Bezier Curves

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Contents

1 The Binomial Theorem and the Bernstein Polynomials

The binomial theorem is

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Therefore

$$1 = ((1-x)+x)^n = \sum_{k=0}^{n} \binom{n}{k} (1-x)^{n-k} x^k.$$

The kth summand is called the kth Bernstein polynomial of order n. The kth Bernstein basis polynomial of order n is written as

$$b_k^n(x) = \binom{n}{k} (1-x)^{n-k} x^k.$$

These polynomials are named after the Russian mathematician Sergei Natanovich Bernstein (March 5, 1880 October 26, 1968) , who used them in an elegant proof of the Wierstrass Approximation Theorem. This theorem says that any continuous function defined on a closed interval $[a, b]$ can be approximated uniformly within a specified distance $\epsilon > 0$ by a polynomial. Actually a Bernstein polynomial is one of the approximating polynomials constructed from sums of the $b_k^n(x)$. See a book on approximation theory such as **Interpolation and Approximation** by Philip Davis, p. 107, or the book **Introduction to Approximation Theory** by E. W. Cheney, p66. It is convenient to define

$$b_k^n(x)$$

to be 0 if $k < 0$ or $k > n$. We can show that a Bernstein polynomial may be computed as a linear combination of lower order Bernstein polynomials. Thus

$$b_i^n(x) = (1-x)b_i^{n-1} + x b_{i-1}^{n-1}$$

Indeed,

$$(1-x)\binom{n-1}{i}(1-x)^{n-1-i}x^i + x\binom{n-1}{i-1}(1-x)^{n-1-(i-1)}x^{i-1}$$

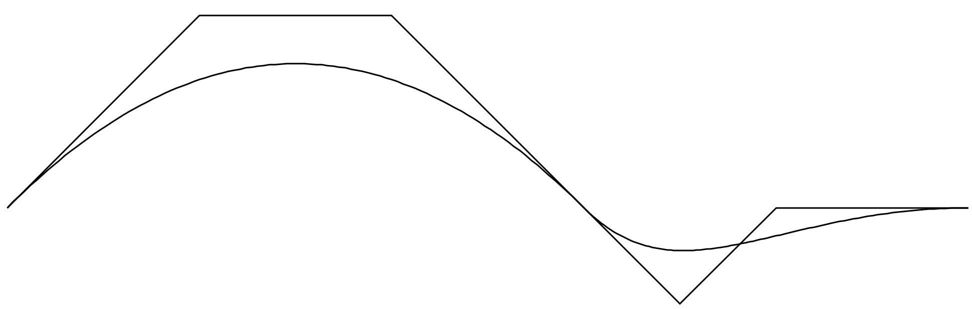

Figure 1: **Bezier Curve.** *This curve consists of two Cubic Bezier segments. The first segment uses the four control points (0,0), (1,1), (2,1), (3,0) . The second segment uses the control points (3, 0), (3.5, -.5) (4 ,0) (5, 0).*

$$= \frac{(n-1)!}{(n-i)!i!}(n-i+i)(1-x)^{n-i}x^i = b_i^n(x).$$

Also we see that $b_0^0 = 1$. Given any $n+1$ points in a vector space $P_0, P_1, ..., P_n$, the sum

$$B(x) = \sum_{i=0}^{n} P_i b_i^n(x),$$

is a curve in the space, where the parameter x takes values in the unit interval. This is called a Bezier curve, and the points are called control points. The Bernstein functions form a partition of unity. That is the sum of the Bernstein functions add to one. This is true because

$$1 = ((1-x)+x)^n = \sum_{i=0}^{n} \binom{n}{i} (1-x)^{n-i}x^i.$$

A set is convex if for every pair of points in the set, the line segment joining the pair of points is also in the set. That is given p_1 and p_2, then for

$$\lambda_1 + \lambda_2 = 1,$$

$$\lambda_1 p_1 + \lambda_2 p_2$$

is in the set.

The convex hull of a set is the smallest convex set containing the original set. This is the set of all linear combinations of points of the set for which the coefficients add to one. We see then that the Bezier curve lies in the convex hull of the control points. The shape of the Bezier curve resembles the shape of the control points.

The Bezier curve of degree three is very popular. It has the form

$$b_0^n(t)p_0 + b_1^n(t)p_1 + b_2^n(t)p_2 + b_3^n(t)p_3.$$

Here is a FORTRAN subroutine for computing a cubic Bezier curve:

```
c+ bez3  bezier plane cubic curve
      subroutine bez3(t,px,py,x,y)
c input:
c  t     variable in the interval [0,1]
c  px,py coordinates of the four control points
c output:
c  x,y   returned point on the bezier curve
```

```
implicit real*8(a-h,o-z)
dimension px(*),py(*)
b0=1-t
x=0.
y=0.
b=b0*b0*b0
x=x+b*px(1)
y=y+b*py(1)
b=3.*b0*b0*t
x=x+b*px(2)
y=y+b*py(2)
b=3.*b0*t*t
x=x+b*px(3)
y=y+b*py(3)
b=t*t*t
x=x+b*px(4)
y=y+b*py(4)
return
end
```

2 Transformation Between a Bezier Bases and A Power Basis

Suppose

$$p(x) = \sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} c_k b_k^n(x).$$

Proposition. Let

$$A_{ij} = \frac{\binom{i}{j}}{\binom{n}{j}}$$

for $i \geq j$ and zero otherwise. Then A is the change of basis matrix from the power representation to the Bernstein representation. That is

$$c = Aa.$$

Proof.

$$x^i = x^i((1-x)+x)^{n-i} = x^i \sum_{k=0}^{n-i} \binom{n-i}{k} (1-x)^{n-i-k} x^k$$

$$= \sum_{k=0}^{n-i} \binom{n-i}{k} (1-x)^{n-i-k} x^{k+i}$$

$$= \sum_{k=i}^{n} \binom{n-i}{k-i} (1-x)^{n-k} x^k$$

$$= \sum_{k=i}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} \binom{n}{k} (1-x)^{n-k} x^k$$

$$= \sum_{k=i}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} b_k^n(x)$$

$$= \sum_{k=i}^{n} A_{ki} b_k^n(x) = \sum_{k=0}^{n} A_{ki} b_k^n(x).$$

Then we have

$$\sum_{j=0}^{n} c_j b_j^n = \sum_{i=0}^{n} a_i x^i$$

$$= \sum_{i=0}^{n} a_i \sum_{j=0}^{n} A_{ji} b_j^n$$

$$= \sum_{j=0}^{n} \sum_{i=0}^{n} A_{ji} a_i b_j^n.$$

This gives

$$c_j = \sum_{i=0}^{n} A_{ji} a_i.$$

This completes the proof.

By expanding b_i^n we have

$$b_i^k = \sum_{k=i}^{n} (-1)^{k-i} \binom{n}{k} \binom{k}{i} x^k.$$

Then letting B be the inverse of A, we have

$$B_{ki} = (-1)^{k-i} \left(\begin{array}{c} n \\ k \end{array} \right) \left(\begin{array}{c} k \\ i \end{array} \right),$$

for $k \geq i$ and zero otherwise.

For $n = 3$ we have

$$\begin{aligned} b_0^3 &= (1-x)^3 \\ b_1^3 &= 3(1-x)^2 x \\ b_2^3 &= 3(1-x)^1 x^2 \\ b_3^3 &= x^3. \end{aligned}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 0 & 0 \\ 1 & 2/3 & 1/3 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Thus

$$\begin{aligned} 1 &= b_0^3(x) + b_1^3(x) + b_2^3(x) + b_3^3(x) \\ x &= 1/3 b_1^3(x) + 2/3 b_2^3(x) + b_3^3(x) \\ x^2 &= 1/3 b_2^3(x) + b_3^3(x) \\ x^3 &= b_3^3(x) \end{aligned}$$

For the inverse B, we have

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}.$$

Then for example

$$b_2^3(x) = 3x^2 + -3x^3.$$

3 The Derivative of a Bernstein Polynomial

We shall show that the derivative of the Bernstein polynomial b_k^n is given as a linear combination of two lower order Bernstein polynomials. We shall show that

$$\frac{db_k^n}{dx} = n(b_{k-1}^{n-1} - b_k^{n-1}).$$

In fact we have

$$\frac{db_k^n}{dx} = \begin{pmatrix} n \\ k \end{pmatrix} [(n-k)(1-x)^{(n-1)-k}x^k + k(1-x)^{(n-1)-(k-1)}x^{k-1}]$$

$$= \begin{pmatrix} n \\ k \end{pmatrix} \left[kb_{k-1}^{n-1} / \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} - (n-k)b_k^{n-1} / \begin{pmatrix} n-1 \\ k \end{pmatrix} \right].$$

Using

$$\begin{pmatrix} m \\ j \end{pmatrix} = \frac{m!}{j!(m-j)!},$$

and cancelling terms, we get the result.

4 The Derivative of a Bezier Curve

We have shown that

$$\frac{db_k^n}{dx} = n(b_{k-1}^{n-1} - b_k^{n-1}).$$

Therefore if

$$p = p_0 b_0^n + p_1 b_1^n + ... + p_n b_n^n,$$

then

$$p' = n[p_0(-b_0^{n-1}) + p_1(b_0^{n-1} - b_1^{n-1}) + p_2(b_1^{n-1} - b_2^{n-1}) + ... + p_n(b_{n-1}^{n-1})]$$

$$= n(p_1 - p_0)b_0^{n-1} + n(p_2 - p_1)b_1^{n-1} + ... + n(p_n - p_{n-1})b_{n-1}^{n-1}.$$

The derivative of a Bezier curve can be computed by combining the control points to get a new Bezier curve of lower order. This can be continued to compute higher order derivatives.

5 The Representation of a Line Segment

Let a line segment have end points P_0 and P_3. The line segment is

$$(1-x)P_0 + xP_3 =$$

$$[(b_0^3 + b_1^3 + b_2^3 + b_3^3) - (\frac{b_1^3}{3} + \frac{2b_2^3}{3} + b_3^3)]P_0 + [\frac{b_1^3}{3} + \frac{2b_2^3}{3} + b_3^3]P_3 =$$

$$[b_0^3 + \frac{2b_1^3}{3} + \frac{b_2^3}{3}]P_0 + [\frac{b_1^3}{3} + \frac{2b_2^3}{3} + b_3^3]P_3 =$$

$$b_0^3 P_0 + b_1^3 P_1 + b_2^3 P_2 + b_3^3 P_3,$$

where

$$P_1 = \frac{2}{3}P_0 + \frac{1}{3}P_3 = P_0 + \frac{1}{3}(P_3 - P_0),$$

and

$$P_2 = \frac{1}{3}P_0 + \frac{2}{3}P_3 = P_0 + \frac{2}{3}(P_3 - P_0).$$

6 The Projective Bezier Curve

A Bezier curve in projective space is defined by

$$c(t) = \sum_{k=0}^{n} P_k b_k^n,$$

for $0 \leq t \leq 1$. Each homogeneous control point P_k is a four dimensional vector. The Euclidean coordinates (Affine coordinates) are given by

$$x(t) = c_1(t)/c_4(t)$$

$$y(t) = c_2(t)/c_4(t)$$

$$z(t) = c_3(t)/c_4(t).$$

The curve defined by these three coordinates is called a rational curve because the coordinates are rational functions of the parameter t.

7 An Example: The Circular Arc

Consider a circle of radius 1 and center at $(1, 0)$. It has the equation

$$(x-1)^2 + y^2 = 1.$$

The equation of the line through the origin with slope t has equation

$$y = tx.$$

We will use t as a parameter for our unit circle. Given a line with slope t, let the intersection point (x, y) of the line and the circle correspond to t. Solving these two equations simultaneously we find that

$$x = \frac{2}{1+t^2},$$

and

$$y = \frac{2t}{1+t^2}.$$

Moving the center to the origin we have

$$x = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2},$$

and

$$y = \frac{2t}{1+t^2}.$$

Thus as t varies from minus infinity to plus infinity we get every point of the circle except the point $(-1, 0)$. We have essentially a rational parameterization of the unit circle. Unfortunately the parametric interval is not finite and the parameterization is not uniform. So in practice we shall use only a portion of this parameterization.

Suppose we want a rational parametric arc of angle 2θ, where θ is less than π . We may use our parameterization of the unit circle. From our original circle centered at $(1, 0)$ we see that if θ is the angle from the center to the point (x, y), then the slope of the intersecting straight line is

$$t = \tan(\theta/2).$$

So let

$$\alpha = \tan(\theta/2),$$

$$t_1 = -\alpha,$$

and

$$t_2 = \alpha.$$

Then as t varies between t_1 and t_2 we get a circular arc of angle 2θ. We shall represent this arc as a Bezier curve. To this end, we change our parameterization from $[t_1, t_2]$ to $[0, 1]$. Let

$$s = \frac{t - t_1}{t_2 - t_1} = \frac{t + \alpha}{2\alpha}.$$

Solving this for t and substituting in the rational representation for the unit circle we get a rational quadratic function of s.

$$x = \frac{-4\alpha^2 s^2 + 4\alpha^2 s + (1 - \alpha^2)}{4\alpha^2 s^2 - 4\alpha^2 s + (1 + \alpha^2)},$$

$$y = \frac{4\alpha s - 2\alpha}{4\alpha^2 s^2 - 4\alpha^2 s + (1 + \alpha^2)}.$$

Then

$$\begin{bmatrix} p_{0x} \\ p_{1x} \\ p_{2x} \\ p_{3x} \end{bmatrix} = A \begin{bmatrix} 1 - \alpha^2 \\ 4\alpha^2 \\ -4\alpha^2 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} p_{0y} \\ p_{1y} \\ p_{2y} \\ p_{3y} \end{bmatrix} = A \begin{bmatrix} -2\alpha \\ 4\alpha \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} p_{0z} \\ p_{1z} \\ p_{2z} \\ p_{3z} \end{bmatrix} = A \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} p_{0w} \\ p_{1w} \\ p_{2w} \\ p_{3w} \end{bmatrix} = A \begin{bmatrix} 1 + \alpha^2 \\ -\alpha^2 \\ 4\alpha^2 \\ 0 \end{bmatrix},$$

The curve c is given by

$$c(s) = p_0 b_0(s) + p_1 b_1(s) + p_2 b_2(s) + p_3 b_3(s).$$

Where

$$p_k = \begin{bmatrix} p_{kx} \\ p_{ky} \\ p_{kz} \\ p_{kw} \end{bmatrix}.$$

If T is an affine transformation then

$$Tc(s) = Tp_0 b_0(s) + Tp_1 b_1(s) + Tp_2 b_2(s) + Tp_3 b_3(s).$$

It follows that the curve C may be transformed by transforming the control points. A general affine transformation has matrix

$$T = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 & q_x \\ -\sin(\theta) & \cos(\theta) & 0 & q_y \\ 0 & 0 & 0 & q_z \\ 0 & 0 & 0 & 1/s \end{bmatrix},$$

where θ is a rotation angle about z, s is a scaling, and q is a translation vector. We have converted to bezier coefficients by using our conversion matrix A to convert from the power basis to the Bernstein bases and we then get a set of Bezier control points: p_0, p_1, p_2, and p_3. In the case that our arc has angle π , $\alpha = -1$.

8 The Bernstein Polynomial On Interval [a,b]

Let

$$t = \frac{s-a}{b-a}.$$

Then

$$B_i^n(t) = B_i^n((s-a)/(b-a) = \left(\begin{array}{c} n \\ i \end{array} \right) \frac{(s-a)^i (b-s)^{n-i}}{(b-a)^n}.$$

We define the ith Bernstein polynomial of degree n, on the interval $[a,b]$, to be

$$B_{i,a,b}^n(s) = \left(\begin{array}{c} n \\ i \end{array} \right) \frac{(s-a)^i (b-s)^{n-i}}{(b-a)^n}.$$

9 A Bezier Curve as a Special B-Spline

The ith B-spline bases function $N_{i,k,\tau}$, of order k (and degree $n = k-1$), on knot set τ, is recursively defined by the Cox-DeBoor algorithm.

The algorithm is expressed as the equation.

$$N_{i,k,\tau}(x) = \frac{x - t_i}{t_{i+k-1} - t_i} N_{i,k-1,\tau}(x) + \frac{t_{i+k} - x}{t_{i+k} - t_{i+1}} N_{i+1,k-1,\tau}(x).$$

A term is understood to be zero if its denominator is zero. $N_{0,1}(x)$, is a square pulse. It takes a positive constant value for $t_0 \le x < t_1$ and is zero elsewhere. It is continuous from the right, but not from the left.

Let τ_n be the special knot set $t_0 = t_1 = .. = t_n = a$, and $t_{n+1} = t_{n+2} = ...t_{2n+1} = b$. Thus in the cubic case

$$\tau_3 = \{a, a, a, a, b, b, b, b\}.$$

We claim that the ith B-spline of degree n is the ith Bernstein polynomial. We take the order of the B-spline to be $k = n+1$. That is the order is the degree plus one. We shall prove that

$$N_{i,k,\tau_n} = B^n_{i,a,b}\chi_{[a,b)}.$$

The characteristic function of a set takes value one on any point in the set, and value zero on any point outside the set. The characteristic function $\chi_{[a,b)}$ of the set $[a,b)$ is equal to 1, if t is in $[a,b)$, and is equal to 0 otherwise.

To prove that the equation is correct, we use induction on the order k. So assume that the equation is true for order $k-1$. We start the induction with the fact that $N_{0,1} = 1 = B^0_{0,a,b}$ on the knot set $\{a,b\}$. So the equation is true for order 1. Using the B-spline recursion formula, and the assumed truth of the equation for order $k-1$, we have

$$N_{i,k,\tau_n}(x) = xN_{i,k-1,\tau_n}(x) + (1-x)N_{i+1,k-1,\tau_n}(x)$$

$$= xN_{i-1,k-1,\tau_{n-1}}(x) + (1-x)N_{i,k-1,\tau_{n-1}}(x)$$

$$= xB^{n-1}_{i-1,a,b}(x) + (1-x)B^{n-1}_{i,a,b}(x) = B^n_{i,a,b}(x).$$

Thus the equation is true for order k, if it is true for order $k-1$. This completes the inductive proof.

We shall now establish a result that allows a continuous composite Bezier curve to be written in a simple form, with a simple knot set, as a B-spline curve. We shall illustrate the property with a specific knot set. Consider the knot set

$$\tau = \{t_0, t_1, ..., t_{10}\} = \{a, a, a, a, b, b, b, c, c, c, c\}.$$

Then

$$\begin{aligned} N_{3,4,\tau} &= \frac{t - t_3}{t_6 - t_3} N_{3,3,\tau} + \frac{t_7 - t}{t_7 - t_4} N_{4,3,\tau} \\ &= \frac{t-a}{b-a} N_{3,3,\tau} + \frac{c-t}{c-b} N_{4,3,\tau} \\ &= \frac{t-a}{b-a} B^2_{3,a,b}\chi_{[a,b]} + \frac{c-t}{c-b} B^2_{4,a,b}\chi_{[b,c]} \\ &= \frac{(t-a)^3}{(b-a)^3}\chi_{[a,b)} + \frac{(c-t)^3}{(c-b)^3}\chi_{[b,c]} \\ &= B^3_{3,a,b}\chi_{[a,b)} + B^3_{4,a,b}\chi_{[b,c]}. \end{aligned}$$

(this needs some checking and editing). A continuous composite Bezier curve with control points $P_0, ..., P_6$, in the interval $[a, c]$ is given by

$$c(t) = (P_0 B^3_{0,a,b} + P_1 B^3_{1,a,b} + P_2 B^3_{2,a,b} + P_3 B^3_{3,a,b})\chi_{[a,b)}$$

$$+(P_3 B^3_{0,b,c} + P_4 B^3_{1,b,c} + P_5 B^3_{2,b,c} + P_6 B^3_{3,b,c})\chi_{[b,c)}.$$

From what we have deduced above, this is equivalent to

$$c(t) = P_0 N_{0,4,\tau} + P_1 N_{1,4,\tau} + ... + P_6 N_{6,4,\tau}.$$

The result can be extended to any continuous composite Bezier curve of degree n.

10 Finding a Cubic Bernstein Interpolant

Let $f(t)$ be a function defined on the interval $[a, b]$. We wish to find the coefficients of a cubic interpolant to f, on equally spaced points in $[a, b]$, in the Bezier basis. We shall first specialize and find the coefficients for a cubic polynomial defined on $[0, 1]$. Consider

$$p(x) = \sum_{j=0}^{3} a_j B^3_j(x),$$

where $0 \leq x \leq 1$. Write

$$p_i = p(i/3) = \sum_{j=0}^{3} a_j B_j^3(i/3), i = 0, 1, 2, 3.$$

Solving this system, we find

$$a_0 = p_0$$

$$a_1 = -\frac{5}{6}p_0 + 3p_1 - \frac{3}{2}p_2 + \frac{1}{3}p_3,$$

$$a_2 = \frac{1}{3}p_0 - \frac{3}{2}p_1 + 3p_2 - \frac{5}{6}p_3,$$

$$a_3 = p_3.$$

Let

$$x = \frac{t-a}{b-a}.$$

If $x_i = \frac{i}{3}$ then

$$t_i = a + \frac{b-a}{3}i.$$

Define

$$p_i = p(x_i) = f(t_i).$$

Then by the uniqueness of polynomial interpolation

$$q(t) = p(x(t)),$$

Is the desired cubic interpolant to f. To calculate $q(t)$, first find the corresponding x, and then evaluate $p(x)$.

We could also establish these facts by making use of the equation

$$B_j^3(x(t)) = \binom{3}{j} \frac{(b-t)^{3-j}(t-a)^j}{(b-a)^3}.$$

Note that when j of these local interpolants are chained together to give a Bezier piecewise polynomial parameterized on the interval [0,j], equally spaced points in this parameterization, do not necessarily correspond to equally spaced points on a chord length parameterization. This is because the chords are not necessarily of uniform length.

11 File Structure

We shall describe a data structure for representing Bezier curves. Let the first row contain a single integers k_1, k_2, k_3. k_1 is the degree. $k_2 = 0$ if each segment ending control point is the same as the next starting control point, and thus appears only once in the file. If $k_2 = 1$ then such control points do not agree in general, which allows for discontinuity of the curve. If $k_3 = 0$ the curve is not rational. It is rational if $k_3 = 1$. Thus the file takes the form

$$k_1 k_2 k_3$$

$$x_1 y_1 z_1 w_1$$

$$x_2 y_2 z_2 w_2$$

$$x_3 y_3 z_3 w_3$$

$$x_4 y_4 z_4 w_4$$

$$x_5 y_5 z_5 w_5$$

....

$$x_m y_m z_m w_m$$

The number of columns may vary from 1 for functions, to 4 for rational curves in 3-space . The first $k_1 + 1$ rows contain the Bezier control points for the first Bezier curve, the next $n + 1$ rows are the control points for the next Bezier curve (n rows if $k_2 = 0$), and so on. If there are k Bezier curve segments joined together, then they may be parameterized on the interval $[0, k]$ in the obvious way. Note also that the segment chord length is given by the length of the affine vector from the first Bezier control point to the last control point of the segment. So the curve can also be parameterized by chord length.

12 The de Casteljau Algorithm

Define $P_{n_0 \ldots n_k}(t)$ to be the point on the Bezier curve, which has control points $P_{n_0}, \ldots, P_{n_k}$, at parameter t.
Proposition.

$$P_{n_0 \ldots n_k}(t) = (1 - t) P_{n_0 \ldots n_{k-1}}(t) + t P_{n_1 \ldots n_k}(t)$$

Proof.

$$(1-t)P_{n_0\dots n_{k-1}}(t)+tP_{n_1\dots n_k}(t)=$$

$$(1-t)\sum_{i=0}^{k-1}B_i^{k-1}P_{n_i}+t\sum_{i=0}^{k-1}B_i^{k-1}P_{n_{i+1}}=$$

$$(1-t)B_0^{k-1}P_{n_0}+(1-t)\sum_{i=1}^{k-1}B_i^{k-1}P_{n_i}+t\sum_{i=1}^{k-1}B_{i-1}^{k-1}P_{n_i}+B_{k-1}^{k-1}P_{n_k}=$$

$$B_0^kP_{n_0}+(1-t)\sum_{i=1}^{k-1}((1-t)B_i^{k-1}+tB_{i-1}^{k-1})P_{n_{i+1}}+B_k^kP_{n_k}=$$

$$\sum_{i=0}^{k}B_i^kP_{n_k}=P_{n_0\dots n_k}(t).$$

This completes the proof.

Define

$$P_j^k=P_{j,\dots,j+k}.$$

Then the recursion formula becomes

$$P_j^k=(1-t)P_j^{k-1}+tP_{j+1}^{k-1}.$$

We may dislay this in a table as

$$\begin{array}{llll}
P_0 & & & \\
P_1 & P_0^1 & & \\
P_2 & P_1^1 & P_0^2 & \\
P_3 & P_2^1 & P_1^2 & P_0^3 \\
\dots & \dots & \dots & \dots
\end{array}$$

13 Subdivision

Proposition. Let

$$0<a<1$$

then for $t \in [0,1]$

$$P_{0...k}(at) = Q_{0...k}(t),$$

where

$$Q_0 = P_0(a) = P_0,$$

$$Q_1 = P_{01}(a),$$

$$Q_2 = P_{012}(a),$$

....

$$Q_k = P_{012..k}(a).$$

Proof. Let $k = 1$.

$$Q_{01}(t) = (1-t)Q_0 + tQ_1 =$$

$$(1-t)P_0 + tP_{01}(a) =$$

$$(1-t)P_0 + t((1-a)P_0 + aP_1) =$$

$$(1-ta)P_0 + taP_1 =$$

$$P_{01}(at).$$

Proposition. Let

$$0 < a < 1$$

then for $t \in [0,1]$

$$P_{0...k}(a(1-t)+t) = R_{0...k}(t),$$

where

$$R_0 = P_{0123...k}(a).$$

....

$$R_{k-2} = P_{(k-2)(k-1)k}(a),$$

$$R_{k-1} = P_{(k-1)k}(a),$$

$$R_k = P_k(a) = P_0,$$

Proof.

14 The WF-curve as a Bezier curve

The Wilson-Fowler curve (WF-curve) is an interpolating curve. Suppose it passes through the interpolation points

$$P_1, P_2, ..., P_n.$$

Let u_i be a unit vector in the direction of the ith line segment joining the points, that is in the direction of $P_{i+1} - P_i$. Let v_i be the unit vector rotated a positive 90 degrees from u_i. These two vectors form a local coordinate system centered at P_i.

Let ℓ_i be the chord length of the ith segment and let s_i be the accumulated chord length to the ith point. The the value $c(s)$ of the Wilson-Fowler curve at a point on the ith segment, whose parameter s satisfies $s_i \leq s \leq s_{i+1}$, is given by

$$c(s) = P_i + (s - s_i)u_i + f_i(s - s_i)v_i,$$

where

$$f_i(x) = \frac{t_i^a x(x - \ell_i)^2 + t_i^b x^2(x - \ell_i)}{\ell_i^2}.$$

Note that

$$\frac{df_i(0)}{dx} = t_i^a,$$

and

$$\frac{df_i(\ell_i)}{dx} = t_i^b.$$

Consider

$$f_i(x) = \frac{t_i^a x(x - \ell_i)^2 + t_i^b x^2(x - \ell_i)}{\ell_i^2}.$$

We shall find the Bezier control points for this curve and then translate and rotate to get the control points for the WF segment. For simplicity of notation we will suppress the i subscript with the understanding that we are dealing with the ith segment of the WF-curve.

Let $w = x/\ell$. Then $f(x) = g(w)$ where

$$g(w) = \ell(t_a + t_b)w^3 - (2t_a + t_b)w^2 + t_a w.$$

Multiplying the column vector of coefficients

$$\ell \begin{bmatrix} 0 \\ t_a \\ -(2t_a + t_b) \\ t_a + t_b \end{bmatrix},$$

by the change of basis matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 0 & 0 \\ 1 & 2/3 & 1/3 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

we find bernstein coefficient vector

$$\begin{bmatrix} 0 \\ \ell t_a/3 \\ -\ell t_b/3 \\ 0 \end{bmatrix}.$$

These are the y components of the control points. The x components of the control points are

$$\begin{bmatrix} 0 \\ \ell/3 \\ 2\ell/3 \\ \ell \end{bmatrix}.$$

This follows because

$$w = 1/3 b_1^3(w) + 2/3 b_2^3(w) + b_3^3(w).$$

So the Bezier control points for this curve are

$$Q_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} \ell/3 \\ \ell t_a/3 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 2\ell/3 \\ -\ell t_b/3 \end{bmatrix},$$

and

$$Q_3 = \begin{bmatrix} \ell \\ 0 \end{bmatrix}.$$

Now we shall rotate and translate the curve and thus rotate and translate the control points, so that we obtain the ith WF curve segment. Define

$$\begin{bmatrix} d_x \\ d_y \end{bmatrix} = P_{i+1} - P_i.$$

Then

$$\ell = \sqrt{d_x^2 + d_y^2}.$$

The proper rotation matrix is

$$\begin{bmatrix} d_x/\ell & -d_y/\ell \\ d_y/\ell & d_x/\ell \end{bmatrix}.$$

After applying the rotation matrix we must translate by P_i. Thus the control points for the ith WF segment are

$$Q_0 = P_i,$$

$$Q_1 = P_i + \begin{bmatrix} d_x/3 - d_y t^a/3 \\ d_y/3 + d_x t^a/3 \end{bmatrix},$$

$$Q_2 = P_i + \begin{bmatrix} 2d_x/3 + d_y t^b/3 \\ 2d_y/3 - d_x t^b/3 \end{bmatrix},$$

and

$$Q_3 = P_i + \begin{bmatrix} d_x \\ d_y \end{bmatrix} = P_{i+1}.$$

Therefore finally we have obtained a Bezier representation of the WF-curve. On the ith WF segment we have

$$c(s) = Q_0 b_0^3(w) + Q_1 b_1^3(w) + Q_2 b_2^3(w) + Q_3 b_3^3(w),$$

where

$$0 \le w = \frac{s - s_i}{\ell_i} \le 1.$$

15 Barycentric Coordinates

Suppose we are given an n-simplex with vertices $v_0, v_1, v_2, ..., v_n$. The barycentric coordinates of a point p sum to one. If the coordinates satisfy

$$0 < \lambda_i < 1,$$

then the point is an interior point of the simplex. If any coordinate is negative, then the point is exterior to the simplex. If

$$0 \le \lambda_i \le 1,$$

then the point is in the interior or on the boundary of the simplex. In the case

$$0 \le \lambda_i \le 1,$$

when a coordinate $\lambda_j = 0$, the point is on the boundary of the simplex opposite the vertex p_j.

To find the barycentric coordinates we may select an arbitrary vertex, say p_n, and solve the linear system

$$\sum_{i=0}^{n-1} \lambda_i (p_i - p_n) = p - p_n,$$

for $\lambda_0, ..., \lambda_{n-1}$. Since the barycentric coordinates sum to 1, this also determines λ_n.

Let us apply this to the problem of determining that a point is in a triangle of the plane. Suppose we are given the triangle vertices

$$p_1 = (1, 2),$$

$$p_2 = (1, 3),$$

$$p_3 = (2, 3).$$

and wish to determine if $p = (1.5, 2.6)$ is in the triangle. Our linear system is

$$\begin{bmatrix} (1-2) & (1-2) \\ (2-3) & (3-3) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} (1.5-2) \\ (2.6-3) \end{bmatrix}.$$

The solution is

$$\lambda_1 = .1, \lambda_2 = .4$$

Then we compute $\lambda_3 = .5$. Therefore, because all coordinates are between 0 and 1, the point is in the triangle.

The general computation to determine an interior point, requires 11 additions or subtractions, 6 multiplications, 2 divisions, and 3 comparisons. The computation may be done as follows.

Let $a_{11} = x_1 - x_3$,$a_{21} = y_1 - y_3$,$a_{12} = x_2 - x_3$,$a_{22} = y_2 - y_3$, and $b_1 = x - x_3$,$b_2 = y - y_3$. Then letting D be the determinant

$$D = a_{11}a_{22} - a_{21}a_{12},$$

we have

$$\lambda_1 = \frac{b_1 a_{22} - b_2 a_{12}}{D},$$

$$\lambda_2 = \frac{a_{11} b_2 - a_{21} b_1}{D},$$

and

$$\lambda_3 = 1 - \lambda_1 - \lambda_2$$