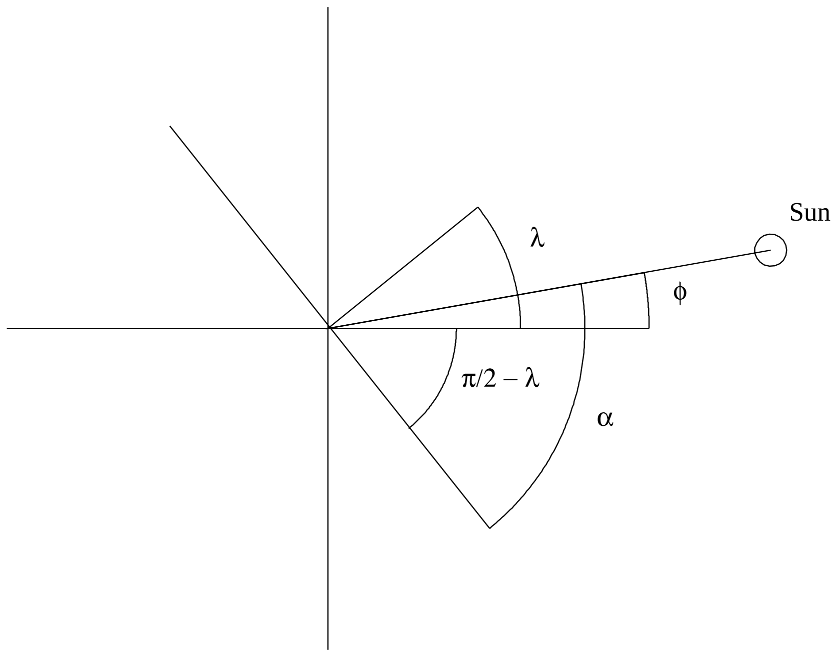

Figure 2: When the Sun is on the meridian, the declination of the sun ϕ, the altitude of the Sun above the horizon α, and the latitude λ, satisfy $\alpha = \phi + (\pi/2 - \lambda)$

meridian, is given by a function called the equation of time. The mean solar could be positioned anywhere in its circular orbit in the equatorial plane. It is positioned so that the average deviation in the time of noon by the mean Sun and the real sun is zero averaged over the year. Such a position can be found, for if the time of the noon crossing of the meridian were increased by a big enough Δt then the cumulative deviation could be made positive at the end of the year, and likewise if the time of the noon crossing were decreased enough, the cumulative deviation could be made negative. A mean solar day is a solar day of 24 hours with respect to the mean sun.

The altitude α of a celestial object is the angle of the object above the horizon. The azimuth ω is the angle of the object measured on the horizontal tangent plane of the observation point on the earth. The zenith is the point vertically overhead. The meridian is the north-south plane through the observation point. When the object is on the meridian the declination ϕ, the latitude λ and the altitude α are related by

$$\phi = \alpha - (\pi/2 - \lambda).$$

That is, the declination equals the altitude minus the colatitude.

2 The Mean Sun and Mean Solar Time

The length of the day, from local noon to local noon on the next day, varies throughout the year. This is because from a coordinate system fixed at the center of the earth, the sun appears to orbit the earth in an elliptical orbit with the focus at the center of the earth. This orbital plane is tilted from the equitorial plane of the earth. So the sun has different velocity in this orbit at different times of the year. So a plane through the axis of the earth which passes through the sun, rotates at different velocities throughout the year.

The mean sun is an ideal object that travels uniformly in a circle in the equatorial plane, with period one year. The time corresponding to this ideal Sun is called mean solar time. The difference between local solar time and mean solar time is given in minutes. This is also the difference between mean solar time and local sundial time. We must make another time correction if we are not at a standard time meridian. The standard meridians are at longitude angles that are multiples of 15 degrees from Greenwich. These define the standard time zones. The time correction for longitude is the time it takes for the earth to rotate from the standard meridian to our meridian.

3 Historical Figures

Claudius Ptolemy, a Roman citizen of Egypt who wrote in Greek, AD 90 AD 168

Nicolaus Copernicus, Polish, 19 February 1473 24 May 1543

Tycho Brahe, Danish, 14 December 14, 1546 24 October 24, 1601

Galileo Galilei, Italian, 15 February 1564 8 January 1642)

Johannes Kepler, German, December 27, 1571 November 15, 1630

Isaac Newton, English, 25 December 1642 20 March 1727

4 The Astronomical Unit, The Parsecond, and The Light Year.

The parsec (parallax of one arcsecond; symbol: pc) is a unit of length, equal to just under 31 trillion kilometers (about 19 trillion miles), or about 3.26 light-years. It is defined as the length of the adjacent side of an imaginary right triangle in space. The two dimensions that specify this triangle are the parallax angle (defined as 1 arcsecond) and the opposite side (defined as 1 astronomical unit (AU), the distance from the Earth to the Sun).

Originally, the AU was defined as the length of the semi-major axis of the Earth's elliptical orbit around the Sun. In 1976, the International Astronomical Union revised the definition of the AU for greater precision, defining it as that length for which the Gaussian gravitational constant (k) takes the value 0.01720209895 when the units of measurement are the astronomical units of length, mass and time.[5] An equivalent definition is the radius of an unperturbed circular Newtonian orbit about the Sun of a particle having infinitesimal mass, moving with a angular frequency of 0.01720209895 radians per day,[2] or that length for which the heliocentric gravitational constant (the product GM?) is equal to (0.01720209895)2 AU3/d2. It is approximately equal to the mean EarthSun distance.

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SI units
149.6010^6 km  149.6010^9 m
Astronomical units
4.8481  10-6 pc    15.81310^-6 ly
US customary / Imperial units
92.95610^6 mi  490.8110^9 ft
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5 The Horizontal Sundial

Let λ be the latitude of the location of the sundial. Let γ be the declination of the sun. Let there be a coordinate system at the center of the earth with unit coordinate vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$. Let $\mathbf{k}$ be in the direction of the north pole. Let $\mathbf{i}$ and $\mathbf{j}$ be in the equatorial plane. Without loss of generality, let the observation point be on the Greenwich meridian. Let $\mathbf{i}$ point from the center of the earth to the Greenwich meridian. Let $\mathbf{j}$ point 90 degrees east of Greenwich. Let P_1 be the tangent plane at the observation point on the earth. Let it have unit vector normal

$$\mathbf{n} = \cos(\lambda)\mathbf{i} + \sin(\lambda)\mathbf{k}.$$

The normal lies in the Greenwich meridian plane. Plane P_1 is the sundial plane. Let there be a plane P_2 at the observation point parallel to the equatorial plane. The angle between P_1 and P_2 is the latitude angle λ. Let a pole be erected on this plane parallel to the north pole. Let h be the length of this sundial pole. This pole forms the edge of the sundial gnomon. This pole casts a shadow on the horizontal sundial plane, which we have called P_2. The end of this shadow then traverses a circle on plane P_2. The ray to the end of the shadow generates a cone as the sun rotates about the earth. The path on the sundial plane is thus a conic section. In the summer at the north pole the path is a circle. At a lower latitude where there is still no sunset the path is an ellipse. At a latitude where the sun sets the path goes to infinity at sunset, therefore the path is a hyperbola. A vector in the direction of the sun ray at time t is

$$\mathbf{R} = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} + \tan(-\gamma))\mathbf{k},$$

where θ is the angle of the rotation of the sun, relative to the earth, corresponding to time t. We have

$$\theta = -(t/12.) * \pi,$$

where t is in hours. θ is negative since the sun travels from east to west. Technically t is the sidereal time of the earths rotation. 24 hours of this time is about 23 hours 56 minutes and 4 seconds of mean solar time. At midnight $\theta = 0$, at 12 noon $\theta = -\pi$.

We construct a vector from the base of the sundial pole (the origin of our coordinate system) by intersecting a line in the direction of $\mathbf{R}$ with the sundial plane P_1. We specify the plane with its normal vector $\mathbf{n}$ and a point on the plane, namely the origin. We specify the line by two points on the line $\mathbf{p}_s$ and $\mathbf{p}_e$. The first point is the end of the sundial pole given by

$$\mathbf{p}_s = h\mathbf{k}$$

the second point is the end of the sun ray vector $\mathbf{R}$ which starts at $\mathbf{p}_s$. Thus

$$\mathbf{p}_e = \mathbf{p}_s + \mathbf{R}.$$

Let $\mathbf{p}_i$ be the intersection point and hence a vector which has the direction and length of the shadow. There will be an intersection point provided $\mathbf{R}$ is not parallel to the sundial plane P_1, that is provided $\mathbf{R}$ is not perpendicular to the plane normal $\mathbf{n}$. Perpendicularity will occur only at the equinoxes. In these cases, the ray vector $\mathbf{R}$ lies in the sundial plane, and is in the direction of the shadow, which has infinite length. Thus $\mathbf{R}$ can play the role of $\mathbf{p}_i$ in determining the angle that the shadow direction makes with the sundial noon direction.

The noon direction is given by unit vector v_n

$$v_n = -\sin(\lambda)\mathbf{i} + \cos(\lambda)\mathbf{k}.$$

At time t and corresponding angle θ, the angle ϕ between this shadow $\mathbf{p}_i$ and the noon shadow is obtained mathematically from the dot product

$$\cos(\phi) = \frac{\mathbf{v}_n \cdot \mathbf{p}_i}{\|\mathbf{p}_i\|}.$$

However, the use of the dot product for finding the angle between vectors, suffers from numerical inaccuracy. See the more accurate computation found in the Fortran subroutine **veca3**.

For example suppose $t = 10$ hours, then $\theta = -\frac{5}{6}\pi$. The time obtained from the sundial must be corrected for the longitude, and for the nonuniform motion of the sun. The latter correction is done using the equation of time. Suppose, as an example, that the location is latitude 39.018167, and longitude 94.59255 (6021 Wornall Road, Kansas City, MO 54113). Because central standard time is measured at longitude 90 degrees west, the time for the sun to travel 4.59255 degrees must be added to the local sundial time. Also a second correction for the nonuniform motion of the sun must be added.

This comes from the equation of time. An approximate representation of the equation of time is

$$E = 9.87\sin(2B) - 7.23\cos(B) - 1.5\sin(B),$$

where

$$B = 2\pi(N - 81)/364,$$

and N is the day number ($N = 1$ on January 1, and so on). For January 28,2006, we find that the first correction for the longitude is 18.2 minutes. The second correction for the equation of time is 14.34 minutes. So local noon occurred at about 12:32 mean solar central standard time. The length s of the noon shadow on the sundial plane can be found from the triangle formed by the sun ray line the shadow line and the sundial pole line. We use the angles and lengths of sides of this triangle and the law of sines to solve for the shadow length s. The relevant angles are σ which is opposite s, and η which is opposite h. We have

$$\sigma = \pi/2. - \gamma.$$

and

$$\eta = \pi - \sigma - \lambda.$$

From the law of sines

$$s = \sin(\sigma) * h/\sin(\eta).$$

This only works of course when we are not at the equinoxes. At the equinoxes the sun declination γ is zero, and so the shadow is infinite.

We can also calculate the time of sunrise from the suns declination. At sunrise vector $\mathbf{R}$ is perpendicular to the sundial plane normal $\mathbf{n}$. Their dot product is zero, that is

$$\cos(\lambda)\cos(\theta) - \sin(\lambda)\tan(\gamma) = 0.$$

Thus

$$\theta = \arccos(\tan(\lambda) * \tan(\gamma)).$$

So at sunrise we find that the time in hours is given by

$$t = \arccos(\tan(\lambda) * \tan(\gamma)) * 12/\pi.$$

By sunrise here, we mean the time when the center of the sun is at the horizon. Normally, I think, sunrise is taken to be when the sun's surface is tangent to the horizon, and just peeks above the horizon. So one must calculate the time for the sun to move from the tangent point to the center point. This clearly depends on the latitude and also on the sun's declination angle, that is the time of year.

Note also that given the time of sunrise we can solve for the declination γ of the sun. Because we now have θ, we know $\mathbf{R}$

$$\mathbf{R} = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} + \tan(-\gamma))\mathbf{k}.$$

So we can calculate the azimuth angle of sunrise. This is the angle between $-\mathbf{R}$, and the north pointing direction at the latitude of observation. We use the minus sign because $\mathbf{R}$ points from the sun to the earth, and we need a vector pointing toward the sun. The azimuth is 90 degrees east of north at the vernal equinox, less than 90 during the summer, and greater than 90 in the winter.

These computations are done in the Fortran program **sundial.ftn**. This program computes a graphical layout of a sundial face in the **eg** format, which is converted to postscript with the program **eg2ps**.

6 The Vertical and Polar Sundials

Let us take a slightly more general approach to the construction of a sundial. Suppose we are at latitude λ. Suppose we have an xyz coordinate system at this latitude, so that the xy plane is a tangent plane to the Earth. And suppose the z axis points vertically upward at this latitude. Suppose the y direction is south, and the x direction is west. A vector $\mathbf{A}$ points in the direction of the north pole, where

$$\mathbf{A} = -\cos(\lambda)\mathbf{i} + \sin(\lambda)\mathbf{k}.$$

We shall consider a plane P through this pole axis $\mathbf{A}$, through the origin, and through the Sun. As the Sun rotates throughout the day, this plane also revolves. The lines on our sundial will be the intersections of this plane P with the sundial plane. In the case of a horizontal sundial this will be the xy plane. In the case of a vertical sundial, mounted on the side of a building, this will be the zx plane. A polar sundial consists of a ring surrounding the polar axis in the direction $\mathbf{A}$, where the shadow of a rod is cast onto the ring. The points on this ring will correspond to intersections of the ring with the plane P. Now let us construct P as a function of the time of day t. First we construct a couple of unit vectors that are perpendicular to $\mathbf{A}$, which we shall use to express the normal $\mathbf{n}$ to the plane P. Let $\mathbf{v}$ be a vector perpendicular to the pole vector $\mathbf{A}$ given by

$$\mathbf{v} = \sin(\lambda)\mathbf{j} + \cos(\lambda)\mathbf{k}$$

Let the second vector $\mathbf{u}$ be the unit x coordinate vector $-\mathbf{i}$. Then we can express the plane normal as

$$\mathbf{n} = \cos(\theta)\mathbf{u} + \sin(\theta)\mathbf{v}.$$

Let t be the time in hours starting from zero at midnight, so t is between 0 and 24. Then

$$\theta = \frac{t-12}{24}2\pi.$$

So at $t = 0$, $\theta = -\pi$ and at $t = 24$, $\theta = \pi$. So this specifies the plane P at time t. So we calculate our sundial layout by intersecting P with the horizontal plane, the vertical plane, or with the concentric ring around $\mathbf{A}$. We can use a general purpose subroutine to intersect pairs of planes. In the case of the polar sundial, the intersection points corresponding to the integral hours, is just a set of point laid out along the ring at 15 degree increments. In the case of the vertical sundial, if the face of the building is inclined to the direction from east to west, we can intersect with the actual plane of the building wall. Programs computing the angles on the faces of the sundials corresponding to local times, for the horizontal and vertical sundials, are respectively **sundialhor.ftn** and **sundialvert.ftn**.

7 The Declination and Right Ascension of the Sun

The declination of the sun varies between -23.44 and 23.44 degrees because the ecliptic plane is tilted by 23.44 degrees with the equatorial plane. The declination on a given day may be determined by measuring the altitude and knowing the latitude. Let the sun be projected in a normal direction to the equatorial plane. The angle from this projected point to the line of the vernal equinox, is called the right ascension of the sun. Let the direction of the vernal equinox be the x direction. As viewed from the north, the earth rotates about the sun in a counterclockwise direction. So also from the point of view of a coordinate system located at the center of the earth, the sun also rotates in a counterclockwise direction. So let a vector **u** in the equatorial plane be directed toward the point of projection of the sun on the equatorial plane. Let

$$\mathbf{u} = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}.$$

Then θ is the right ascension. Let the ecliptic plane pass through the x axis and have normal

$$\mathbf{n} = -\sin(\lambda)\mathbf{j} + \cos(\lambda)\mathbf{k},$$

where λ is the tilt angle of the ecliptic equal to 23.44 degrees. The vector

$$\mathbf{v} = \mathbf{u} + \mathbf{w},$$

where

$$\mathbf{w} = w\mathbf{k},$$

points toward the sun provided it lies in the ecliptic plane. Because **u** is a unit vector, this will be true if

$$w = \tan(\phi)$$

where ϕ is the declination of the sun. The condition for **w** to lie in the ecliptic plane is that it be perpendicular to the ecliptic normal **n**. So

$$\mathbf{n} \cdot \mathbf{v} = (-\sin(\lambda)\mathbf{j} + \cos(\lambda)\mathbf{k}) \cdot (\cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} + \tan(\phi)\mathbf{k}) = 0$$

That is

$$\sin(\theta) = \frac{\tan(\phi)}{\tan(\lambda)}.$$

There are two solutions for θ in the range $[0, 2\pi]$. So for every ϕ there are two values of the right ascension θ. For example, if the declination is positive, then the right ascension could be before the summer solstice, or could be after the summer solstice, where each one is equally displaced from the summer solstice right ascension value, which is $\theta = 90$ degrees. Right ascension is usually given in hours of earth rotation, so right ascension of the summer solstice is 6 hours.

Here is an example of this calculation. Let θ be the right ascension of the sun, ϕ the Declination of the sun, and $\lambda = 23.44$ degrees, is the angle between the equatorial plane and the ecliptic plane. Now ϕ is between -23.44

and 23.44 degrees. It must lie between plus or minus λ. Suppose we are in February and the declination is $\phi = -11.1$ degrees. This is $-.194$ radians. $\lambda = .409$ radians. We write θ in hours. Let

$$\eta = \arcsin(\frac{\tan(\phi)}{\tan(\lambda)})\frac{12}{\pi}.$$

We get

$$\eta = -1.794.$$

We shall always have

$$-6 \leq \eta \leq 6.$$

There are two possible values for the right ascension, which we shall call θ_1 and θ_2. These are symmetric values about the winter solstice, which has right ascension 18. Hence for the winter case of negative declination we have

$$\theta_1 = 24 - |\eta| = 22.206,$$

and

$$\theta_2 = 12 + |\eta| = 13.794.$$

Since we are in February, the actual right ascension is

$$\theta_1 = 22.206.$$

These are sidereal hours, 1 hour corresponds to 15 degrees of earth rotation. If the sun declination were positive, which occurs in summer, the two possible right ascensions would be

$$\theta_1 = |\eta|,$$

and

$$\theta_2 = 12 - |\eta|.$$

A MathCAD file for this Sun Right Ascension calculation is called **sunra.mcd**. A table of the sun declination and right ascension angle throughout the year is given in the book by Muirden on page 73.

8 The Time for the Sun to Rise

The sun has an apparent daily motion in a circle about the axis of the earth. Let the distance from the earth to the sun be r_1. Let the distance from the sun to the Earth's axis be r_2. Then if ϕ is the declination of the sun

$$\frac{r_2}{r_1} = \cos(\phi).$$

The angular velocity of the sun relative to the earth is

$$\omega = \frac{ds}{dt}\frac{1}{r_1},$$

where ds is the arclength change in the Sun's orbit in time dt.

The sun travels around its apparent orbit of radius r_2 in 24 hours. So

$$\frac{ds}{dt}\frac{1}{r_2} = \frac{2\pi}{24}.$$

Therefore the angular velocity of the sun relative to the earth is

$$\omega = \frac{ds}{dt}\frac{1}{r_1} = \frac{ds}{dt}\frac{1}{r_2}\frac{r_2}{r_1} = \frac{2\pi}{24}\cos(\phi) = \frac{\pi\cos(\phi)}{12}.$$

From Figure xxx, we see that the angle of sunrise is the colatitude $\mu = \pi/2 - \lambda$. At a given latitude λ, the sun rises if the sun's declination ϕ is less than the colatitude. Thus if the sun's declination is 10 degrees, then at latitudes above 80 degrees the sun does not rise or set but is visible during the entire day. The sun's diameter varies as its distance from the earth varies. On average it is about 32 minutes or .533333 degrees. The angular radius R of the sun as seen from the earth is on the average about .2667 degree, which we take to be approximately 1/4 degree. So the time Δt for the sun to move from a position just below the horizon to fully above the horizon is the time for the sun to travel angular distance

$$\frac{R}{\sin(\mu)},$$

where R in radians is given by

$$R = \frac{1}{4}\frac{\pi}{180} = \frac{\pi}{720}$$

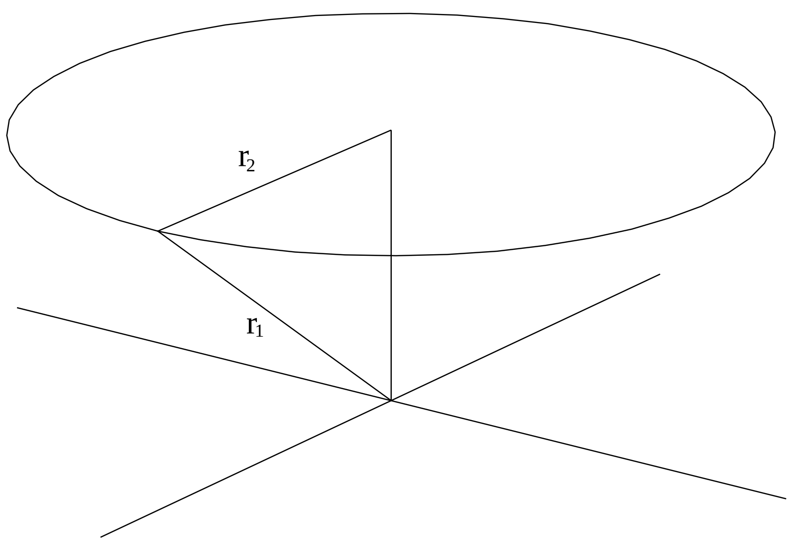

Figure 3: The angular velocity of the sun in its apparent orbit, as observed from the earth, depends on the radii r_1 and r_2, where r_1 is the distance from the earth to the sun, and r_2 is the distance from the Earth's axis to the sun.

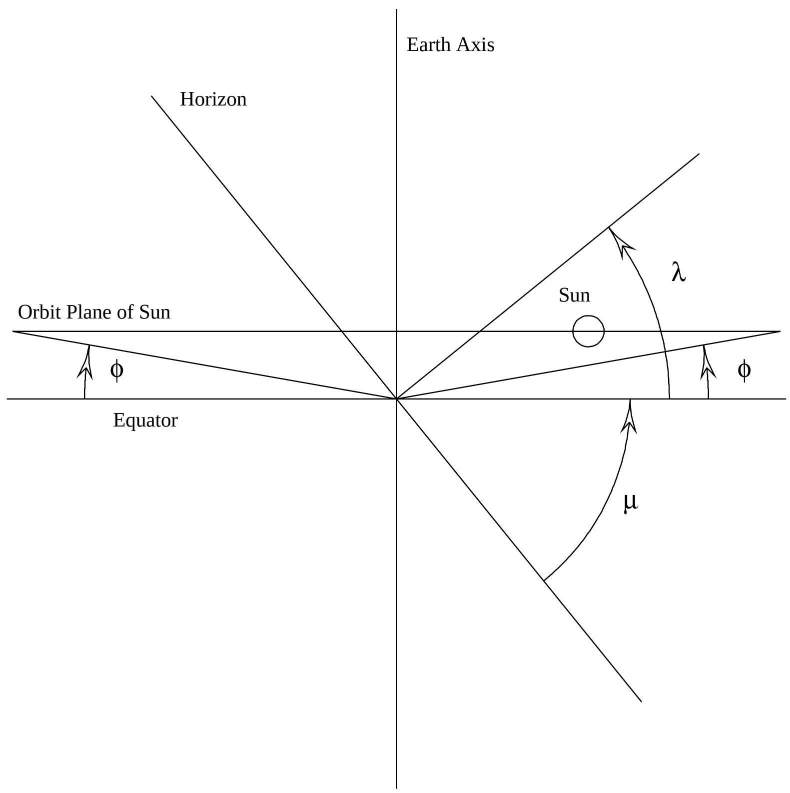

Figure 4: The time for the sun to rise above the horizon depends on the declination of the sun ϕ, and the latitude λ. The sun appears to orbit in a circle on a plane parallel to the equator. The angle of sunrise is the colatitude $\mu = \pi/2 - \lambda$.

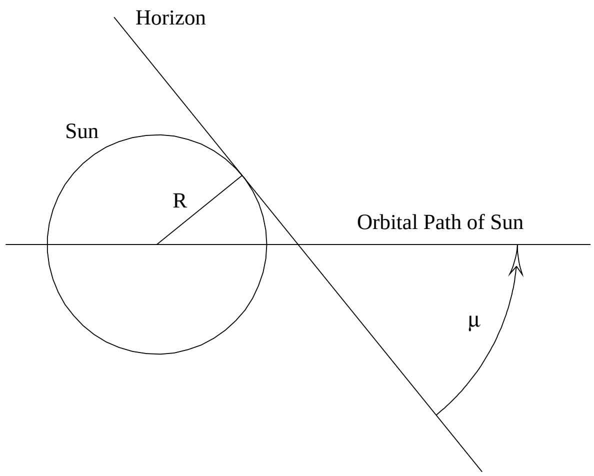

Figure 5: The sun just touching the horizon and about to rise. If the radius of the sun is R, the sun must move a distance $R/\sin(\mu)$ for its center to be at the horizon. The angle of sunrise is μ. μ equals the colatitude $\pi/2 - \lambda$.

radians. So

$$\omega \Delta t = \frac{R}{\sin(\mu)} = \frac{\pi}{720 \sin(\mu)}$$

So Δt in hours is

$$\Delta t = \frac{\pi}{720 \sin(\mu) \omega} = \frac{12\pi}{720 \sin(\mu) \pi \cos(\phi)} = \frac{1}{60 \sin(\mu) \cos(\phi)}.$$

Take for example a place where the latitude is 39 degrees, and when the sun's declination is $\phi = 10$ degrees. The colatitude is $\mu = 51$ degrees. The time Δt for the sun to rise, so that its center is at the horizon, is given by

$$\Delta t = \frac{1}{60(.771)(.985)}.$$

This is .0218 hours or 1.3 minutes.

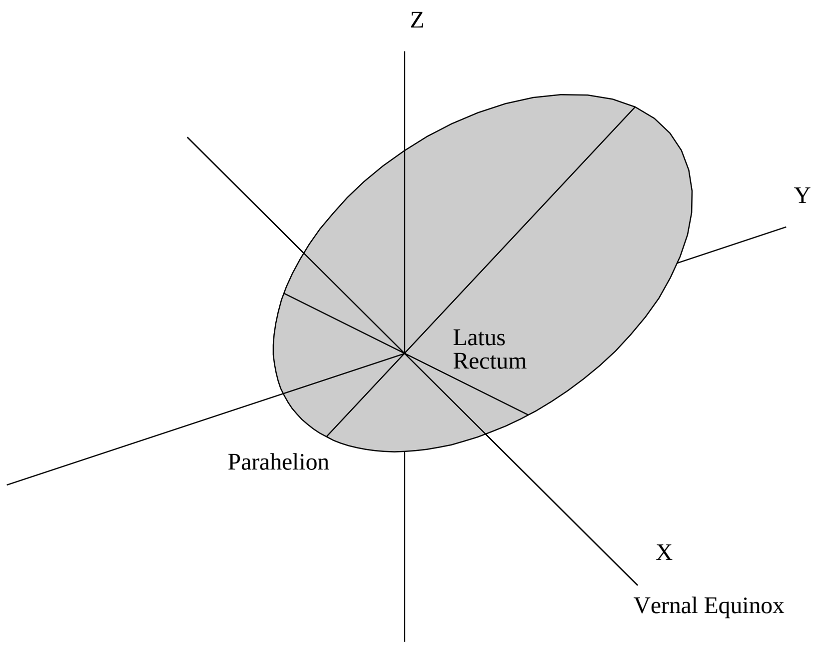

Figure 6: The XY plane is the equatorial plane (also called the celestial equatorial plane) of the earth. The ellipse is in the ecliptic plane and is the apparent path of the sun. Viewed from the north or Z direction, the sun orbits the earth in a counterclockwise direction. The eccentricity of the orbit is exaggerated here. The right ascension of the perihelion is about 289(24/360) hours, the right ascension of the vernal equinox is 0 by definition.

9 A Table of the Sun's Declination and Right Ascension

There is an approximate table on page 77 of **How to Use an Astronomical Telescope** by James Muirden, 1988. Values for the declination and the right ascension of the sun for the year 2010 are listed in a section below also.

10 Planetary Motion

The planets move about the sun in ellipses, with the sun at a focus. The book by Smart is a good source for information on this motion. Kepler's three laws of motion are consequences of this motion.

11 An Approximation To The Equation of Time

For a coordinate system located and fixed at the center of the earth, the sun appears to travel about the earth in an ellipse in the ecliptic plane. The earth is at a focus. The perihelion (the closest distance to the earth) of the Sun's orbit occurs around the 3rd or 4th day of January. At this time the radial distance to the sun is about 1.67 per cent closer than the average radial distance. The angular velocity of the sun at this time is about 3.37 per cent faster than the average angular velocity. The normal projection to the equatorial plane gives us the local solar time. The length of the local solar day is the time for successive sun transit of the local meridian. Such a solar day is not uniform throughout the year. A hypothetical mean sun travels uniformly in a circle on the equatorial plane. This mean sun is defined to have the same period as the the actual sun. The yearly average time difference between the real Sun and the Mean Sun is zero. At any day of the year the difference between the angular position of the mean sun and the projection of the actual sun to the equatorial plane, is called the equation of time. This difference is given in minutes. A plot of the equation of time is given in the book by Muirden on page 54. An Fourier approximation to the equation of time is

$$E = 9.87\sin(2B) - 7.23\cos(B) - 1.5\sin(B),$$

where

$$B = 2\pi(N - 81)/364,$$

and N is the day number, that is, $N = 1$ for January 1, and so on. To compute the exact equation of time we need to specify the elliptical orbit of the sun in the ecliptic plane as a function of time. At a given time t we project the sun to the equatorial plane, and compute the angle between the projected sun and the mean sun. Converting this angle to time, we get the time difference between solar time and mean solar time called $E(t)$, which is the value at t of the equation of time. The nonuniformity of the length of the solar day has two causes. First the Sun moves in an ellipse at nonuniform velocity. Secondly, the sun moves in the ecliptic plane that is tilted with the earth's equatorial plane. Although the Sun moves in an ellipse, the eccentricity is .017, so that this elliptical motion is very nearly circular. Even if the Sun were to travel at a uniform speed in an exact circle in the ecliptic, its projection to the equatorial plane, would become an ellipse. So there would be nonuniform motion because of this tilted ecliptic plane.

Day	Date	Minutes	Day	Date	Minutes
	Jan			July	
1	1	3.5502520	186	5	4.3129781
6	6	5.7072001	191	10	5.1108500
11	11	7.7062895	196	15	5.7365692
16	16	9.5022028	201	20	6.1575417
21	21	11.055469	206	25	6.3475137
26	26	12.333532	211	30	6.2874543
31	31	13.311619		Aug	
	Feb		216	4	5.9662213
36	5	13.973378	221	9	5.3809877
41	10	14.311266	226	14	4.5374189
46	15	14.326686	231	19	3.4495905
51	20	14.029862	236	24	2.1396529
56	25	13.439451	241	29	.63724794
	March			Sept	
61	2	12.581916	246	3	-1.0213026
66	7	11.490676	251	8	-2.7940145
71	12	10.205053	256	13	-4.6344346
76	17	8.7690499	261	18	-6.4929752
81	22	7.2300000	266	23	-8.3183495
86	27	5.6371274	271	28	-10.059067
	April			Oct	
91	1	4.0400564	276	3	-11.664943
96	6	2.4873203	281	8	-13.088578
101	11	1.0249104	286	13	-14.286766
106	16	-.30508916	291	18	-15.221783
111	21	-1.4657424	296	23	-15.862525
116	26	-2.4263397	301	28	-16.185458
	May			Nov	
121	1	-3.1632826	306	2	-16.175350

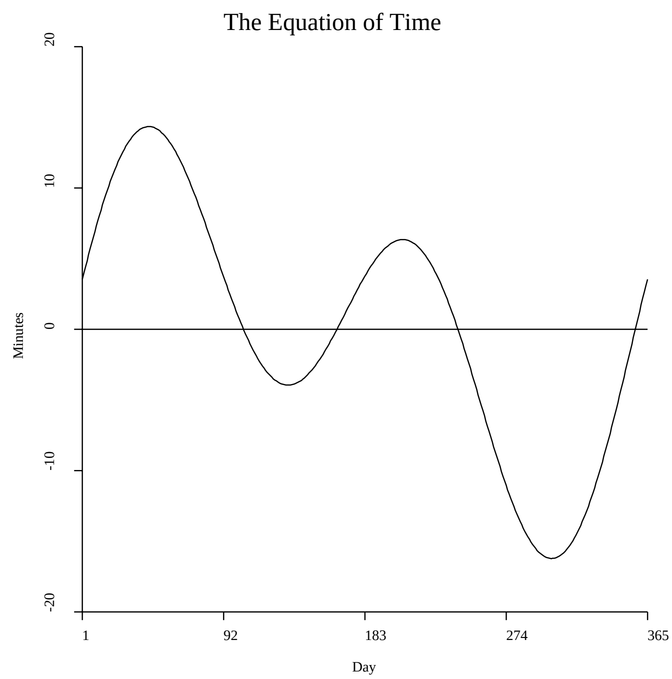

Figure 7: The equation of Time. This gives the minutes by which the mean sun leads the real Sun.

126	6	-3.6607482	311	7	-15.825765
131	11	-3.9111141	316	12	-15.139304
136	16	-3.9151318	321	17	-14.127591
141	21	-3.6818432	326	22	-12.810983
146	26	-3.2282439	331	27	-11.218048
151	31	-2.5787029		Dec	
	June		336	2	-9.3847869
156	5	-1.7641573	341	7	-7.3536507
161	10	-.82110636	346	12	-5.1723686
166	15	.20956490	351	17	-2.8926232
171	20	1.2838967	356	22	-.56861194
176	25	2.3562601	361	27	1.7444625
181	30	3.3808154			

12 Stellar Structure

The Sun's gravitational field increases the pressure and temperature of its gases and produces an extremely high density, so that hydrogen fusion occurs. The resulting thermonuclear reaction generates the radiation and the heat of the Sun and stars. From the details of this reaction and the observation of the Sun, it is possible to deduce much information about its structure. Hans Bethe won a Nobel prize for his work on stellar thermonuclear reactions. The great physicist S. Chandrasekhar did a lot of work on stellar structure. See his **An Introduction to the study of Stellar Structure** Dover, 1967. A modern book on the subject is **Stellar Structure and Evolution** by R. Kippenhahn and A. Weigert.

13 The Big Bang

The measured doppler red shift of the light from the stars, showed that the objects in the universe are moving away at higher velocities at greater distances. These facts were first discovered by the astronomer Hubble. Extrapolating backwards in time, using General Relativity theory, one can deduce that the universe originated at a single point in time and space. This origin of the universe is called the big bang. There is no center of the universe, rather the three dimensional expansion of the universe is analogous to the two dimensional expansion of a rubber balloon that is being inflated. Points on the surface of the expanding balloon are moving apart, but no point on the surface of the balloon can be considered the center of expansion. The details of the big bang are a consequence of the gravitational theory known as the General Theory of Relativity. The equations of General Relativity lead to the Friedmann equation

$$(\frac{dR}{dt})^2 = \frac{8\pi}{3}G\rho R^2 - kc^2$$

$R(t)$ is the scale factor of the universe at time t, k is the assumed curvature of the universe. We get

$$R(t) = At^{2/3}.$$

et cetera, ... See Krane chapter 16. Also see a book on General Relativity such as the book **Gravitation** by Wheeler, Kip Thorne et al. Also see the references for chapter 15 of Krane.

14 The Sidereal Period of the Earth

The time T_1 for a complete rotation of the earth is called the sidereal period. The time T_2 for a complete rotation of the earth relative to the mean sun is greater than T_2. This is the time from noon to the next noon. The mean sun moves uniformly in a circle in the equatorial plane and completes one revolution in 365.26 mean solar days. The earth rotates about the sun in a counterclockwise direction. The earth also rotates about its axis in a counterclockwise direction. So the mean sun rotates through angle $\frac{360}{365.26}$ degree in each solar day. So the earth must rotate through this extra angle each day in order for the mean sun to be at the meridian at noon each day. So

$$\frac{T_2}{T_1} = \frac{1 + 1/365.26}{1}.$$

Thus

$$T_1 = \frac{365.26}{366.26}T_2.$$

The mean solar day T_2 is equal to 24 hours by definition. Thus

$$T_1 = 24\frac{365.26}{366.26} = 23.93447278$$

hours. This is 23 hours, 56 minutes, and 4.1 seconds. This is the sidereal period of the earth, the time of a complete rotation of the earth with respect to the fixed stars.

15 Telescopes

The two traditional astronomical optical telescopes, are the refracting telescope, which uses spherical lenses, and the Newtonian reflecting telescope, which uses a spherical mirror. A Radio telescope is a very large parabolic antenna that receives radiation from stellar radio sources. Other specialized telescopes are employed to study for example, the radiation from the sun, and X-ray sources. The modern CCD (Charge Coupled Device) allows images from very dim optical sources to be captured and is a great improvement to photographic film. See the book **Telescopes** by Thornton Page and Lou Williams Paige. This is a collection of articles from the magazine **Sky and Telescope**.

16 The Vernal Equinox, and the Perihelion

The precession of the earth has a period of 26000 years, so the Equinox precesses by 1 degree every 72.22 years. The perihelion occurred in 2006, at the 4th of January at 15:30 UT (Universal Time, this is the modern version of mean solar time). The vernal equinox occurred at Mar 20, 2006, 18h 26m UT. This is about the $31 + 28 + 20 = 79$ day of the year. More precisely, the time of the year is $79 + 18.433333/24 = 79.768055$ day. The time of the vernal equinox is written as

$$t_{VE} = 79.768055.$$

The perihelion occurs each year, on the third or fourth of January. Perihelion occurred January 4th 2006 at 15:30 UT (reference NASA). This is the 4th day of the year. The time of year is $4 + 15.5/24 = 4.6458$ day. A reference for this data is the Google Web page: **Calculating exact Longitude from Right Ascension 19h 1m 22s**. At perihelion, the right ascension was 19h 1m 22s. This is 19.02277778 hours, and corresponds to 285.3416667 degrees. The perihelion occurs when the projection of the sun to the celestial equator has rotated 285.3416667 degrees from the vernal equinox.

The time from the perihelion to the vernal equinox is

$$79.768055 - 4.6458 = 75.122255$$

Let R_s be the right ascension of the Sun. Let H_s be the hour angle of the

Sun. Let T_{st} be the sidereal time. Recall that

$$T_{st} = H_s + R_s.$$

Consider the vernal equinox. Then $R_s = 0$ and so the sidereal time is the hour angle

$$T_{st} = H_s.$$

Let T be the UT (Universal Time). The mean sun is at the meridian at 12 pm, when its hour angle is zero. In a day the mean sun rotates through an angle of 360 degrees, that is an angle of 24 hours, from noon to noon. So T is the hour angle of the mean sun minus 12 hours. So

$$T_{st} = H_m + R_m = (T - 12) + R_m,$$

where R_m is the right ascension of the Mean Sun. Here T is known, since it is the U.T. at the Vernal Equinox. We do not know the hour angle H_s of the Sun itself, and thus we do not know the sidereal time T_{st} from this data. We could of course measure this by observation with a telescope. The selection of the position of the Mean Sun, an imaginary body, is somewhat arbitrary. If its position were changed, this would be equivalent to changing the U.T. time of the vernal equinox. However, one assumes that the position of the Mean Sun was chosen so that the average deviation of the Mean Sun from the real Sun is zero. This deviation is known as the Equation of time. From a published table, the approximate difference between the hour angle of the mean Sun and the hour angle of the real Sun, on March 20, is about 6 minutes, or about 1/10 of an hour. Hence

$$H_m - H_s = (T - 12) - H_s = 1/10,$$

hour. This means that the Sun transits the meridian at about 6 minutes after noon on March 20. Because

$$0 = T_{st} - T_{st} = H_m + R_m - (H_s + R_s),$$

this difference in hour angle is also equal to the difference in Right Ascension of the two bodies,

$$R_s - R_m = H_m - H_s = 1/10.$$

So at the Vernal Equinox, we have

$$T_s = H_s - R_s = H_s + 0 = H_s = (T - 12) - 1/10 = T - 12.1.$$

Now we can compute the equation of time at each day of the year, using this 1/10 hour difference at the Vernal Equinox. Given the normal anomaly ν, which is the angle formed by the focus of the ellipse, the perihelion point, and the current sun location, we can calculate the time for the sun to go from the perihelion to the current sun location. From the angle ν we know the right ascension S of the sun, using the right ascension at perihelion. So this calculation gives us the change in UT time at ν, from the UT time at perihelion. So we now know the time it takes the mean sun to get to the vernal equinox, and the rate of change of the right ascension of the mean sun with U.T. time. So we know the change ΔR_m of right ascension of the mean sun from its value at ν to its value at the vernal equinox. But we know the right ascension of the mean sun at the vernal equinox, so we know the right ascension of the mean sun at ν. Thus we know the difference

$$R_s - R_m$$

at the ν point, where the time T is known from the calculation. So we have the equation of time as a function of ν. And this function can be inverted to give the equation of time as a function of time. We can check that the average deviation of the mean sun and the sun is zero. The equation of time value is

$$(R_s - R_m)60 = (H_m - H_s)60,$$

in minutes. This is the time between local noon when the sun crosses the meridian, and the time when the mean sun crosses the meridian. By definition the later time will always be 12PM UT. One can see how this information is used in the orbit calculation by examining the listing of the program **orbit.ftn**.

17 Conics

This section is a version of the document conics.tex with title **Conics**. Any updates will be made to the document conics.tex itself. Not all of this material is used in this **Astronomy** document. The principal parts used are in the sections with titles **The Focus Directrix Form of a Conic** and **The Polar Coordinate Form of a Conic**. This material is used in the proofs of **Kepler's Laws of Planetary Motion**.

The document conics.tex is related to the document **An Elliptical Treatise** by James D. Emery (ellipse.tex). Some of the computations are realized in the C Plus Plus class **ellipse**, and in programs that use that class, and also in the programs **conics**.**ftn** and **pltconics**.**ftn**.

17.1 Computing A Canonical Representation From a General Algebraic Representation

Suppose the conic is given in matrix form as

$$E = \{p : p^* Ap = 0\},$$

where A is a symmetric matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We shall put E into canonical form by mapping the locus with a rotation through an angle θ, followed by a translation by (t_1, t_2). The combined transformation has matrix

$$T = \begin{bmatrix} c & -s & t_1 \\ s & c & t_2 \\ 0 & 0 & 1 \end{bmatrix}$$

The inverse of T is

$$R = \begin{bmatrix} c & s & e \\ -s & c & f \\ 0 & 0 & 1 \end{bmatrix}$$

where

$$e = -(t_1 c + t_2 s)$$

$$f = t_1 s - t_2 c$$

The mapping of the conic locus is

$$T(E) = \{Tp : p^* Ap = 0\}$$

$$= \{q : (Rq)^* ARq = 0\}$$

$$= \{q : q^* R^* A R q = 0\}$$

$$= \{q : q^* B q = 0\}$$

B is a symmetric matrix. The coefficients of the matrix

$$B = R^* A R$$

are

$$b_{11} = (ca_{11} - sa_{12})c - (ca_{12} - sa_{22})s$$

$$b_{12} = (ca_{11} - sa_{12})s + (ca_{12} - sa_{22})c$$

$$b_{13} = (ca_{11} - sa_{12})e + (ca_{12} - sa_{22})f + ca_{13} - sa_{23}$$

$$b_{22} = (sa_{11} + ca_{12})s + (sa_{12} + ca_{22})c$$

$$b_{23} = (sa_{11} + ca_{12})e + (sa_{12} + ca_{22})f + sa_{13} + ca_{23}$$

$$b_{33} = (ea_{11} + fa_{12} + a_{13})e + (ea_{12} + fa_{22} + a_{23})f + ea_{13} + fa_{23} + a_{33}$$
$$= a_{11}e^2 + 2a_{12}ef + a_{22}f^2 + 2a_{13}e + 2a_{23}f + a_{33}.$$

Notice that

$$b_{33} = (e, f, 1)A(e, f, 1)^*.$$

We shall choose θ so that b_{12} is zero. We have

$$b_{12} = (c^2 - s^2)a_{12} + sc(a_{11} - a_{22}) = 0.$$

That is

$$0 = a_{12}\cos(2\theta) + (1/2)(a_{11} - a_{22})\sin(2\theta).$$

So the rotation angle is determined by

$$\tan(2\theta) = \frac{2a_{12}}{a_{22} - a_{11}}.$$

Thus

$$2\theta = \text{atan2}(2a_{12}, a_{22} - a_{11}).$$

Let us write

$$b_{13} = c_{11}e + c_{12}f - g_1$$

$$b_{23} = c_{21}e + c_{22}f - g_2,$$

where

$$c_{11} = (ca_{11} - sa_{12})$$

$$c_{12} = (ca_{12} - sa_{22})$$
$$c_{21} = (sa_{11} + ca_{12})$$
$$c_{22} = (sa_{12} + ca_{22})$$
$$g_1 = -(ca_{13} - sa_{23})$$
$$g_2 = -(sa_{13} + ca_{23})$$

For purpoaes of classification, the transformation can be done in two steps, a pure rotation where,

$$e = 0, f = 0.$$

This is followed by a pure translation, where the rotation angle is zero.

If the rotation angle is zero, then

$$c_{11} = a_{11}$$

$$c_{12} = a_{12}$$
$$c_{21} = a_{12}$$
$$c_{22} = a_{22})$$
$$g_1 = -a_{13}$$
$$g_2 = -a_{23}.$$

After such a pure rotation, it is easy to see the conic type from the properties of the coefficients, where the xy cross term is absent.

Here let us return to the single transformation. We can make $b_{13} = 0$ and $b_{23} = 0$, if we can find e and f so that

$$c_{11}e + c_{12}f = g_1$$

$$c_{21}e + c_{22}f = g_2.$$

We have a linear equation to solve for e and f.

The determinant of this system is

$$d_2 = a_{11}a_{22} - a_{12}^2.$$

Because the upper 2 by 2 subdeterminant of R and R^T is 1, d_2 also equals

$$b_{11}b_{22} - b_{12}^2 = b_{11}b_{22}.$$

If $d_2 = 0$, then the quadric is neither an ellipse or nor a hyperbola.

If $d_2$is not zero, then using the values of e and f that make b_{13} and b_{23} zero, we

get that the matrix of the transformed canonical form to be

$$B = \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}.$$

There are several types of conics for the cases where d_2 is not zero.

Conic Type 1, Ellipse. All coefficients b_{11}, b_{22} and b_{33}, are not zero. The coefficients b_{11} and b_{22} have the same sign, which differs from the sign of b_{22}. Then the canonical equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where the radii of the ellipse are

$$a = \sqrt{-b_{33}/b_{11}}$$

and

$$b = \sqrt{-b_{33}/b_{22}}.$$

Conic Type 2, Hyperbola. All coefficients b_{11}, b_{22} and b_{33}, are not zero. The coefficients b_{11} and b_{22} have different signs. There are two subcases: $b_{11}b_{33} < 0$ and $b_{11}b_{33} > 0$. If $b_{11}b_{33} < 0$, then the canonical equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where

$$a = \sqrt{-b_{33}/b_{11}}$$

and

$$b = \sqrt{b_{33}/b_{22}}.$$

If we define the axis of the hyperbola to be the line of symmetry that meets the hyperbola at two points, then the axis of the original conic is at angle $-\theta$, where θ is the rotational angle of the canonical transformation.

If $b_{11}b_{33} > 0$, then the canonical equation is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1,$$

where

$$a = \sqrt{b_{33}/b_{11}}$$

and

$$b = \sqrt{-b_{33}/b_{22}}.$$

Then the axis of the original conic is at angle $-(\theta + \pi/2)$, where θ is the rotational angle of the canonical transformation.

Conic Type 3, Intersecting lines.

If b_{11} and b_{22} have different signs, and $b_{33} = 0$, then the conic has equation

$$|b_{11}|x^2 = |b_{22}|y^2,$$

which is equivalent to two equations of a line through the origin

$$y = \frac{\alpha}{\beta}x,$$

and

$$y = -\frac{\alpha}{\beta}x,$$

where

$$\alpha = \sqrt{|b_{11}|}, \beta = \sqrt{|b_{22}|}$$

Conic Type 4, Point.

If b_{11} and b_{22} have the same signs, and $b_{33} = 0$, then the conic has equation

$$|b_{11}|x^2 = -|b_{22}|y^2,$$

whose only solution is

$$x = 0, y = 0.$$

Conic Type 5, No real points.

If b_{11} and b_{22} have the same signs, and b_{33} a different sign, then the conic has equation

$$|b_{11}|x^2 + |b_{22}|y^2 = -|b_{33}|,$$

which has no real solution.

Notice that the determinant of A is preserved by the transformation and so is also equal to the determinant of B. We shall call this determinant d_3.

Then for an ellipse, type 1, $d_3 < 0$. For a hyperbola, type 2, $d_3 > 0$, and for type 5, no real points, $d_3 < 0$. If $d_3 = 0$ then the conic is a point or a pair of intersecting lines.

This exhausts the cases where d_2 is not zero.

If $d_2 = 0$, then we can not necessarily choose the translation (e, f) so that $b_{13} = 0$ and $b_{23} = 0$. In these cases, either b_{11} or b_{22} is zero.

Suppose b_{11} is not zero, and $b_{22} = 0$.

In this case we have three equations

$$b_{13} = c_{11}e + c_{12}f - g_1$$

$$b_{23} = c_{21}e + c_{22}f - g_2$$

$$b_{33} = (e, f, 1)A(e, f, 1)^*.$$

Consider a second set of equations obtained by setting the b coefficients to zero.

$$0 = c_{11}e + c_{12}f - g_1$$

$$0 = c_{21}e + c_{22}f - g_2$$

$$0 = (e, f, 1)A(e, f, 1)^*.$$

We claim that if b_{11} is not zero, then the 1st and 3rd equation of the second set have a solution. If b_{22} is not zero, then the 2nd and 3rd equations have a solution. This follows by examining the component transformations, by first applying the rotation, then the translation. Finding the translation is equivalent to completing squares. For example, it is equivalent to transforming the terms like

$$k_1 x^2 + k_2 x$$

into terms like

$$k_1 (x + m_1)^2 + m_2.$$

Solving the first and third equation is equivalent to finding a finite intersection point

$$(e, f, 1).$$

This is the intersection of the line with coordinates $(c_{11}, c_{12}, -g_1)$, with the conic A.

Thus our canonical matrix becomes

$$B = \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & 0 & b_{23} \\ 0 & b_{23} & 0 \end{bmatrix}.$$

The equation is

$$b_{23} y = -b_{11} x^2,$$

$$4Fy = x^2.$$

If b_{23} is not zero, then this represents a parabola.

The focal distance is

$$F = -\frac{b_{23}}{4b_{11}}.$$

Otherwise equation

$$b_{11} x^2 = -b_{33},$$

is two parallel lines, given by

$$x = \pm \frac{b_{33}}{b_{11}}.$$

If $b_{33} b_{11} > 0$, then there is no real solution.

Suppose b_{22} is not zero, and $b_{11} = 0$.

Thus our canonical matrix becomes

$$B = \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & b_{22} & 0 \\ b_{13} & 0 & b_{33} \end{bmatrix}.$$

The equation is

$$b_{13}x = -b_{22}y^2 - b_{33}$$

$$4Fx = y^2 + c$$

which if b_{13} is not zero is a parabola.

$$F = -\frac{b_{13}}{4b_{22}}.$$

Otherwise

$$b_{22}y^2 = -b_{33}$$

is two parallel lines

$$y = \pm\frac{b_{33}}{b_{22}}$$

or has no real roots if $b_{33}b_{22} > 0$.

Finally if

$$b_{11} = b_{22} = 0$$

Then we have the straight line

$$b_{13}x + b23y + b_{33} = 0$$

Note that if $b_{33} = 0$, then the ellipse degenerates into a point.

If an ellipse has a center $C = (c_x, c_y)$, then the original transformation T maps it to the origin, which is given in homogeneous coordinates as

$$(0, 0, 1)^*$$

Hence R, which is the inverse of T, maps the origin to the original ellipse center.

Therefore

$$\begin{bmatrix} c_x \\ c_y \\ 1 \end{bmatrix} = \begin{bmatrix} c & s & e \\ -s & c & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e \\ f \\ 1 \end{bmatrix}$$

So (e, f) is the center of the original conic, if it is an ellipse, or a hyperbola. If it is a parabola, then (e, f) is the vertex. The center of a parabola is at infinity.

Alternately, for the case of the ellipse or hyperbola, the first two rows of matrix A are homogeneous coordinates of diameter lines (being the polars of points at infinity), so their intersection is the center.

If the matrix A is nonsingular, then the unique center (c_x, c_y) is the solution of

$$A \begin{bmatrix} c_x \\ c_y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This follows because the polar of every point at infinity

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

is a diameter given by

$$[x, y, 0]A.$$

So every diameter meets the center if and only if

$$[x, y, 0]A \begin{bmatrix} c_x \\ c_y \\ 1 \end{bmatrix} = 0$$

if and only if

$$A \begin{bmatrix} c_x \\ c_y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

17.2 The Focus Directrix Form of a Conic

Consider the locus of points whose distance from a point and from a line have a constant ratio. Suppose we are given a point $f = (c, 0)$ and a line ℓ with equation $x = s$. The point f is called the focus and the line ℓ is called the directrix. The locus of points is defined by

$$\{p : d(f, p) = \epsilon d(l, p)\},$$

where ϵ is called the eccentricity. We shall show that this represents a conic. We get the canonical conics when special conditions are met. These conditions will make the parameters c, s, and ϵ dependent. We may consider s to be dependent on c and ϵ. Let us require that $c > 0, \epsilon > 0$. Our equation is

$$(x-c)^2 + y^2 = \epsilon^2(x-s)^2.$$

Let us first assume that ϵ is not 1. Expanding we get

$$x^2 - 2cx + c^2 + y^2 = \epsilon^2(x^2 - 2sx + s^2).$$

By requiring that the x term vanish, we shall get the canonical form of the ellipse, or the canonical form of the hyperbola. This requires that

$$-2cx = -\epsilon^2 2sx,$$

and thus that the x coordinate of the directrix is given by

$$s = \frac{c}{\epsilon^2}.$$

Then

$$x^2(1-\epsilon^2) + c^2 + y^2 = \frac{c^2}{\epsilon^2}.$$

So

$$x^2 + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{\epsilon^2}.$$

Letting

$$a = \frac{c}{\epsilon},$$

we have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-c^2/a^2)} = 1.$$

Then

$$\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1.$$

If $c < a$ then $\epsilon < 1$ and we take

$$b^2 = a^2 - c^2,$$

and get the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If $c > a$ then $\epsilon > 1$ and we take

$$b^2 = c^2 - a^2,$$

and get the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Now we consider the case of $\epsilon = 1$. Our expanded equation is

$$x^2 - 2cx + c^2 + y^2 = x^2 - 2sx + s^2.$$

So the x^2 term does not appear. We have

$$-2cx + c^2 + y^2 = -2sx + s^2.$$

We choose to eliminate the constant term to make the equation simple. We take $c^2 = s^2$. Specifically we take $s = -c$. Then we get the canonical equation of the parabola

$$x = \frac{1}{4c}y^2.$$

We have shown that each canonical conic has the focus directrix property, and hence that every ellipse, hyperbola, or parabola has this property.

Notice that for the ellipse, the focus lies between the center and the directrix. For the the hyperbola, the directrix lies between the center and the focus, and for the parabola the center is at infinity, and the vertex lies halfway between the directrix and the focus.

For a circle, a special case of the ellipse, the eccentricity is zero, the focus lies at the center, and the directrix is at infinity.

17.3 The Polar Coordinate Form of a Conic

Let us use the focus directrix property, with the focus at the coordinate origin. Let us express the curve in polar coordinates. Let the directrix be at

$x = q$ with q positive. Let the eccentricity be ϵ. Then a point on the curve locus is $p = (x, y) = (r\cos(\theta), r\sin(\theta),$

$$r = \sqrt{x^2 + y^2},$$

and the focus point is $f = (0, 0)$. Then we have

$$r = d(f, p) = \epsilon d(\ell, p) = \epsilon|q - r\cos(\theta)|.$$

Assuming that

$$q > r\cos(\theta),$$

then the quantity between the absolute value signs is positive. So we have

$$r = \frac{\epsilon q}{1 + \epsilon\cos(\theta)}.$$

Hence at least part of the conic coincides with the locus of this polar equation. Now we consider the complete locus of this equation as θ varies from 0 to 2π. Then r may take on negative values, specifically it will do that when ϵ is greater than 1.

We have $p = (x, y) = (r\cos(\theta), r\sin(\theta)$, where r can be negative. Then

$$d(f, p) = |r| = |\frac{\epsilon q}{1 + \epsilon\cos(\theta)}|$$

$$d(\ell, p) = |q - x| = |q - rcos(\theta)| = |q - \frac{\epsilon q}{1 + \epsilon\cos(\theta)}\cos(\theta)|$$

$$= |\frac{q(1 + \epsilon\cos(\theta)) - \epsilon q\cos(\theta)}{1 + \epsilon\cos(\theta)}|$$

$$= |\frac{q}{1 + \epsilon\cos(\theta)}|.$$

And so

$$d(f, p) = |r| = |\frac{\epsilon q}{1 + \epsilon\cos(\theta)}| = \epsilon d(\ell, p).$$

Thus every point of the locus

$$r = \frac{\epsilon q}{1 + \epsilon\cos(\theta)},$$

for $0 <= \theta <= 2\pi$, lies on the conic with focus at the origin, eccentricity ϵ, and directrix $x = q$ for $q > 0$. Note that if ϵ is greater than 1, then r goes to infinity at the asymptotes where the denominator in the expression for r goes to zero.

In the case of the canonical forms of the ellipse and the hyperbola, where the center is at the origin, the directrix lies at x coordinate

$$s = \frac{c}{\epsilon^2},$$

where c is the distance from the center to the focus. Hence the distance from the focus to the directrix, which is q is

$$q = |s - c| = |\frac{c}{\epsilon^2} - c|$$

So

$$c = \frac{q}{|1/\epsilon^2 - 1|}.$$

Thus we can locate the centers from the polar form parameters. For $\epsilon > 1$ the x coordinate of the center of the hyperbola is c, and $c > q$, so that the center is to the right of the directrix. The angular directions θ of the asymptotes satisfy

$$1 + \epsilon \cos(\theta) = 0$$

For $\epsilon < 1$ the x coordinate of the center of the ellipse is $-c$. The center lies to the left of the focus.

For $\epsilon = 1$ the vertex of the parabola, has x coordinate $q/2$.

The locus for

$$r = \frac{\epsilon q}{1 - \epsilon \cos(\theta)}$$

is a reflection through the origin of the previous locus, because it may be written as

$$r = \frac{\epsilon q}{1 + \epsilon \cos(\theta - \pi)}.$$

The locus for

$$r = \frac{\epsilon q}{1 + \epsilon \sin(\theta)}$$

is similarly a conic, as is the locus for

$$r = \frac{\epsilon q}{1 - \epsilon \sin(\theta)}.$$

These conics are oriented along the y axis rather than the x axis.

17.4 The Two Body Problem for an Inverse Square Force

The two body problem is reduced to the one body problem by introducing the reduced mass μ as in the Berkely Physics book **Mechanics** by Kittle, Knight, and Rudrman.

Let there be two particles, a particle at $\mathbf{r_1}$ of mass M_1, and a particle at $\mathbf{r_2}$ of mass M_2. Let $\mathbf{r}$ be the vector from M_2 to M_1.

$$\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}.$$

The equations of motion are

$$M_1 \frac{d^2\mathbf{r_1}}{dt} = -\frac{GM_1M_2}{\|\mathbf{r}\|^3}\mathbf{r}$$

and

$$M_2 \frac{d^2\mathbf{r_2}}{dt} = \frac{GM_1M_2}{\|\mathbf{r}\|^3}\mathbf{r}.$$

Subracting equation two from equation one, we find that a single equation for the relative motion, in vector form, in cartesian coordinates, is

$$\frac{d^2\mathbf{r}}{dt^2} = -(1/M_1 + 1/M_2)\frac{GM_1M_2}{\|\mathbf{r}\|^3}\mathbf{r}.$$

If we introduce the reduced mass μ by

$$\frac{1}{\mu} = \frac{1}{M_1} + \frac{1}{M_2},$$

the equation for the relative motion of M_1 about M_2 becomes

$$\mu\frac{d^2\mathbf{r}}{dt^2} = -\frac{GM_1M_2}{\|\mathbf{r}\|^3}\mathbf{r}.$$

The center of mass of the system is given by

$$\mathbf{R}_{cm} = \frac{M_1\mathbf{r_1} + M_2\mathbf{r_2}}{M_1 + M_2}.$$

Let a coordinate system be centered at M_2. Using polar coordinates (r, θ), where

$$r = \|\mathbf{r}\|,$$

the potential giving the central force

$$-\frac{\partial V}{\partial r} = -\frac{GM_1M_2}{r^2},$$

for this one body system is

$$V = -\frac{GM_1M_2}{r}.$$

The kinetic energy of the equivalent one body system is

$$T = \frac{1}{2}\mu(r^2 + r^2\omega^2),$$

where ω is the angular velocity $\dot{\theta}$. We might compare this to the sum of the kinetic energies of the pair of bodies. So the Lagrangian of the one body system is

$$L = T - V = \frac{1}{2}\mu(r^2 + r^2\dot{\theta}^2) + \frac{GM_1M_2}{r}.$$

Hence the pair of Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0,$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0,$$

are (See Solkolnikov **Tensor Analysis**)

$$\mu\ddot{r} - \mu r\dot{\theta}^2 + \frac{GM_1M_2}{r^2} = 0,$$

and

$$\frac{d}{dt}(\mu r^2\dot{\theta}) = 0.$$

The second Lagrange equation shows that the angular momentum

$$J = \mu r^2\dot{\theta}$$

is constant. It can be shown that this angular momentum for the equivalent one body problem, equals the angular momentum of the original two body system (See the Berkeley **Mechanics**).

From this we deduce a relationship between operators

$$\frac{d}{dt} = \frac{J}{\mu r^2}\frac{d}{d\theta}.$$

Therefore the first Lagrange equation may be written as

$$\mu\frac{J}{\mu r^2}\frac{d}{d\theta}\left(\frac{J}{\mu r^2}\frac{dr}{d\theta}\right) - \mu r\left(\frac{J}{\mu r^2}\right)^2 = -\frac{GM_1M_2}{r^2}.$$

We can get rid of the troublesome $1/r^2$ inside the parentheses of the first term by letting

$$u = \frac{1}{r}.$$

Then

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}.$$

We get a simple differential equation in u which has constant coefficients, namely

$$\frac{d^2u}{d\theta^2} + u = \frac{GM_1M_2\mu}{J^2}.$$

Let

$$K = \frac{GM_1M_2\mu}{J^2}$$

The general solution of this differential equation is

$$u = K + e\cos(\theta+\alpha)$$

for some constants e and α. Letting

$$se = \frac{1}{K}$$

we have

$$u = \frac{1}{se}(1 - e\cos(\theta+\alpha)),$$

and so

$$r = \frac{se}{1 - e\cos(\theta+\alpha)}.$$

So the first body moves in a conic curve relative to the second body. In **Mechanics** several properties of the motion are calculated. For example, in the case of an ellipse with $e < 1$, integrating the angular momentum expression, we get an expression for the period.

$$T = \frac{2\pi ab\mu}{J},$$

where a is the semimajor axis and b is the semiminor axis, and πab is the area of the ellipse. Because $2a = r_{max} + r_{min}$ we have

$$2a = \frac{2}{1-e^2}\frac{J^2}{GM_1M_2\mu}.$$

Solving this for J^2 and substituting in the square of the expression for the period, we obtain Kepler's third law

$$T^2 = \frac{4\pi^2 a^3}{G(M_1+M_2)}.$$

From this, knowing T,a,and the mass of the earth M_1 from the acceleration of gravity, we can deduce the mass of the sun.

The center of mass of the system lies on the line joining the two masses, where the ratio, of the distance to M_1, to the distance to M_2, is constant. So the first body rotates about the center of mass with polar coordinate equation

$$r = k\frac{B}{1 - e\cos(\theta+\alpha)},$$

for a coordinate system through the center of mass, and for some constant k, which depends on M_1 and M_2. Thus each body moves in a conic section about the center of mass of the pair of bodies.

Some books do not use Lagranges equations to get the equation of motion. They introduce the centrifugal force directly and mention a rotating coordinate system. These mechanics books introduce this so called rotating coordinate system in order to obtain the centrifugal force, and the Coriolis force. This is usually done while considering motion on the rotating earth. But these somewhat confusing fictional forces are no more than the components of acceleration in cylindrical coordinates, with respect to a nonrotating inertial system. When say a missile is fired on the earth, its coordinate velocity $\dot{\theta}$ is with respect to an inertial coordinate system. Its velocity and position are compared to earth coordinates, using the constant angular velocity of the earth, to give the Coriolis deflection.

17.5 Conic Sections: The Ice Cream Cone Proof

Given a cone containing a small ball, and a larger ball. A plane section through the cone that is tangent to the two balls intersects the cone surface in a curve that is an ellipse, where the tangent points are the focii of the ellipse. So suppose a point P is on this intersection curve. Let a line on the cone be drawn through P. Then this line through P is tangent to each the spheres. Consider say the larger sphere. A second line through P and the contact point with the sphere is also a tangent to the sphere, so the length of these two tangent lines are equal. So the distance from P to the contact point or focus equals the distance from P to the circle of contact of the cone and sphere. This property is also true of the second sphere, so the sume of the distances from P to the two focii is equal to the sum of these two distances $d_{P1} + d_{P2}$ from P to the two sphere contact circles. But this sum is constant, being the distance between the two sphere contact circles on the cone surface. It follows that the sum of the distance from P to the two foci also equals this distance, therefore this intersection curve is an ellipse.

A similar argument establishes that by increasing the angle of intersection of the cone we get the parabola and the hyperbola.

References: Apostle, **Calculus** Volume 1, or Morrey, **Calculus**.

17.6 Derivation of the Equation of the Ellipse From the Focal Property

An ellipse is the locus of points (x, y), such that the sum of the distances to two fixed points is a constant. Let the distance between the fixed points be $2c$. These points are called the focal points of the ellipse. Suppose these points have coordinates $(-c, 0)$ and $(c, 0)$, with $c > 0$. Let point (x, y) be a point on the ellipse. Let d_1 be the distance from (x, y) to the first focal point, and d_2 the distance from (x, y) to the second focal point. Then

$$d_1 = \sqrt{(x-c)^2 + y^2},$$

and

$$d_2 = \sqrt{(x+c)^2 + y^2}.$$

Consider the special case of a point on the ellipse, where the point is on the x-axis, with coordinates $(a, 0)$, where $a > c > 0$. The sum of distances to the focal points is $2a$ here, because

$$d_1 + d_2 = (a - c) + ((a - c) + 2c) = 2a.$$

So a point is on the ellipse if and only if

$$d_1 + d_2 = 2a.$$

Now suppose point $(0, b)$ is on the ellipse, where $b > 0$. Then

$$d_1 + d_2 = \sqrt{c^2 + b^2} + \sqrt{c^2 + b^2} = 2\sqrt{c^2 + b^2} = 2a.$$

So

$$c^2 + b^2 = a^2.$$

The numbers a and b are called the semi-axes of the ellipse.
Proposition. The canonical equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Proof. Squaring

$$d_1 = 2a - d_2,$$

we find

$$(x - c)^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2.$$

Then

$$(x - c)^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2,$$

so

$$-4xc = 4a^2 - 4a\sqrt{(x + c)^2 + y^2}.$$

So

$$xc + a^2 = a\sqrt{(x + c)^2 + y^2}.$$

Squaring and dividing by a^2, we find

$$\frac{x^2c^2 + 2xca^2 + a^4}{a^2} = (x + c)^2 + y^2,$$

$$\frac{x^2c^2}{a^2} + 2xc + a^2 = (x + c)^2 + y^2.$$

$$\frac{x^2c^2}{a^2} + 2xc + a^2 = x^2 + 2xc + c^2 + y^2$$

Then

$$\frac{x^2c^2}{a^2} + a^2 = x^2 + c^2 + y^2$$

$$\frac{x^2(a^2 - b^2)}{a^2} + a^2 = x^2 + (a^2 - b^2) + y^2$$

$$\frac{x^2(a^2 - b^2)}{a^2} = x^2 + -b^2 + y^2$$

$$x^2 - \frac{x^2b^2}{a^2} = x^2 + -b^2 + y^2$$

$$\frac{x^2b^2}{a^2} = b^2 - y^2.$$

Dividing by b^2, we have

$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}.$$

Then we have the desired equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

17.7 Parabolic Mirrors

Consider the standard parabola with axis in the direction of the x axis.

$$y^2 = 4cx,$$

where the focal point is at $(c, 0)$. A vector representation with parameter y of the upper portion of the parabola is

$$\mathbf{r}(y) = x\mathbf{i} + y\mathbf{j} = \frac{y^2}{4c}\mathbf{i} + y\mathbf{j}$$

A tangent vector is

$$\frac{d\mathbf{r}}{dy} = \frac{y}{2c}\mathbf{i} + \mathbf{j},$$

or

$$\mathbf{T} = y\mathbf{i} + 2c\mathbf{j}.$$

A unit normal vector is

$$\mathbf{N} = \frac{1}{\sqrt{4c^2 + y^2}}(2c\mathbf{i} - y\mathbf{j}).$$

The cosine of the angle θ_1 between this normal and the unit vector in the x direction is

$$\cos\theta_1 = \mathbf{N} \cdot \mathbf{i} = \frac{2c}{\sqrt{4c^2 + y^2}}.$$

A vector from the point $\mathbf{r}$ on the parabola to the focus is

$$c\mathbf{i} - (\frac{y^2}{4c}\mathbf{i} + y\mathbf{j}).$$

Multiplying by $4c$, a parallel vector is

$$(4c^2 - y^2)\mathbf{i} - 4cy\mathbf{j}.$$

So a unit vector in this direction is

$$\mathbf{v} = \frac{1}{\sqrt{(4c^2 - y^2)^2 + (4cy)^2}}(4c^2 - y^2)\mathbf{i} - 4cy\mathbf{j}.$$

Expanding the expression under the radical, we have

$$(4c^2 - y^2)^2 + (4cy)^2 = 16c^4 + 8c^2y^2 + y^4 = (4c^2 + y^2)^2,$$

So the unit vector becomes

$$\mathbf{v} = \frac{(4c^2 - y^2)\mathbf{i} - 4cy\mathbf{j}}{4c^2 + y^2}.$$

The cosine of the angle θ_2 between this vector and the normal vector is

$$\cos\theta_2 = \mathbf{N} \cdot \mathbf{v}$$

$$= \frac{2c(4c^2 - y^2) + 4cy^2}{(4c^2 + y^2)^{3/2}}$$

$$= \frac{8c^3 + 2cy^2}{(4c^2 + y^2)^{3/2}}$$

$$= \frac{2c(4c^2 + y^2)}{(4c^2 + y^2)^{3/2}}$$

$$= \frac{2c}{\sqrt{4c^2 + y^2}}$$
$$= \cos\theta_1.$$

Thus any ray from the focal point is reflected into a ray parallel to the axis of the ellipse.

This is a solution to problem 21 on page 313, of Apostle's **Calculus**, volume 1, first edition.

The point at infinity in the direction of the x axis may be considered the second focal point of the parabola, because as the cone sectioning plane is tilted towards a plane parallel to the side of the cone, an elliptical section becomes long and thin and the second focal point goes to infinity as the ellipse turns into a parabola. So a ray from the first focal point is reflected to the second focal point in the same way as for an ellipse.

For parallel rays not in the direction of the parabolic axis, we get an envelope curve near the focal point. This is called a caustic curve.

17.8 Bibliography for Conics

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18 The Polar Equation of an Ellipse

The polar coordinate form of a conic section takes the form

$$r = \varepsilon(d - r\cos(\theta)),$$

or

$$r = \frac{\varepsilon d}{1 + \varepsilon\cos(\theta)}.$$

The origin of the coordinate system is at a focus of the ellipse. Polar coordinates are r and θ. The distance from the focus to a point on the conic is r. The directrix is a vertical line located at distance d from the focus. The eccentricity is ε. The equation says that the distance from the focus to a point on the conic is equal to the eccentricity times the distance from the point to the directrix. For an ellipse the eccentricity is less than 1. We shall convert to Cartesian coordinates to find a standard form for the ellipse. Thus

$$r = \sqrt{x^2 + y^2},$$

and

$$x = r\cos(\theta).$$

So

$$\begin{aligned} x^2 + y^2 &= \varepsilon^2(d - x)^2 \\ &= \varepsilon^2(d^2 - 2dx + x^2) \end{aligned}$$

Then

$$(1 - \varepsilon^2)x^2 + y^2 = \varepsilon^2 d^2 - 2d\varepsilon^2 x.$$

Knowing how the calculation will come out, let us simplify the algebra a bit, by introducing symbols a, b, and c, and by taking

$$\varepsilon = c/a,$$

$$d = b^2/c$$

and

$$a^2 = b^2 + c^2.$$

From this we have

$$1 - \varepsilon^2 = b^2/a^2,$$

$$\varepsilon^2 = c^2/a^2,$$

$$\frac{\varepsilon^2}{1-\varepsilon^2} = \frac{c^2}{b^2}.$$

Dividing by

$$1-\varepsilon^2,$$

we get

$$x^2 + \frac{y^2}{1-\varepsilon^2} = \frac{\varepsilon^2 d^2}{1-\varepsilon^2} - \frac{2d\varepsilon^2 x}{1-\varepsilon^2}.$$

This becomes

$$(x + \frac{d\varepsilon^2}{1-\varepsilon^2})^2 - (\frac{d\varepsilon^2}{1-\varepsilon^2})^2 + \frac{y^2}{b^2/a^2} = d^2\frac{\varepsilon^2}{1-\varepsilon^2}.$$

$$(x + \frac{b^2}{c}\frac{c^2}{b^2})^2 + \frac{y^2}{b^2/a^2} = \frac{c^2}{b^2}\frac{b^4}{c^2} + (\frac{d\varepsilon^2}{1-\varepsilon^2})^2.$$

$$(x+c)^2 + \frac{y^2}{b^2/a^2} = b^2 + (\frac{b^2}{c}\frac{c^2}{b^2})^2 = b^2 + c^2 = a^2.$$

We get a standard form of the ellipse, where the curve has been translated to the left by c,

$$\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1.$$

That is the origin is at a focus rather than at a center.

The polar form occurs naturally in describing planetary motion. From this derivation it is easy to transform the polar form to the parametric form. That is, given the position of a focus $\mathbf{p}$, a unit vector $\mathbf{n}$ pointing toward the directrix, the distance d from the focus to the directrix, and an eccentricity ε, we may compute a, b, c. The center point is at

$$\mathbf{p} - c\mathbf{n},$$

and so on.

The chord of the ellipse that passes through the focus and is parallel to the directrix is called the latus rectum, literally meaning "straight side." From the polar equation of the ellipse we see that the length of the latus rectum is $2ed$. Sometimes the polar equation is given as

$$r = \frac{p}{1 - \varepsilon \cos(\theta)},$$

where $2p$ is the length of the latus rectum. If we know ε and d, we may compute a, b and c as follows. We have

$$a^2 = \frac{c^2}{\varepsilon^2}$$

$$b^2 = dc.$$

$$c^2 = a^2 - b^2 = \frac{c^2}{\varepsilon^2} - dc.$$

So

$$c = \frac{c}{\varepsilon^2} - d.$$

From which

$$c = d\frac{\varepsilon^2}{1 - \varepsilon^2}.$$

Then

$$a = \frac{c}{\varepsilon}$$

and

$$b = \sqrt{dc}.$$

If $\varepsilon = 1$ the polar conic is a parabola, and if $\varepsilon > 1$ it is a hyperbola. We may show this using calculations similar to the one we have just done above for the ellipse.

19 Kepler's Three laws of Planetary Motion

The three laws are

1. **Planets move in elliptical orbits around the Sun with the Sun at a focus.**

2. **A line joining the Sun to the planet sweeps out equal areas in equal times.**

3. **The period of the orbit is proportional to the $3/2$ power of the major diameter of the ellipse.**

We shall prove these laws below. The motion of a planet about the sun is in an elliptical orbit about the center of gravity of the planet and sun. But the sun is so massive comparred to the planet, that we may assume that the sun is fixed. Let a planet have velocity $\mathbf{v}$ then it has momentum $m\mathbf{v}$, and angular momentum (moment of momentum) $m\mathbf{r} \times \mathbf{v}$, where $\mathbf{r}$ is the position vector of the planet and m is its mass.

Proposition. A planet moves in a plane about the sun, and the planet has constant angular momentum.

Proof. We have

$$\begin{aligned} \frac{d(\mathbf{r} \times \mathbf{v})}{dt} &= \frac{d\mathbf{r}}{dt} \times \mathbf{v} + \mathbf{r} \times \frac{d\mathbf{v}}{dt} \\ &= \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \frac{d\mathbf{v}}{dt} \\ &= \mathbf{r} \times \mathbf{a}. \\ &= 0. \end{aligned}$$

This is zero because the force and thus the acceleration are radial in the direction of $\mathbf{r}$, and the cross product of parallel vectors is zero. So

$$\mathbf{r} \times \mathbf{v}$$

is a constant vector, and the angular momentum is constant. The velocity v is always perpendiculat to this constant vector, so motion is in the plane that is normal to

$$\mathbf{r} \times \mathbf{v}.$$

We shall use polar coordinates. The unit coordinate tangent vectors are

$$\mathbf{u}_r = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j},$$

and

$$\mathbf{u}_\theta = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}.$$

The derivatives are

$$\begin{aligned} \frac{d\mathbf{u}_\theta}{d\theta} &= -\mathbf{u}_r \\ \frac{d\mathbf{u}_r}{d\theta} &= \mathbf{u}_\theta. \end{aligned}$$

The time derivatives of the unit vectors are then

$$\frac{d\mathbf{u}_r}{dt} = \frac{d\theta}{dt}\frac{\mathbf{u}_r}{d\theta} = \frac{d\theta}{dt}\mathbf{u}_\theta,$$

and

$$\frac{d\mathbf{u}_\theta}{dt} = \frac{d\theta}{dt}\frac{\mathbf{u}_\theta}{d\theta} = -\frac{d\theta}{dt}\mathbf{u}_r.$$

We have then

$$\mathbf{r} = r\mathbf{u}_r.$$

and

$$\mathbf{v} = \frac{d(r\mathbf{u}_r)}{dt} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\mathbf{u}_r}{dt}$$
$$= \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta.$$

Then

$$\mathbf{r} \times \mathbf{v} = r^2\frac{d\theta}{dt}\mathbf{u}_r \times \mathbf{u}_\theta = r^2\frac{d\theta}{dt}\mathbf{u}_z,$$

where $\mathbf{u}_z$ is a unit vector in the z direction perpendicular to the plane of motion. Then $\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_z$ form a right handed system of orthogonal unit vectors. And

$$\mathbf{u}_r \times \mathbf{u}_z = -\mathbf{u}_\theta.$$

19.1 Kepler's First Law

Kepler's First Law. Planets move in elliptical orbits around the Sun with the Sun at a focus.
Proof. Let

$$\boldsymbol{\Upsilon} = \mathbf{r} \times \mathbf{v}.$$

It is constant and essentially the angular momentum We have

$$\boldsymbol{\Upsilon} = r^2\frac{d\theta}{dt}\mathbf{u}_z.$$

Now the acceleration depends on the inverse square of r,

$$\mathbf{a} = -\frac{GM}{r^2}\mathbf{u}_r.$$

So if we multiply $\boldsymbol{\Upsilon}$ by the acceleration, we can cancel the r^2. We have

$$\frac{d(\mathbf{v} \times \boldsymbol{\Upsilon})}{dt} = \mathbf{a} \times \boldsymbol{\Upsilon}$$

$$= GM \frac{d\theta}{dt} \mathbf{u}_\theta$$

$$= GM \frac{d\theta}{dt} \frac{d\mathbf{u}_r}{d\theta}$$

$$= GM \frac{d\mathbf{u}_r}{dt}$$

$$= \frac{d(GM\mathbf{u}_r)}{dt}.$$

So integrating we have

$$\mathbf{v} \times \boldsymbol{\Upsilon} = GM\mathbf{u}_r + \boldsymbol{\Gamma},$$

where $\boldsymbol{\Gamma}$ is a constant vector. We are looking for a scalar equation, so we take the dot product with $\mathbf{r}$. We get

$$\|\boldsymbol{\Upsilon}\|^2 = (\mathbf{r} \times \mathbf{v}) \cdot \boldsymbol{\Upsilon}$$

$$= \mathbf{r} \cdot (\mathbf{v} \times \boldsymbol{\Upsilon})$$

$$= \mathbf{r} \cdot (GM\mathbf{u}_r + \boldsymbol{\Gamma})$$

$$= rGM + \mathbf{r} \cdot \boldsymbol{\Gamma}$$

$$= r(GM + \|\boldsymbol{\Gamma}\| \cos(\phi)),$$

where ϕ is the angle between $\mathbf{r}$ and $\boldsymbol{\Gamma}$. We can write this as

$$r = \frac{\|\boldsymbol{\Upsilon}\|^2/GM}{1 + (\|\boldsymbol{\Gamma}\|/GM)\cos(\phi)}.$$

This has the form

$$r = \frac{\varepsilon d}{1 + \varepsilon \cos(\phi)},$$

with the eccentricity

$$\varepsilon = \|\boldsymbol{\Gamma}\|/GM,$$

and distance from the focus to the directrix

$$d = \frac{\|\boldsymbol{\Upsilon}\|^2}{\varepsilon GM}.$$

This is the equation of a conic section in polar coordinates with the origin at a focus. So actually Kepler's law is not quite correct, for bodies not only can orbit in ellipses but also in hyperbolas and parabolas.

But in the case of the planets of our solar system, the orbits are periodic, so the conic section must be an ellipse. We have thus proven Kepler's first law.

19.2 Kepler's Second Law

The second law follows informally from Newton's second law of motion. Consider a system of two particles and a coordinate system located at their center of mass. Suppose they exert forces on each other in a direction along the line joining them. Forces are proportional to the accelerations by the second law. Taking cross products with the radii, we find that the rate of increase of angular momentum of the system of particles is proportional to the external torque on the system. But this torque is zero. Therefore we have conservation of angular momentum. This change of angular momentum of a particle is proportional to the derivative of the cross product of the radius and the velocity. Since the Sun is relatively massive, we may consider it to be at rest, and that the center of gravity of the system coincides with that of the Sun. Then the angular momentum is that of the planet, and is clearly proportional to the rate at which the radius sweeps out area. But let us throw a bit more rigor at the problem.

Kepler's Second Law. A line joining the Sun to a planet sweeps out equal areas in equal times.

Proof. We use polar coordinates. The unit coordinate tangent vectors are

$$\mathbf{u}_r = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j},$$

and

$$\mathbf{u}_\theta = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}.$$

The derivatives are

$$\frac{d\mathbf{u}_\theta}{d\theta} = -\mathbf{u}_r$$

$$\frac{d\mathbf{u}_r}{d\theta} = \mathbf{u}_\theta.$$

The time derivatives of the unit vectors are then

$$\frac{d\mathbf{u}_r}{dt} = \frac{d\theta}{dt}\frac{\mathbf{u}_r}{d\theta} = \frac{d\theta}{dt}\mathbf{u}_\theta,$$

and

$$\frac{d\mathbf{u}_\theta}{dt} = \frac{d\theta}{dt}\frac{\mathbf{u}_\theta}{d\theta} = -\frac{d\theta}{dt}\mathbf{u}_r.$$

We write

$$\mathbf{r} = r\mathbf{u}_r.$$

We shall show that if the acceleration is radial, then the rate that area is swept out by the radius vector is a constant. To this end the velocity is

$$\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{u}_r + r\frac{\mathbf{u}_r}{dt} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta.$$

The acceleration is

$$\frac{d^2\mathbf{r}}{dt^2} = (\frac{d^2r}{dt^2}\mathbf{u}_r + \frac{dr}{dt}\frac{d\mathbf{u}_r}{dt}) + (\frac{d(rd\theta/dt)}{dt}\mathbf{u}_\theta + r\frac{d\theta}{dt}\frac{d\mathbf{u}_\theta}{dt}).$$

$$= \frac{d^2r}{dt^2}\mathbf{u}_r + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + (\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2})\mathbf{u}_\theta - r\frac{d\theta}{dt}\mathbf{u}_r$$

$$= (\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2)\mathbf{u}_r + (2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2})\mathbf{u}_\theta.$$

Now if the acceleration is radial, then the coefficient of $\mathbf{u}_\theta$ is zero. We shall show that this requires that the rate that area is swept out by the radius vector, is a constant. So notice that

$$\frac{d(r^2d\theta/dt)}{dt} = 2r\frac{dr}{dt}\frac{d\theta}{dt} + r^2\frac{d^2\theta}{dt^2}$$

$$= r(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2})$$

So

$$\frac{1}{r}\frac{d(r^2(d\theta/dt))}{dt}$$

$$= (2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}).$$

Therefore we may write the acceleration as

$$\frac{d^2\mathbf{r}}{dt^2} = (\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2)\mathbf{u}_r + \frac{1}{r}\frac{d(r^2(d\theta/dt))}{dt}\mathbf{u}_\theta.$$

Because the force and hence the acceleration is purely radial, we have

$$\frac{d(r^2(d\theta/dt))}{dt} = 0.$$

Therefore

$$r^2\frac{d\theta}{dt} = C,$$

where C is a constant. We relate this to the area swept out. Let Ω be the area swept out by the radius vector in time t. A differential element of area swept out by the radius vector is

$$d\Omega = \frac{1}{2}r^2 d\theta.$$

Hence, dividing by dt, the rate of area swept out is

$$\frac{d\Omega}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}.$$

We have just shown that this is a constant. This proves Kepler's second law.

19.3 Kepler's Third Law

Let α be the area of an ellipse with major axis a and miner axis b. We have

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}.$$

$$\alpha = 4\int_0^a y dx$$

$$= 4\int_0^b b\sqrt{1 - \frac{x^2}{a^2}}dx.$$

Let $u = x/a$ and $du = dx/a$. So

$$\alpha = 4ab\int_0^1 \sqrt{1 - u^2}du$$

The integral equals one forth the area of the unit circle. Hence

$$\alpha = ab\pi.$$

Now we shall derive kepler's third law relating the period T and the major semi-axis of the ellipse a.

$\upsilon = \|\mathbf{\Upsilon}\|$ is equal to twice the rate of area sweep, so if T is the period then the area of the ellipse is

$$ab\pi = T\frac{\upsilon}{2}.$$

Let us eliminate b using

$$b = a\sqrt{1-\varepsilon^2}.$$

The major axis of the ellipse is $2a$, so using the polar form of the ellipse, we get

$$2a = r(0) + r(\pi) = \varepsilon d(\frac{1}{1+\varepsilon} + \frac{1}{1-\varepsilon}$$
$$= \varepsilon d\frac{2}{1-\varepsilon^2}.$$

So

$$a = \frac{\varepsilon d}{1-\varepsilon^2}.$$

Then

$$d = a\frac{1-\varepsilon}{\varepsilon}$$

Also above we had

$$d = \frac{\upsilon^2}{GM\varepsilon}$$

This gives

$$\upsilon = \sqrt{dGM\varepsilon}$$
$$= \sqrt{\frac{a(1-\varepsilon^2)}{\varepsilon}}\sqrt{GM\varepsilon}$$
$$= \sqrt{GMa}\sqrt{1-\varepsilon^2}$$

Now we can write

$$T = \frac{2\pi ab}{\upsilon} = \frac{2\pi a^2\sqrt{1-\varepsilon^2}}{\upsilon}$$

$$= \frac{2\pi a^2\sqrt{1-\varepsilon^2}}{\sqrt{GMa}\sqrt{1-\varepsilon^2}}$$
$$= \frac{2\pi}{\sqrt{GM}}a^{3/2}.$$

This is Kepler's third law: *The period is proportional to the three halves power of the semi-major axis of the ellipse.*

20 Methods of Finding the Planetary Orbits in the Time of Kepler

Kepler made use of the wealth of observational data accumulated by Tycho Brahe. Various kinds of triangulations were used to establish orbits. For example, suppose four observations of Mars are known on four different days of the earth year. Then we would know four points on the earths orbit, and lines from these points are determined by the observation angles are then known that pass through points on the orbit of Mars. We know the period of the earth, namely 365 days. We may find the period of Mars by various observations. We make the simplifying assumption that both orbits are circular, suppose we observe Mars from two points on the earth, at time intervals of the martian period. Then we may assume that Mars is at the same position in its orbit. Intersecting these two lines we find a point on the orbit of Mars. From three such points we can find a circle. If the orbit were circular, the orbit of Mars would be this circle. To locate the elliptical orbit we would need 5 points. The size of this orbit would be relative to the size of the orbit of earth, one astronomical unit, at the time of unknown actual size. We could also locate points on the Earth's orbit in this manner by taking observations of Mars at different times on a given day of the Martian orbit. See **The Astronomical Revolution** by Alexandre Koyre, for an explanation of how these planet determinations were actually done by Kepler and others.

21 The Ptolemaic System

An informative diagram of the Earth centered Ptolemaic System, where the Sun, Moon, and the Planets, moved in circles and epicycles is given in **The**

Planet Observer's Handbook p28. The Copernican system was Sun centered, but relied on epicycles to explain deviations from circular motion.

22 The Derivation of the Elliptical Path Using Vector Analysis and the Conservation of Angular Momentum

This derivation is given in **Fundamentals of Astrodynamics** pages 19-21, and **Fundamental Astronomy** pages 132-136.

23 The Elliptical Orbit as a Function of Time: The Method of Kepler

The method of Kepler will be described here. The normal anomaly is the angle ν in the polar form of the ellipse. The eccentric anomaly is a corresponding angle on the circle circumscribing the ellipse. This term arises because of the deviation of the ellipse from a circle. Circular uniform motion is the traditional motion ascribed to the planets first by Aristotle. So any deviation from such motion was considered non-regular, and so was called an anomaly.

The Kepler method is described in the book **Fundamentals of Astrodynamics**, pp 182-188, and also in **Fundamental Astronomy**, pp142-144. Here we present an outline of the method. Given an ellipse with major semi-axis a. Suppose we specify a position P on the ellipse, at polar angle ν, (called the normal anomaly). We wish to find the time the planet took to go from a position at angle $\nu = 0$ to the current point. Because the rate at which area is swept out by the radius vector from the focus is constant, if we can find the area, we will essentially have the time. In order to do this we construct the circle of radius a circumscribing the ellipse, and which has the same center as the ellipse. Through the point P on the ellipse, we construct a vertical line intersecting the circle at Q. The angle of the line from the center of the circle (and the ellipse) to point Q is called E, known as the eccentric anomaly. The areas to the right of the vertical line on the ellipse and on the circle are in the ratio a/b. This comes from the area integrals of the ellipse and circle, because

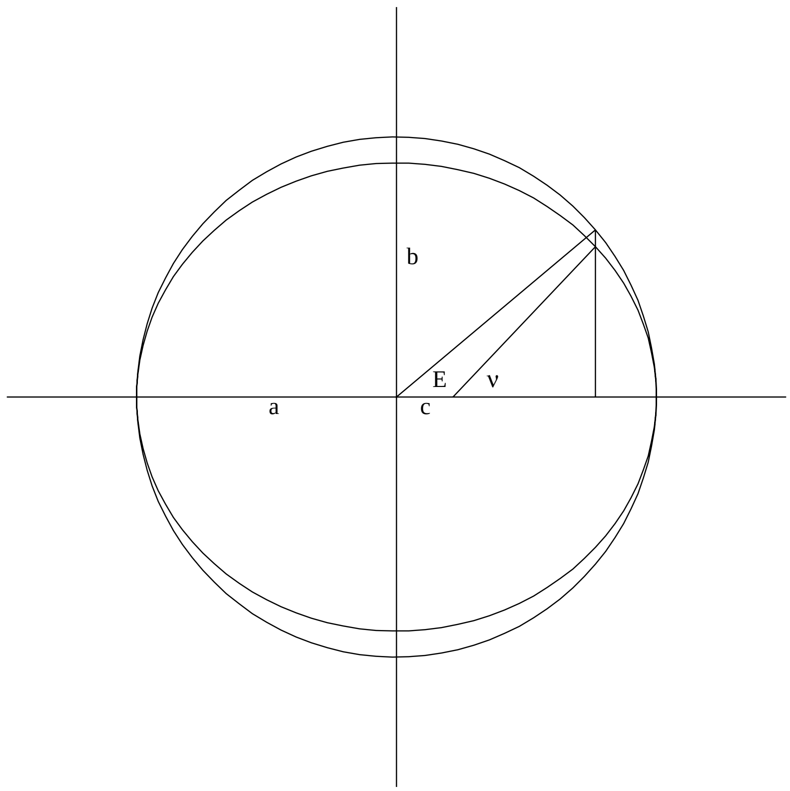

Figure 8: The Kepler method of computing an orbit. The Figure shows the relation between the two angles ν and E , which are called respectively the normal anomaly and the eccentric anomaly. These two angles allow us to compute the area swept out by the radius vector moving from the perihelion, to a point P on the ellipse. This leads to the time it takes to move in the orbit from the perihelion to the point P

their respective equations are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and

$$x^2 + y^2 = a^2.$$

This fact, along with some trigonometry results in a formula for the area A swept out by the radius vector from the focus of the ellipse to the point P on the ellipse. We get the formula

$$A = \frac{ab}{2}(E - \varepsilon \sin(E).$$

We have

$$\frac{t - t_0}{A} = \frac{T}{\pi ab},$$

where t_0 is the time at angle $\nu = 0$. We also find an equation relating the eccentric anomaly E, and the normal anomaly ν, we find that

$$\cos(E) = \frac{\varepsilon + \cos(\nu)}{1 + \varepsilon \cos(\nu)}.$$

We may write a computer program to compute the function $t = f(\nu)$. To compute ν given t we compute values of the inverse function f^{-1}. So the method of computing the equation of time given on the program **orbit.ftn** is the following. For each value ν of a set of values of the polar angles ν, we find the time it takes to go from the perihelion, (the closest point to the focus) where $\nu = 0$, to ν. Given ν, from the last equation we compute the $\cos(E)$, and so the eccentric anomaly E. Then from the formula above for the area swept out, we compute the area A. Finally, from the equation

$$\frac{t - t_0}{A} = \frac{T}{\pi ab},$$

we find the time t for the Sun to reach the position at angle ν. Because the Mean Sun travels uniformly in a circle in the equatorial plane, knowing the time t we may compute its position if we know its position at the perihelion. The real Sun is in the ecliptic plane, whereas the Mean Sun is in the equatorial plane. We must take this into account also. So what is the position of the mean Sun at perihelion? Now if it's starting position is shifted, then the equation of time difference, (see the figure) will be shifted up or down without changing shape. But for the Mean Sun to deserve it's name, the average difference between the Mean Sun and the real Sun throughout the year, must be zero. This means that the position of the Mean Sun at perihelion, must be chosen so that this average is zero. There is only one such position as one sees from the figure showing the equation of time function. For the details of the calculation, see the listing of the program **orbit.ftn**.

The period T for the earth is given by Kepler's third law,

$$T = \frac{2\pi}{\sqrt{\mu_s}} a^{3/2},$$

where

$$\mu_s = GM_s.$$

The mass of the sun is given in **Fundamental Astronomy** as

$$1.989(10^{30})\text{Kg},$$

and the gravitational constant is

$$G = 6.67(10^{-11})M^3/(KgS^2),$$

so

$$\mu_s = 13.267(10^{19})M^3/S^2.$$

The value of a is about

$$1.49(10^{11})M.$$

The period of the earth's orbit is one year. This is about

$$3.16(10^7)$$

seconds. The perihelion each year occurs at a date near January 3rd or 4th. The mean distance to the sun is about $1.5(10^{11})$ meters. The orbital eccentricity is $\varepsilon = .017$. At this date the right ascension of the Sun is about 19 hours. So the major axis of the Sun's orbit is at about 1 hour, or 15 degrees from the vernal equinox in the ecliptic plane. The right ascension of an object is the angle to the Vernal Equinox in hours of earth rotation, measured in the ecliptic plane.

The ecliptic plane is tilted to the equator by 23.44 degrees. This changes slowly over time. The Vernal Equinox is represented by the symbol Υ because this direction points towards the constellation Aries, which is "The

Ram." The letter Υ (upsilon)looks like a pair of ram's horns. From these facts, we can compute the equation of time. This is the time difference in minutes between the position of the Mean Sun, and the real Sun, as a function of the time of the year. Given a day of the year, that is the time, we compute the normal anomaly ν of the Sun. This is measured in the ecliptic plane. We project this angle to the equator, and compare it with the angle of the mean Sun. This gives us the difference between the time of local noon corresponding to ν and the mean solar time of noon. The mean sun is an ideal object that travels uniformly in a circle in the equatorial plane, with period one year. The time corresponding to this ideal Sun is called mean solar time. The difference between local solar time and mean solar time is given in minutes. This is also the difference between mean solar time and local sundial time. We must make another time correction if we are not at a standard time meridian. The standard meridians are at longitude angles that are multiples of 15 degrees from Greenwich. These define the standard time zones. The time correction for longitude is the time it takes for the earth to rotate from the standard meridian to our meridian.

24 The Distance to the Sun, the Astronomical Unit

The astronomical unit is the average distance to the sun. Edmund Halley in his 1716 paper, outlined a method of calculating this distance by measuring the time of transit of Venus at different locations on the earth. See Fernie, Halley, Pannekoek, and Schaefer in the bibliography for more on the transit of Venus method. The distances between the planets in terms of the astronomical unit are known via Kepler's laws. But an accurate determination of the actual distance to the Sun was not known accurately for a long time. One crude method was the observation, with a telescope, of the shadow of Venus on the Sun, while making the assumption that the diameter of Venus is comparable to that of the Earth. A modern technique of determining the astronomical unit is to bounce a radar signal off of Venus. The book by Smart describes a method that superseded the transit of Venus method. Also see the html documents: **halleyTransitofVenus.mht,** This is Edmund Halley's paper on the transit of Venus. and **Lecture26HowFartotheSun.mht** This is about the determination of the distance to the sun, the transit of Venus, and the size of the astronomical unit. The astronomical unit (AU) has value 149597870700 Meters. Because there are 1.609344 Km per Mile, this gives 92.9556 million miles to the Sun.

The polar equation of the ellipse is of the form

$$r(\theta) = \frac{b}{1 + \epsilon \cos(\theta)}$$

By the average distance we mean the average value of the radius from the focus to the elliptical curve. This is

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{b}{1 + \epsilon \cos(\theta))} d\theta.$$

We have

$$\int \frac{1}{1 + a\cos(x)} dx = \frac{2}{\sqrt{1 - a^2}} \tan^{-1}(\sqrt{\frac{1-a}{1+a}} \tan(x/2)) + C.$$

For the definite integral we have

$$\int_0^{2\pi} \frac{1}{1 + a\cos(x)} dx = \frac{2\pi}{\sqrt{1 - \epsilon^2}}$$

The average radius is then

$$\bar{r} = \frac{b}{\sqrt{1 - \epsilon^2}}.$$

The smallest radius (perihelion) is

$$r(0) = \frac{b}{1 + \epsilon}.$$

The largest radius (aphelion) is

$$r(\pi) = \frac{b}{1 - \epsilon}.$$

Multiplying these together, we get

$$r(0)r(\pi) = \frac{b^2}{1 - \epsilon^2},$$

or

$$b = \sqrt{1-\epsilon^2}\sqrt{r(0)r(\pi)}.$$

So the average radius is

$$\bar{r} = \frac{b}{\sqrt{1-\epsilon^2}} = \sqrt{r(0)r(\pi)}.$$

That is, the average radius (Astronomical Unit) is the geometric mean of the perihelion and the aphelion of the Earth's orbit about the Sun.

Let us review how to calculate the integral

$$\int \frac{1}{1+a\cos(x)}dx = \frac{2}{\sqrt{1-a^2}}\tan^{-1}(\sqrt{\frac{1-a}{1+a}}\tan(x/2)) + C.$$

Whenever we have an integral of a rational expression involving sines and cosines, we may use the substitution

$$u = \tan x/2$$

to convert the integrand to a rational expression in u. Then we can apply the method of partial fractions to evaluate the integral. So we have

$$\tan_1 u = \frac{x}{2},$$

so

$$dx = \frac{2}{1+u^2}du.$$

Also

$$\sin(x) = 2\sin(x/2)\cos(x/2) = \frac{2\tan(x/2)}{1+\tan^2(x/2)} = \frac{2u}{1+u^2}$$

And

$$\begin{aligned}\cos(x) = \cos^2(x/2) - \sin^2(x/2) &= 2\cos^2(x/2) \\ &= \frac{2}{\sec^2(x/2)} - 1 \\ &= \frac{2}{1+\tan^2(x/2)} - 1 \\ &= \frac{2}{1+u^2} - 1\end{aligned}$$

$$= \frac{1-u^2}{1+u^2}$$

So our integral becomes

$$\int \frac{1}{1+a\cos(x)}dx = \frac{1}{1+a}\int \frac{2}{1+\frac{1-a}{1+a}u^2}du$$

Making the substitution

$$v = \sqrt{\frac{1-a}{1+a}}u,$$

$$dv = \sqrt{\frac{1-a}{1+a}}du,$$

our integral becomes

$$\frac{2}{\sqrt{(1+a)(1-a)}}\int \frac{1}{1+v^2}dv$$

$$= \frac{2}{\sqrt{(1+a)(1-a)}}\tan^{-1}(v) + C$$

$$= \frac{2}{\sqrt{1-a^2}}\tan^{-1}(\sqrt{\frac{1-a}{1+a}}\tan(x/2)) + C.$$

25 Declination of the Sun

Here is a Fourier approximation for the declination of the Sun data. Although the data could be calculated from the program **orbit.ftn**, this data at 5 day increments comes from the book **How To Use An Astronomical Telescope**. The approximation was computed using the program **lsqfourier.ftn**, which is a linear least squares program for trigonometric basis functions. The functions used are

$$f_1(t) = 1., f_2(t) = \sin(t(2\pi/T)), f_3(t) = \cos(t(2\pi/T)),$$

$$f_4(t) = \sin(t2(2\pi/T)), f_5(t) = \cos(t2(2\pi/T)).$$

Here is a listing of the output of **lsqfourier.ftn**.

```
f(x)= .392075983063474     f( 1) +
       4.12729073245691     f( 2) +
      -22.8801439188754     f( 3) +
       .471399044566739E-01 f( 4) +
      -.381041415709779     f( 5)
```

Day x	Angle y	Angle fit
1.0000000	-23.000000	-22.869109
6.0000000	-22.600000	-22.415903
11.000000	-21.900000	-21.785695
16.000000	-21.000000	-20.983697
26.000000	-18.800000	-18.892577
31.000000	-17.500000	-17.621052
36.000000	-16.100000	-16.212735
41.000000	-14.500000	-14.679437
46.000000	-12.900000	-13.033956
51.000000	-11.100000	-11.289935
56.000000	-9.3000000	-9.4617240
61.000000	-7.4000000	-7.5642274
66.000000	-5.5000000	-5.6127540
71.000000	-3.5000000	-3.6228666
76.000000	-1.6000000	-1.6102337
81.000000	.40000000	.40951400
86.000000	2.4000000	2.4209211
91.000000	4.3000000	4.4088365
96.000000	6.3000000	6.3585338
101.00000	8.1000000	8.2558198
106.00000	9.9000000	10.087133
111.00000	11.700000	11.839630
116.00000	13.300000	13.501258
121.00000	14.900000	15.060818
126.00000	16.400000	16.508015
131.00000	17.700000	17.833497
136.00000	19.000000	19.028886
141.00000	20.100000	20.086791
146.00000	21.000000	21.000828
151.00000	21.800000	21.765621
156.00000	22.500000	22.376797
161.00000	23.000000	22.830986
166.00000	23.300000	23.125810
171.00000	23.400000	23.259869
176.00000	23.400000	23.232731
181.00000	23.200000	23.044918
186.00000	22.800000	22.697892
191.00000	22.300000	22.194043
196.00000	21.600000	21.536676
201.00000	20.800000	20.729998
206.00000	19.800000	19.779110
211.00000	18.600000	18.689990
216.00000	17.400000	17.469483
221.00000	16.000000	16.125289
226.00000	14.500000	14.665938
231.00000	12.900000	13.100777
236.00000	11.300000	11.439938
241.00000	9.5000000	9.6943095
246.00000	7.7000000	7.8754960
251.00000	5.9000000	5.9957713

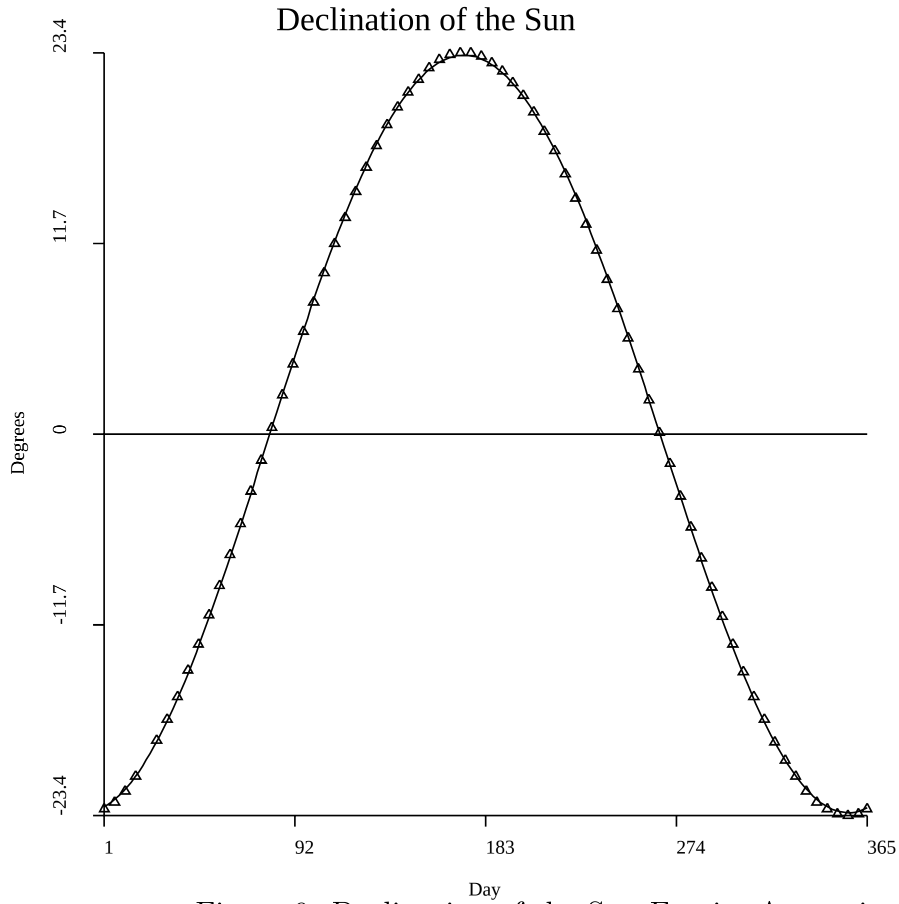

Figure 9: Declination of the Sun Fourier Approximation

256.00000	4.0000000	4.0680229
261.00000	2.1000000	2.1056871
266.00000	.10000000	.12267414
271.00000	-1.8000000	-1.8667164
276.00000	-3.8000000	-3.8478901
281.00000	-5.7000000	-5.8060624
286.00000	-7.6000000	-7.7263704
291.00000	-9.4000000	-9.5939949
296.00000	-11.200000	-11.394287
301.00000	-12.900000	-13.112901
306.00000	-14.600000	-14.735932
311.00000	-16.100000	-16.250051
316.00000	-17.500000	-17.642644
321.00000	-18.900000	-18.901949
326.00000	-20.000000	-20.017182
331.00000	-21.000000	-20.978663
336.00000	-21.900000	-21.777926
341.00000	-22.600000	-22.407829
346.00000	-23.000000	-22.862633
351.00000	-23.300000	-23.138085
356.00000	-23.400000	-23.231470
361.00000	-23.300000	-23.141656
365.00000	-23.000000	-22.938161

sigma = .12857902951942

26 A Link to a Siderial Time Calculator

```
http://tycho.usno.navy.mil/sidereal.html

Local Apparent Sidereal Time--------------------------------------------------------

KANSAS CITY , MO Longitude -94.55 degrees

05:34 LST

Current UTC (or GMT/Zulu)-time used: Saturday, July 10, 2010 at 16:37:18
UTC is Coordinated Universal Time, GMT is Greenwich Mean Time.
Great Britain/United Kingdom is one hour ahead of UTC during summer.
```

27 Finding Sirius

```
Data:
Constellation: Canis Major
Right ascension: (06h 45m 08.9173s) = 90.7525 degrees
Declination: -(16d  42m 58.017s) = - 16.7161158333 degrees
Apparent magnitude:  -1.46
```

Sirius is the brightest star in the night sky. With a visual apparent magnitude of -1.46. it is almost twice as bright as Canopus, the next brightest star. The name "Sirius" is derived from the Ancient Greek Seirios ("scorcher"). What the naked eye perceives as a single star is actually a binary star system, consisting of a white main sequence star termed Sirius A, and a faint white dwarf companion termed Sirius B. Sirius appears bright due to both its intrinsic luminosity and its proximity to Earth. At a distance of 2.6 parsecs (8.6 light years), the Sirius system is one of our near neighbors. Sirius A is about twice as massive as the Sun and has an absolute visual magnitude of 1.42. It is 25 times more luminous than the Sun[8] but has a significantly lower luminosity than other bright stars such as Canopus or Rigel. The system is between 200 and 300 million years old. It was originally composed of two bright bluish stars. The more massive of these, Sirius B, consumed its resources and became a red giant before shedding its outer layers and collapsing into its current state as a white dwarf around 120 million years ago.

Sirius is also known colloquially as the "Dog Star", reflecting its prominence in its constellation, Canis Major (English: Big Dog).[14] The heliacal rising of Sirius marked the flooding of the Nile in Ancient Egypt and the "Dog Days" of summer for the Ancient Greeks, while to the Polynesians it marked winter and was an important star for navigation.

28 The Right Ascension and Declination of the Sun for 2010

18 hours Universal Time corresponds to local apparent noon in the Central time zone of the United States when standard time is in effect. During the spring and summer months, and part of autumn it corresponds to 1 PM Central Daylight Time. OBJECT = Sun (18.0H UT)

```
Right Ascension and Declination of Sun for 2010
Kevin Krisciunas
http://faculty.physics.tamu.edu/krisciunas/ra_dec_sun.html

     DATE      RA       DEC
JAN 1, 2010 18:48:47 -22:57:54
JAN 2, 2010 18:53:12 -22:52:31
JAN 3, 2010 18:57:35 -22:46:43
JAN 4, 2010 19: 1:59 -22:40:25
JAN 5, 2010 19: 6:23 -22:33:41
JAN 6, 2010 19:10:45 -22:26:32
JAN 7, 2010 19:15: 8 -22:18:55
JAN 8, 2010 19:19:30 -22:10:51
JAN 9, 2010 19:23:51 -22: 2:23
JAN 10, 2010 19:28:12 -21:53:27
JAN 11, 2010 19:32:33 -21:44: 6
JAN 12, 2010 19:36:52 -21:34:21
JAN 13, 2010 19:41:12 -21:24: 9
JAN 14, 2010 19:45:30 -21:13:32
JAN 15, 2010 19:49:48 -21: 2:34
JAN 16, 2010 19:54: 5 -20:51: 9
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JAN 21, 2010 20:15:20 -19:48:17
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JAN 30, 2010 20:52:46 -17:32:23
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```

29 Computer Programs

The Fortran program **orbit.ftn**, calculates the "orbit" of the Sun around the Earth, and the values of the equation of time. The program **sundial.ftn** does the calculations to define the layout of a horizontal sundial. Other relevant programs are: **lsqfourier.ftn**, **sundialhor.ftn**, **eot.ftn**, **eg2ps.c**, and **sundialvert.ftn**.

30 When Lost In The Forest, How to Find True North With A Sundial

Assuming a watch, knowing the date and the approximate longitude, we look up the equation of time value for that date. Using the standard time from the watch, we are able to use the equation of time value to find the local sundial time. We level our sundial and rotate it until it agrees with the computed sundial time. Then the 12 O'clock position of the sundial points to the true north. If this forest has trees, there may be a difficulty.

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