

Controlling a Rolling Ball On a Tilting Plane

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1 Introduction

We are given a ball on a tilted table moving with a specified velocity. We are to change the velocity and position in a specified way by changing the tilt angles of the table. If you know about the law of cosines and the inner (dot product) of vectors, you may skip the next two sections.

2 The Law of Cosines and the Inner Product of Vectors

We shall prove the law of cosines. Suppose we have three points

$$p_0 = (0, 0), p_1 = (b, 0), p_2 = (x, y) = (a \cos(\theta), a \sin(\theta)).$$

These points form a triangle with sides p_0p_2, p_0p_1, p_2p_1 . These sides have lengths a, b, c . The angle between side p_0p_1 and side p_0p_2 is θ . We have

$$\begin{aligned} c^2 &= (x - b)^2 + y^2 \\ &= (x - b)^2 + a^2 - x^2 \end{aligned}$$

$$\begin{aligned}
&= x^2 - 2xb + b^2 + a^2 - x^2 \\
&= a^2 + b^2 - 2xb = a^2 + b^2 - 2ab \cos(\theta).
\end{aligned}$$

Thus we have the law of cosines, namely the square of the side opposite an angle of a triangle, is equal to the sum of the squares of the adjacent sides, minus two times the product of the sides and the cosine of the angle. That is,

$$c^2 = a^2 + b^2 - 2ab \cos(\theta).$$

The inner product (dot product) of two vectors, A and B , is defined as

$$A \cdot B = a_1b_1 + a_2b_2 + a_3b_3.$$

Then the dot product of a vector with itself is the square of its length. That is,

$$A \cdot A = a_1a_1 + a_2a_2 + a_3a_3 = \|A\|^2.$$

Let

$$C = B - A.$$

Then

$$\begin{aligned}
\|C\|^2 &= (B - A) \cdot (B - A) \\
&= B \cdot B - B \cdot A - A \cdot B + A \cdot A \\
&= \|B\|^2 - 2A \cdot B + \|A\|^2.
\end{aligned}$$

From which it follows that

$$2A \cdot B = \|A\|^2 + \|B\|^2 - \|C\|^2.$$

But the right hand side is, by the law of cosines,

$$2\|A\|\|B\| \cos(\theta),$$

where θ is the angle between vectors A and B . Hence

$$A \cdot B = \|A\|\|B\| \cos(\theta).$$

Thus if the the dot product is zero, then the cosine is zero, and so the angle between the vectors is plus or minus $\pi/2$, and the vectors are perpendicular.

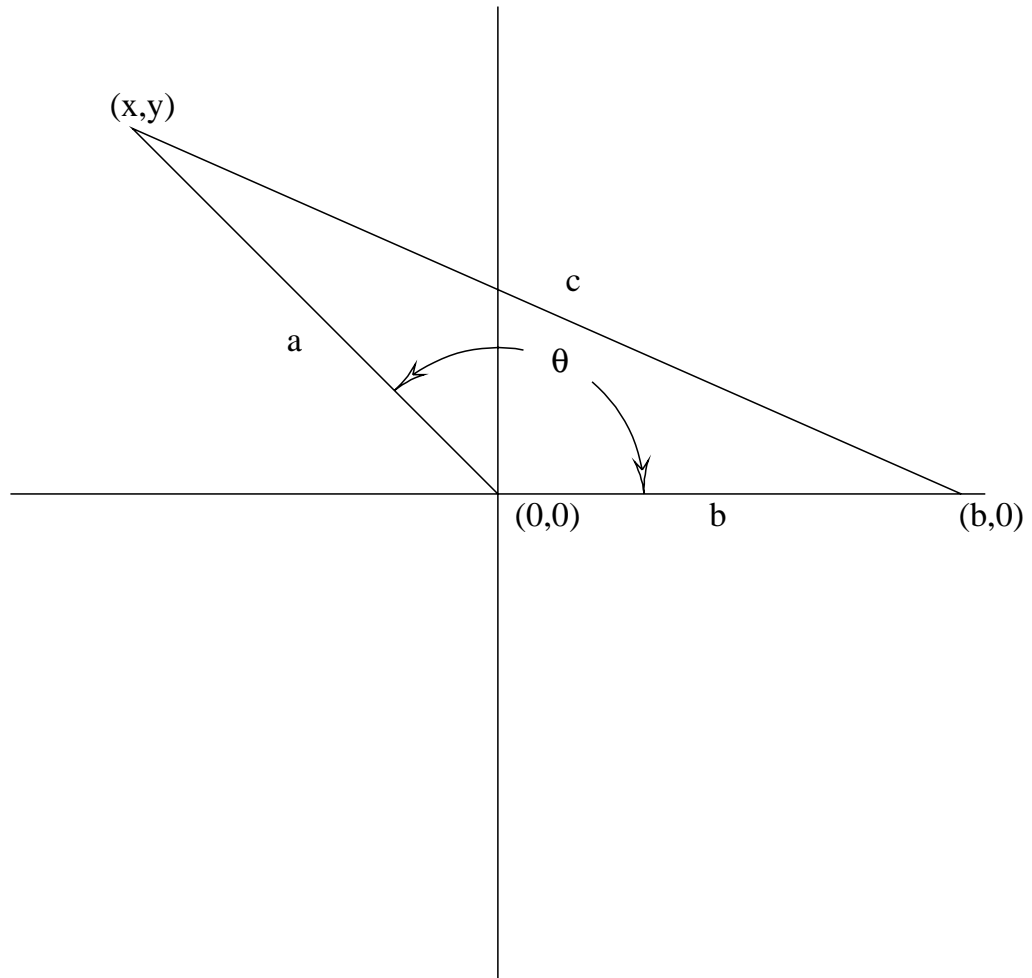


Figure 1: Derivation of the law of cosines. $x = \cos(\theta)$, $y = \sin(\theta)$. Computing c^2 , we find that $c^2 = a^2 + b^2 - 2ab \cos(\theta)$.

3 The Equation of a Plane

Suppose a plane has a unit normal \mathbf{n} . Let a point on the plane be given by a vector \mathbf{p} . Then

$$\mathbf{n} \cdot \mathbf{p} = \|\mathbf{p}\| \cos(\theta),$$

where θ is the angle between \mathbf{n} and \mathbf{p} . This is the perpendicular distance d from the origin to the plane. Therefore the equation of the plane is

$$\mathbf{n} \cdot \mathbf{p} = d,$$

where

$$\mathbf{p} = (x, y, z).$$

that is

$$n_1x + n_2y + n_3z = d.$$

4 A Description of the Tilted Plane

We consider a table elevated from the floor by a distance h . That is the pivot point on the table plane is located at a distance h . This pivot point in the center of the table is supported by a column of length h . We assume here for simplicity that the pivot point is on the surface of the table, neglecting certain offsets that will be needed later. The table is tilted by two actuators. We take our coordinate system so that the xy plane is the floor, and the origin is directly below the table pivot point. An x actuator is a line segment of variable length that always lies in the $y = 0$ plane. It is connected to the center supporting column at a point a distance c_x from the pivot point in the table plane. It is connected to the table plane at a point that is at a distance t_x from the pivot point. Similarly, a y actuator always lies in the $x = 0$ plane. It is connected to the center supporting column at a point a distance c_y from the pivot point in the table plane. It is connected to the table plane at a point that is at a distance t_y from the pivot point. Let the x actuator have variable length α , and the y actuator have variable length β .

5 The Plane Normal as a Function of Actuator Lengths

The tilt of the table is controlled by two actuator lengths, as specified in the previous section. Let us first consider the x actuator with length α .

We want to compute a vector \mathbf{u}_1 that lies in the intersection of the $y = 0$ plane and the table plane. This vector is given by

$$\mathbf{u}_1 = (\cos(\theta_x), 0, \sin(\theta_x)),$$

where θ_x is the angle between a line that is the intersection of the table plane with the $y = 0$ plane, and the horizontal plane. This is

$$\theta_x = \phi_x - \pi/2,$$

where angle ϕ_x is given by the law of Cosines

$$\cos(\phi_x) = \frac{c_x^2 + t_x^2 - \alpha^2}{2c_x t_x}.$$

Similarly, we compute a vector \mathbf{u}_2 that lies in the intersection of the $x = 0$ plane and the table plane. This vector is given by

$$\mathbf{u}_2 = (0, \cos(\theta_y), \sin(\theta_y)),$$

where θ_y is the angle between a line that is the intersection of the table plane with the $x = 0$ plane, and the horizontal plane. This is

$$\theta_y = \phi_y - \pi/2,$$

where angle ϕ_y is given by the law of Cosines

$$\cos(\phi_y) = \frac{c_y^2 + t_y^2 - \beta^2}{2c_y t_y}.$$

Now the plane normal is

$$\mathbf{n} = \mathbf{u}_1 \times \mathbf{u}_2.$$

The equation of our table plane then is

$$\mathbf{n} \cdot \mathbf{p} = h,$$

where h is the vertical distance from the floor to the table pivot point. We will use this equation to compute the z coordinate of the center of the ball rolling on the tilted plane, where the x and y coordinates are specified by the camera image.

6 The Mapping of Camera Coordinates to Horizontal x and y Coordinates

This is the problem of aerial photography in miniature. But let us assume that there are two functions f_x and f_y that map the video pixel coordinates (i, j) to x and y coordinates. That is

$$x = f_x(i, j),$$

and

$$y = f_y(i, j).$$

Perhaps these are given by OpenCV functions. For more precision we might have to consider the optics and focal properties of the camera lenses. Now to compute the z coordinate we need the equation of the table plane for a given tilt.

7 Projecting to the Tilted Plane

So in the previous section we assume we have the x and y coordinates computed from the camera image as

$$x = f_x(i, j),$$

and

$$y = f_y(i, j).$$

Now we need the z coordinate of the point on the plane. We find z using the equation of the plane found from the actuator lengths which define the plane normal \mathbf{n} . The equation of the plane is

$$xn_x + yn_y + zn_z = h,$$

from which we have

$$z = \frac{h - (xn_x + yn_y)}{n_z}.$$

8 Sampling the Ball Velocity

We compute the ball velocity by getting two ball positions in a small time interval Δt .

$$\mathbf{v} = \frac{\mathbf{p}_2 - \mathbf{p}_1}{\Delta t}$$

9 The Motion of the Ball With Fixed Table Tilt

Suppose we measure the position and velocity of the ball using the camera. We wish to compute the trajectory of the ball on the plane with fixed actuator lengths α and β . So suppose the ball is at position \mathbf{p} with velocity \mathbf{v} . First we intersect the table plane with the horizontal plane getting a line direction unit vector \mathbf{w}_1 . Then we compute a perpendicular unit vector \mathbf{w}_2 . We resolve the velocity vector into components in these directions, say

$$\mathbf{v} = v_1\mathbf{w}_1 + v_2\mathbf{w}_2$$

So the motion is constant in the direction of \mathbf{w}_1 , and in the direction of \mathbf{w}_2 it is pure rolling down an incline plane. So in the next section we derive the motion of the ball down an incline plane.

10 The Motion of a Ball on an Incline Plane

To compute the motion of a ball rolling down an incline plane we need to derive some properties of uniform linear and also circular motion since the ball is rolling. It turns out that we need to consider the rotational moment of inertia of the ball. The next few sections treat this material.

11 Uniform Acceleration

Suppose a mass point undergoes uniform acceleration a . Then

$$a = \frac{d^2s}{dt^2},$$

where s is the location coordinate of the point. Integrating, the velocity becomes

$$v = \int a dt = v_0 + at$$

where v_0 is the initial velocity. Integrating again, the position is

$$s = \int v dt = s_0 + v_0t + a\frac{t^2}{2},$$

where s_0 is the initial position. The change in kinetic energy is

$$m\left(\frac{v^2}{2} - \frac{v_0^2}{2}\right) = f(s - s_0) = ma(s - s_0),$$

where f is a constant force of acceleration. Dividing by $m/2$ we get

$$v^2 - v_0^2 = 2a(s - s_0).$$

The velocity increases linearly with the time, so the average velocity multiplied by the time is the distance traveled

$$\begin{aligned}\frac{v + v_0}{2}t &= \frac{v_0 + at + v_0}{2}t \\ &= \left(v_0 + \frac{at}{2}\right)t \\ &= v_0t + \frac{at^2}{2} \\ &= (s - s_0).\end{aligned}$$

The four equations

$$\begin{aligned}v &= v_0 + at, \\ s &= s_0 + v_0t + a\frac{t^2}{2}, \\ v^2 - v_0^2 &= 2a(s - s_0),\end{aligned}$$

and

$$\frac{v + v_0}{2}t = (s - s_0),$$

usually suffice to solve problems of this type.

Example 1. Suppose a ball is rolled off of the edge of a table top. Suppose the table is at height h and the ball lands on the floor at a distance s from a point on the floor below the table edge. What was the horizontal velocity of the ball as it left the table edge?

Solution The ball is accelerated by the acceleration of gravity g downward. The initial vertical velocity is zero, and the initial vertical position is zero. So using the second formula

$$h = g\frac{t^2}{2},$$

from which we find the time at which the ball hits the floor to be

$$t = \sqrt{2h/g}.$$

Then again from the first equation with the horizontal acceleration $a = 0$, we have

$$v_0 t = s.$$

So we find the initial horizontal velocity to be

$$v_0 = \frac{s}{t} = s\sqrt{\frac{g}{2h}}.$$

In a real experiment let a be the distance up a ramp, so that the ball starting at a gains velocity v , in rolling down the ramp. Let $h = 0.9$ meters. We measure the following values and compute v .

a	s	v
10cm	.48m	1.11m/s
20	.68	1.59
30	.79	1.84
40	.89	2.08
50	1.08	2.52
60	1.13	2.63
70	1.18	2.77
80	1.29	3.01

Example 2. Suppose a stone is dropped from a one hundred meter tall bridge. What is the velocity of the stone as it hits the water, neglecting air resistance?

Solution The acceleration of gravity is $g = 9.81$ meters per second squared. Then from the third equation

$$v^2 = 2gs,$$

$$v = \sqrt{2gs} = \sqrt{2(100)(9.81)} = 44.29$$

meters per second, or 159.5 kilometers per hour.

12 A Mass Point in Circular Motion

Let a mass point be specified in polar coordinates (θ, r) . Let the point be constrained to lie on a circle of radius r . Let the polar coordinate unit vectors be

$$\begin{aligned}\mathbf{u}_r &= \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}, \\ \mathbf{u}_\theta &= -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}.\end{aligned}$$

The first vector is perpendicular to the circle and the second is tangent to it. Let the position vector of the point be

$$\mathbf{p} = r\mathbf{u}_r.$$

The velocity is

$$\mathbf{v} = \frac{d\mathbf{p}}{dt} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\mathbf{u}_r}{dt} = r\frac{d\mathbf{u}_r}{dt},$$

because here r is constant. We have

$$\begin{aligned}\frac{d\mathbf{u}_r}{dt} &= \frac{d\mathbf{u}_r}{d\theta} \frac{d\theta}{dt} \\ &= \mathbf{u}_\theta \frac{d\theta}{dt}.\end{aligned}$$

So

$$\mathbf{v} = r\frac{d\theta}{dt}\mathbf{u}_\theta = r\omega\mathbf{u}_\theta,$$

where ω is the angular velocity. The acceleration is

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} = r\frac{d\omega}{dt}\mathbf{u}_\theta + r\omega\frac{d\mathbf{u}_\theta}{dt} \\ &= r\frac{d\omega}{dt}\mathbf{u}_\theta - r\omega^2\mathbf{u}_r \\ &= r\frac{d\omega}{dt}\mathbf{u}_\theta - \frac{v^2}{r}\mathbf{u}_r,\end{aligned}$$

where $d\omega/dt$ is the angular acceleration, and $v = r\omega$ is the tangential velocity. If the angular acceleration is zero then v^2/r is the magnitude of the centripetal acceleration directed toward the center of the circle.

13 The Kinetic Energy of a Mass Point

Let a mass point of mass m be accelerated from an initial velocity 0 to a velocity v by a force f through a distance s . The work done and thus the kinetic energy of the mass point is

$$\begin{aligned}\int f ds &= \int m \frac{dv}{dt} ds \\ &= \int m \frac{ds}{dt} dv \\ &= \int m v dv \\ &= m \frac{v^2}{2}.\end{aligned}$$

The kinetic energy of a rigid body of mass M moving with linear velocity v is the sum of the kinetic energies of its mass points,

$$M \frac{v^2}{2}.$$

If a mass point is revolving uniformly with angular velocity ω in a circle of radius r , then its kinetic energy is

$$m \frac{v^2}{2} = m \frac{r^2 \omega^2}{2}.$$

The moment of inertia of such a mass point is

$$I = m r^2,$$

hence its kinetic energy is

$$I \frac{\omega^2}{2}.$$

It follows that a body rotating about a fixed point with moment of inertia I has a kinetic energy of rotation

$$I \frac{\omega^2}{2}.$$

14 The Simple Rotation of a Rigid Body

Although the general motion of a rigid body is rather complicated, the case of rotation about a fixed axis is simple, so we shall derive the needed formulas here.

Suppose a mass point, with mass m_i , is constrained to move on a circle with radius r_i . Suppose a force F_i is applied to this point. The forces of constraint may be ignored, since they do no work. Then the work done is

$$dW = r_i d\theta F_{ti} = d\theta \tau_i,$$

where F_{ti} is the tangential component of the force and τ_i is the torque. Then the rate of doing work is

$$\frac{dW}{dt} = \omega \tau_i.$$

This is equal to the change in kinetic energy of the particle

$$\omega \tau_i = \frac{d((1/2)r_i^2 \omega^2 m_i)}{dt} = r_i^2 m_i \frac{d\omega}{dt}.$$

Thus

$$\tau_i = r_i^2 m_i \frac{d\omega}{dt}.$$

Summing over all particles in the rotating rigid body, we get

$$\tau = \sum \tau_i = \left(\sum r_i^2 m_i \right) \frac{d\omega}{dt} = I \frac{d\omega}{dt},$$

where I is the rotational moment of inertia.

15 The Acceleration of a Body Rolling Down an Incline Plane

Let a sphere roll down a ramp that is inclined to the horizontal by angle α . Let s be the distance from the bottom of the ramp to the contact position of the ball. Let the ball have radius r and let θ be the angle of rotation of the ball. Then let

$$s = r\theta.$$

The kinetic energy of the ball is

$$\begin{aligned} T &= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}Mv^2 + \frac{1}{2}I(v/r)^2 \\ &= \frac{1}{2}(M + I/r^2)v^2 \end{aligned}$$

The potential energy is

$$V = hgm,$$

where

$$h = s \sin(\theta)$$

is the height of the ball. The Lagrangian is

$$L = T - V = \frac{1}{2}(M + I/r^2)v^2 - s \sin(\theta)gm.$$

$$\begin{aligned} \frac{\partial L}{\partial v} &= (M + I/r^2)v \\ \frac{d}{dt} \frac{\partial L}{\partial v} &= (M + I/r^2) \frac{dv}{dt} \\ \frac{\partial L}{\partial s} &= -\sin(\theta)gm. \end{aligned}$$

The equation of motion is

$$\frac{d}{dt} \frac{\partial L}{\partial v} - \frac{\partial L}{\partial s} = 0$$

So

$$(M + I/r^2) \frac{dv}{dt} + \sin(\theta)gm = 0$$

or

$$\frac{dv}{dt} = -g \sin(\theta) \frac{M}{M + I/r^2}$$

So the effective acceleration has been reduced by the factor

$$\frac{M}{M + I/r^2}$$

In the case of a sphere

$$I = \frac{2}{5}Mr^2,$$

so the factor becomes

$$\frac{M}{M + (2/5)M} = \frac{5}{7}.$$

16 Bringing the Ball to Rest in a Given Distance by Tilting the Plane.

Given a position \mathbf{p} and a velocity \mathbf{v} , compute the pure incline plane tilt that would give the ball a velocity $-\mathbf{v}$ after having traveled distance d . Then since the initial velocity was \mathbf{v} , the ball will come to a zero velocity in distance d . In practice the motion will not be exact, so the method can be iterated until the desired result is obtained.

17 Given a Ball at Rest at Point P, Compute the Tilt Motion to Bring It to Rest at Point Q

Let d be the distance between P and Q. Give it a reasonable starting tilt in the direction of Q, compute the time it takes to reach the halfway point. Use the technique of the previous section to bring the ball to rest in distance $d/2$.

18 Making a Ball Follow a Specified Path

Given a curved path, compute sample points along the path. Do the tilting to bring the ball to rest at each of the sample points.

19 Optimization

Not only must we move the ball in a specified path, we must do it in a short amount of time. So more optimal methods of moving the ball will no doubt be required.

20 Practical Considerations: Defining a Path for the Ball to Follow

Suppose we want the ball to follow a path that avoids holes in the plane. We can construct a sequence of linear paths to follow. However, if there are objects that will be bumped into, we will have to consider rebounds. Practical versions of the tilting must be constructed after behavior is determined by experiment.

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