Quick Calculus Theory

James D Emery

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Contents

1 Introduction

The title of this work is Quick Calculus Theory. It is meant to give much of Calculus theory in a quick and abbreviated form. The ideas are presented in a way to give only the essence of the ideas, techniques, and proofs. This can be further developed and completed by the reader. Careful statements of the conditions under which theorems are true is mostly not present, so must be added for full rigor. Normally A big weighty Calculus book is full of examples, comments, and guidance. It is not the purpose of this document to attempt an immitation of such a book. Rather it is presented mostly for review and recall. We sometimes make an informal requirement that a function is "nice." By a nice function we usually mean that it is a function that has at least a continuous derivative.

The document called **What is Calculus?** is related to this one. It gives an introduction to the ideas of Calculus, with a few examples and applications. It is reprinted here as **Appendix B**.

`www.stem2.org/je/calcwhat.pdf`

2 The Limit of a Function

Let f be a real valued function of a real variable. The limit of $f(x)$ as x goes to x_0 is c, if and only if, for every positive number $\epsilon > 0$ there exists a number $\delta > 0$ such that, if $|x - x_0| < \delta$, then $|f(x) - c| < \epsilon$. This is written as

$$\lim_{x \to x_0} f(x) = c.$$

Example 1. It is intuitive that if $f(x) = x^2$, then

$$\lim_{x \to 3} f(x) = 9.$$

We must prove this fact using the definition. We must show that

$$|x^2 - 9|$$

can be made small, when

$$|x - 3|$$

is sufficiently small. To do this we shall find a relationship between these two expressions. Suppose δ is some positive number, and suppose $|x - 3| < \delta$. Then

$$|x| = |x - 3 + 3| \leq |x - 3| + |3| < \delta + 3.$$

Then

$$|x + 3| \leq |x| + 3 < \delta + 3 + 3 = \delta + 6.$$

Now we can find an inequality for the difference of the squares.

$$|x^2 - 9| = |x - 3||x + 3| \leq \delta(\delta + 6).$$

Now given an arbitrary $\epsilon > 0$, we can find a proper δ. Indeed, choose δ to be less than 1 and less than $\epsilon/7$, then

$$|x^2 - 9| = |x - 3||x + 3| \leq \delta(\delta + 6) < \frac{\epsilon}{7}(1 + 6) = \epsilon.$$

3 Limit Theorems

$$\lim_{x \to a} f(x) + \lim_{x \to a} g(x) = \lim_{x \to a}(f(x) + g(x)).$$

$$\lim_{x \to a} f(x) \lim_{x \to a} g(x) = \lim_{x \to a}(f(x)g(x)).$$

$$\lim_{x \to a}(\alpha f(x)) = \alpha \lim_{x \to a} f(x).$$

If $\lim_{x \to a} f(x)$ is not zero, then

$$\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{\lim_{x \to a} f(x)}.$$

Example. We shall prove the product formula. Let

$$\lim_{x \to a} f(x) = b,$$

and

$$\lim_{x \to a} g(x) = c.$$

Then

$$\begin{aligned} &|f(x)g(x) - bc| \\ &= |f(x)g(x) - f(x)c + cf(x) - bc| \\ &\leq |f(x)||g(x) - c| + |c||f(x) - b|. \end{aligned}$$

Given $\epsilon_1 > 0 \ \exists \ \delta_1$, such that if $|x - a| < \delta_1$, then $|f(x) - b| < \epsilon_1$, and given $\epsilon_2 > 0 \ \exists \ \delta_2$, such that if $|x - a| < \delta_2$, then $|g(x) - c| < \epsilon_2$. We have

$$|f(x)| = |f(x) - b + b| \leq \epsilon_1 + |b|,$$

so

$$|f(x)g(x) - bc| \leq (\epsilon_1 + |b|)\epsilon_2 + |c|\epsilon_1.$$

Given ϵ, we may choose ϵ_1 and ϵ_2 so that

$$(\epsilon_1 + |b|)\epsilon_2 + |c|\epsilon_1 < \epsilon.$$

Let δ be the smaller of δ_1 and δ_2. Then if $|x - a| < \delta$, then

$$|f(x)g(x) - bc| < \epsilon.$$

4 Continuity

A function is continuous at a point a iff (if and only if)

$$\lim_{x \to a} f(x) = f(a).$$

Example. Define function f by, $f(x) = 0$, if $x < 0$, and $f(x) = 1$ if $x \geq 0$. Then f is not continuous at 0. To prove this, we only need to find one $\epsilon > 0$ for which there is no $\delta > 0$, so that there is some x whose distance to 0 is less than delta, but for which $|f(x)| > \epsilon$. Let us choose $\epsilon = 1/2$. Let δ be any positive number. Let $x = -\delta/2$. Then $|x - 0| < \delta$, but

$$|f(x) - f(0)| = |0 - 1| = 1 > \epsilon.$$

We can find no δ that works for this $\epsilon = 1/2$. Hence f is not continuous at 0.

5 The Derivative

Given a function $y = f(x)$, the slope of the secant line, which passes through the curve at $(x, f(x))$ and at $(x + h, f(x + h)$, is

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}.$$

As $h \to 0$, this secant line approaches the tangent line at $(x, f(x))$. The slope of this limiting tangent line is called the derivative of f at x. We write the derivative as

$$f'(x) = \frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Example Given two differentiable functions f and g, the derivative of the product is

$$\frac{d(fg)}{dx} = \frac{df}{dx}g + f\frac{dg}{dx}.$$

Proof.

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}g(x) \\
&= f\frac{dg}{dx} + \frac{df}{dx}g.
\end{aligned}$$

Example. The derivative of a constant is zero. Let $f(x) = c$. Then

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = 0.$$

Example. The derivative of the identity is one. Let $f(x) = x$. Then

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x + h - x}{h} = 1.$$

Example. Let $f(x) = x^n$. Then

$$\frac{df}{dx} = nx^{n-1}.$$

Proof. This is true for the identity where $n = 1$. Assume it is true for n. Let $f(x) = x^{n+1}$ and $g(x) = x^n$. Then

$$g'(x) = nx^{n-1}.$$

$$f'(x) = (xg(x))' = 1g(x) + xg'(x) = x^n + xnx^{n-1} = (n+1)x^n.$$

So it is true for $n+1$. By induction it is true for all positive integers.

6 Maxima and Minima

If a nice function f has a relative maxima, or a relative minima at a point a, then $f'(a) = 0$.

Proof. Suppose there is a relative maxima at a, then for small $h > 0$,

$$f(a+h) - f(a) \leq 0$$

and

$$\frac{f(a+h) - f(a)}{h} \leq 0.$$

Similarly for $h < 0$,

$$\frac{f(a+h) - f(a)}{h} \geq 0.$$

Therefore both

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \leq 0,$$

and

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \geq 0.$$

Therefore $f'(a) = 0$.

7 Rolle's Theorem

Let f be a differential function on $[a, b]$. Suppose $f(a) = f(b)$. Then there is a number c, $a < c < b$ so that $f'(c) = 0$.

Proof. There must be a relative maximum or a relative minimum of f at some point c between a and b, so a point c where the derivative is zero.

Michel Rolle, who lived from April 21, 1652 to November 8, 1719, was a French mathematician. He provided a formal proof of the theorem, now called Rolle's theorem (1691). Also he is said to be the co-inventor of Gaussian elimination (1690).

8 The Mean Value Theorem

Theorem Let f be a nice function. There exists a point c, $a < c < b$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Let function g be formed by subtracting the straight line that passes through the points $(a, f(a))$ and $(b, f(b)$. Then

$$g(x) = f(x) - [\frac{b - x}{b - a} f(a) + \frac{a - x}{a - b} f(b)].$$

Then $g(a) = g(b) = 0$, so by Rolle's Theorem there is a c, $a < c < b$, so that $g'(c) = 0$. Then

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Hence

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

9 The Intermediate Value Theorem

Theorem. If $f(x)$ is continuous on $[a, b]$ and u is between $f(a)$ and $f(b)$ then there exists a c in $[a, b]$ so that

$$f(c) = u$$

Proof. This follows from the completness of the real numbers.

Also see the Bolzano Theorem.

As a second proof we can use the fact that a connected set is mapped by a continuous function to a connected set. So assuming there were a u between $f(a)$ and $f(b)$ not in the image of $[a, b]$, then the image would not be connected, which is a contradiction.

note There is also an intermediate value theorem for derivatives.

10 Taylor's Formula

Functions can be approximated by polynomials. Consider the polynomial

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + c_3(x - x_0)^3 + ... + c_n(x - x_0)^n$$

The kth derivative at x_0 is

$$p^{(k)}(x_0) = k!c_k.$$

So

$$c_k = \frac{p^{(k)}(x_0)}{k!}.$$

So the polynomial can be written as

$$p(x) = \sum_{k=0}^{n} \frac{p^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Now consider an arbitrary function with n derivatives. The polynomial

$$p(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,$$

has derivatives that agree with the derivatives of function f at x_0, that is

$$f^k(x_0) = p^k(x_0)$$

for $k = 0, 1, 2, 3, ..., n$.

So in a neighborhood of x_0, $f(x)$ is approximated by the polynomial $p(x)$. $p(x)$ is called the nth degree Taylor polynomial for function f. This is an approximation at x in general so there is an error term. The folowing theorem gives an expression for the error.

Theorem *Taylor's Formula With Remainder*. Let f be a function that has n derivatives at each point in the interval (a, b). Then given x_0 and x in (a, b) there is a number c, between x_0 and x, so that

$$f(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + f^{(2)}(x_0)\frac{(x - x_0)^2}{2!} + f^{(3)}(x_0)\frac{(x - x_0)^3}{3!} + ... +$$

$$f^{(n-1)}(x_0)\frac{(x - x_0)^{n-1}}{(n-1)!} + f^{(n)}(c)\frac{(x - x_0)^n}{n!}$$

.

This c depends on x, x_0 and n

Proof. Let p be the Taylor polynomial of degree $n-1$

$$p(x) = f(x_0)+f^{(1)}(x_0)(x-x_0)+f^{(2)}(x_0)\frac{(x-x_0)^2}{2!}+...+f^{(n-1)}(x_0)\frac{(x-x_0)^{n-1}}{(n-1)!}.$$

The strategy is to take the difference between $f(x)$ and the right side of the Taylor formula, where an unknown constant M replaces the derivative value of $f^{(n)}(c)$ in the error term. The difference is set equal to a function $g(x)$. Function $g(x)$ is differentiated n times, applying Rolles Theorem repeatedly. This will allow us to find the constant c, and determine the constant M so that

$$M = f^{(n)}(c).$$

Without loss of generality we shall assume $x_0 < x$.

Let M be defined by

$$f(x) = p(x) + \frac{M(x-x_0)^n}{n!}.$$

We define a function g by

$$g(t) = f(t) - [p(t) + \frac{M(t-x_0)^n}{n!}].$$

This will allow us to apply Rolles theorem to g on the interval $[x_0, x]$.

So by our definitions we have

$$g(x_0) = f(x_0) - f(x_0) + 0 = 0,$$

and

$$g(x) = 0.$$

By Rolle's' Theorem there is a number x_1, $x_0 < x_1 < x$, so that $g'(x_1) = 0$. Because $g'(x_0) = 0$, we may apply Rolle's Theorem again to $g' = g^{(1)}$ and obtain a number $x_2, x_0 < x_2 < x_1$, so that $g^{(2)}(x_2) = 0$.

Continuing in this way, after n steps, we find that there is an x_n so that $x_0 < x_n < x$, and letting $c = x_n$, so that

$$f^{(n)}(c) - M = g^{(n)}(c) = 0$$

(Notice that we have annihilated the polynomial p by n differentiations). Therefore $M = f^{(n)}(c)$ and so we have

$$f(x) = p(x) + f^{(n)}(c)\frac{(x-a)^n}{n!}.$$

This is Taylor's Formula.

If we let n go to infinity, we get the formal power series

$$\sum_{k=0}^{\infty} f^{(k)}(a)\frac{(x-a)^k}{k!}$$

known as the Taylor series for the function $f(x)$. Such a series may or may not converge and may or may not represent the function $f(x)$.

Theorem If the error term in Taylor's Formula goes to zero as n goes to infinity then the Taylor Series for $f(x)$ converges and represents the function $f(x)$.

Proof.

See the section on Taylor Series.

The converse of this theorem is false. That is there exists a function $f(x)$ which has derivatives of all orders (called a C^∞ function), that has a Taylor Series that converges, but the series does not represent the function. See the section on Taylor Series.

Examples of Taylor's Formula and Taylor's series.

Let the nth degree Taylor polynomial for the function $f(x)$ be

$$p_n(x) = \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!}$$

Then by Taylor's formula, the Taylor series converges pointwise at x provided

$$f(x) - p_n(x) = f^{(n)}(c(x,n))\frac{(x-x_0)^{n+1}}{(n+1)!} \to 0.$$

We write $c(x,n)$ because the constant c in Taylors formula depends upon x and n.

The Taylor Series for the exponential function $\exp(x)$.

Each derivative of the exponential function $\exp(x)$ equals the function itself. So the Taylor series for the exponential function developed about 0 is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Given $a > 0$, for all $x \in [-a, a]$

$$|\exp(x) - p_n(x)| \leq \exp(c(x,n))\frac{a^{n+1}}{(n+1)!} \leq \exp(a)\frac{a^{n+1}}{(n+1)!}.$$

The right side goes to zero as n goes to ∞. Thus the series represent the function for all real $x \in [-a, a]$. And the convergence is uniform on this interval (See a later section on uniform convergence of a sequence of functions). Since $a > 0$ is arbitrary, the series converges and represents the function $\exp(x)$ for all real x. However, the convergence of the polynomial functions p_n is not uniform on the whole real line. It will take more terms for the series to converge to $\exp(x)$ for large x.

Approximating the Sine Function With a Taylor Polynomial.

The derivative of $\sin(x)$ is $\cos(x)$, of $\cos(x)$ is $-\sin(x)$. It follows that the Taylor series about 0 for $\sin(x)$ is

$$\begin{aligned}\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + ... \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)!}.\end{aligned}$$

Keeping the 19 nonzero terms in Taylors Formula up to degree $n = 37$ we find that the Taylor approximation is accurate to a full 14 decimal places over the interval $[0., 2\pi]$, as is verified in the following table produced by computer program **sine.ftn**.

x	$\sin(x)$	Taylor Approximation
.00000000000000	.00000000000000	.00000000000000
.26179938779915	.25881904510252	.25881904510252
.52359877559830	.50000000000000	.50000000000000
.78539816339745	.70710678118655	.70710678118655
1.0471975511966	.86602540378444	.86602540378444
1.3089969389957	.96592582628907	.96592582628907
1.5707963267949	1.0000000000000	1.0000000000000
1.8325957145940	.96592582628907	.96592582628907
2.0943951023932	.86602540378444	.86602540378444
2.3561944901923	.70710678118655	.70710678118655
2.6179938779915	.50000000000000	.50000000000000
2.8797932657906	.25881904510252	.25881904510252
3.1415926535898	0.0	0.0
3.4033920413889	-.25881904510252	-.25881904510252
3.6651914291881	-.50000000000000	-.50000000000000
3.9269908169872	-.70710678118655	-.70710678118655
4.1887902047864	-.86602540378444	-.86602540378444
4.4505895925855	-.96592582628907	-.96592582628907
4.7123889803847	-1.0000000000000	-1.0000000000000
4.9741883681838	-.96592582628907	-.96592582628907
5.2359877559830	-.86602540378444	-.86602540378444
5.4977871437821	-.70710678118655	-.70710678118655
5.7595865315813	-.50000000000000	-.50000000000000
6.0213859193804	-.25881904510252	-.25881904510252
6.2831853071796	0.0	0.0

The figure called **Taylor Approximation** shows the approximation for the two polynomials of degree 5 and 11 on the interval $[0, 2\pi]$. For x near zero the approximation is quit good with a low degree polynomial. Because of the nature of the periodic function $\sin(x)$, any value is determined by values on the interval $[0, \pi/2]$.

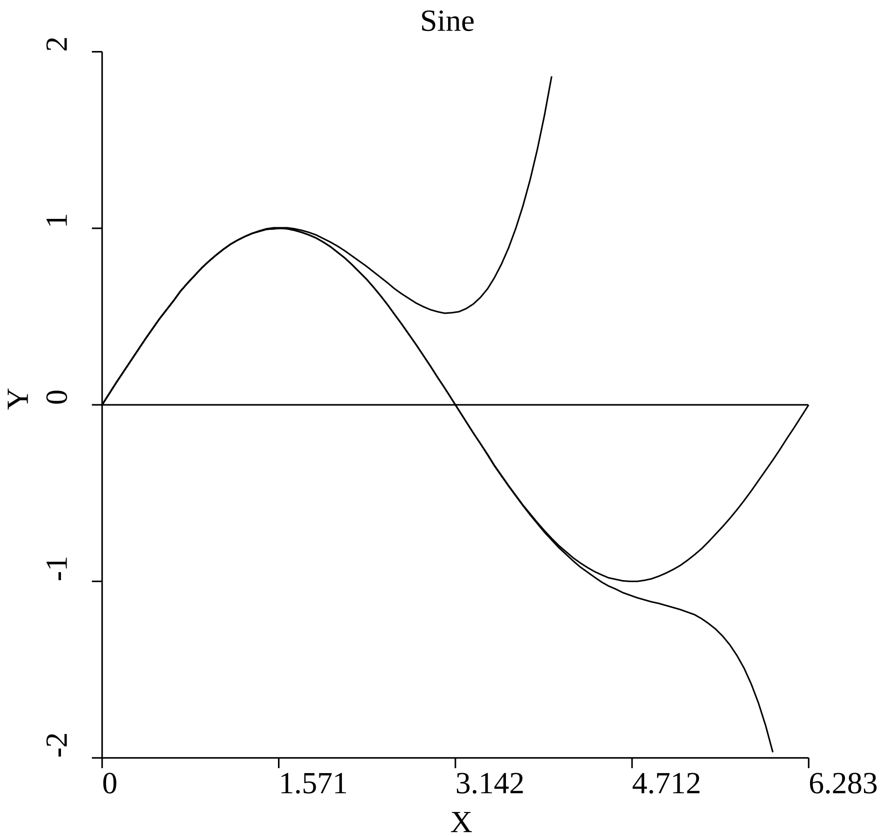

Figure 1: **Taylor Approximation**. The approximation of the function $y = \sin(x)$ on the interval $[0, 2\pi]$ by the 5th degree Taylor polynomial (upper curve), and by the 11th degree polynomial (lower curve).

11 L'Hospital's Rule

Suppose

$$\lim_{x \to a} f(x) = 0$$

and

$$\lim_{x \to a} g(x) = 0.$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Proof. Certain conditions must be specified, for example suppose the limit is from the right, the derivatives exist in the interval (a, b), and $g'(x)$ is not zero in this interval. It is not required that f and g be defined at a. But extend functions f and g to F and G defined on $[a, b)$, by defining $F(a) = 0$ and $G(a) = 0$. Then F and G are continuous in $[a, b)$. Let

$$h(x) = F(x)(G(b) - G(a)) - G(x)(F(b) - F(a)).$$

Then $h(a) = h(b)$. So by Rolle's Theorem there exists a c, $a < c < b$ so that

$$0 = h'(c) = F'(c)(G(b) - G(a)) - G'(c)(F(b) - F(a)) = f'(c)g(b) - g'(c)f(b).$$

So

$$\frac{f'(c)}{g'(c)} = \frac{f(b)}{f(b)}.$$

Taking limits as b and hence c go to a, we get the result.

L'Hospital's Rule also holds when $x \to \infty$ and when the limits in the numerator and denominator are infinity.

L'Hospital, Guillaume de (1661-1704) was a French mathematician who, at age 15, solved a difficult problem about cycloids posed by Pascal. He published the first book ever on differential calculus, L'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes (1696). In this book, l'Hospital included l'Hospital's rule. l'Hospital's name is commonly seen spelled both "l'Hospital" and "l'Hopital" (e.g., Maurer 1981, p. 426), the two being equivalent in French spelling.

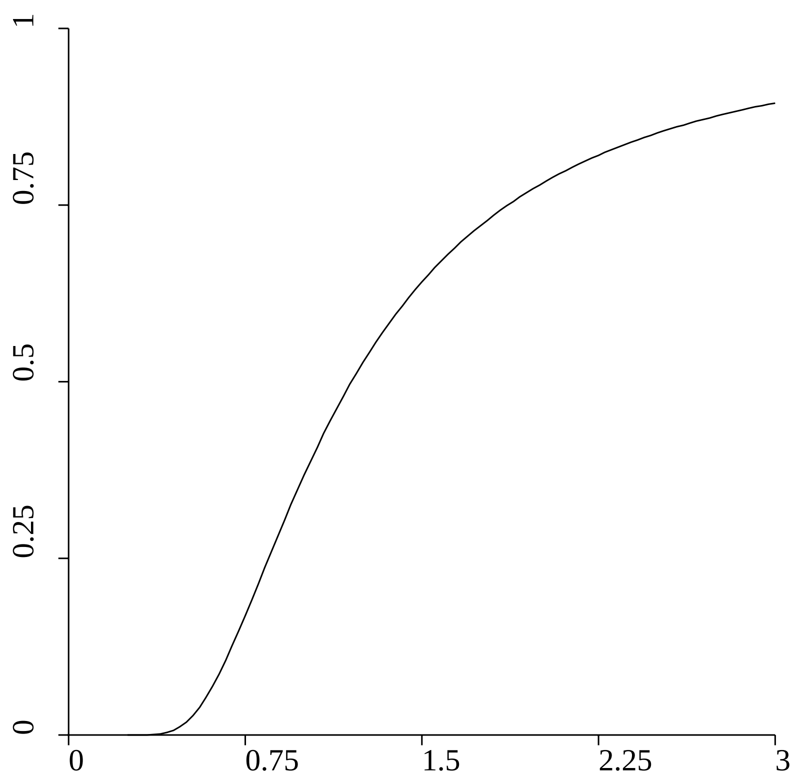

Figure 2: **A Function not represented by its Taylor Series**. $f(x) = \exp(-1/x^2)$. All derivatives at zero are zero, so the Taylor Series is the zero power series.

12 A Non-analytic Smooth Function, A C^∞ Function Without a Taylor Series

Cauchy discovered the function

$$\exp(-1/x^2)$$

which has continuous derivatives of all orders, and whose nth derivative values at 0 are all 0, thus which has a zero Taylor series about zero, so does not represent the function, except at 0. We have

$$\lim_{x\to 0} \exp(-1/x^2) = 0$$

and all derivatives, using an induction argument, are equal to a finite sum of terms like

$$\exp(-1/x^2)\left[\frac{c}{x^k}\right],$$

where $k \geq 1$, and so which all go to zero as $x \to 0$.

$$f(x) = \exp(-1/x^2)$$

$$Df(x) = \exp(-1/x^2)[2/x^3]$$

$$D^2 f(x) = \exp(-1/x^2)[4/x^6 - 6/x^4]$$

$$D^3 f(x) = \exp(-1/x^2)[24/x^5 - 36/x^7 + 8/x^9]$$

$$D^4 f(x) = \exp(-1/x^2)[300/x^8 - 120/x^6 - 144/x^{10} + 16/x^{12}]$$

Now

$$D\exp(-1/x^2)/x^k) = 2\exp(-1/x^2)/x^{k+3} - k\exp(-1/x^2)/x^{k+1}$$

So all derivatives will be a sum of terms, each of which is a product of some constant c_k, $\exp(-1/x^2)$, and $1/x^k$, for some positive integer k. Now

$$\lim_{x\to 0}(\exp(-1/x^2)/x^k) = 0,$$

because writing $y = 1/x$ this is equivalent to

$$\lim_{y\to\infty}(y^k/\exp(y^2)) = 0.$$

Another function of this sort is one that equals $\exp(-1/x)$ for $x > 0$ and zero elsewhere.

13 The Chain Rule

Suppose $k(x) = g(f(x))$. Then

$$\frac{dk}{dx} = g'(f(x))f'(x).$$

$$\frac{dk}{dx} = \lim_{h\to 0} \frac{k(x+h) - k(x)}{h}$$
$$= \lim_{h\to 0} \frac{g(f(x+h)) - g(f(x))}{h}$$

Let

$$f(x) = y, f(x+h) = y + j(h),$$

where

$$\lim_{h\to 0} j(h) = 0.$$

We have

$$j(h) = f(x+h) - f(x).$$
$$\frac{dk}{dx} = \lim_{h\to 0} \left[\frac{g(f(x+h)) - g(f(x))}{j(h)} \frac{j(h)}{h}\right]$$
$$= \lim_{j\to 0} \frac{g(y+j) - g(y)}{j} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= g'(y)f'(x)$$
$$= g'(f(x))f'(x).$$

This proof is valid provided j is bounded away from zero, in a neighborhood of zero.

We shall present a second proof. By the mean value theorem

$$g(f(x+h)) - g(f(x)) = g'(f(\chi_1))(f(x+h) - f(x)) = g'(f(\chi_1))f'(\chi_2)h.$$

We divide by h, and then let h go to zero. If g' and f' are continuous, then

$$k'(x) = \lim_{h\to 0} \frac{g(f(x+h)) - g(f(x))}{h}$$
$$= \lim_{h\to 0} g'(f(\chi_1))f'(\chi_2)$$
$$= g'(f(x))f'(x).$$

Example

Let $f(x) = 1/x$, $1 = xf(x)$ We differentiate this last equation

$$0 = f(x) + xf'(x),$$

so

$$f'(x) = \frac{-1}{x^2}.$$

Example Let $f(x) = x^{1/n}$, let $g(x) = x^n$, then

$$g(f(x)) = x.$$

Thus

$$g'(f(x))f'(x) = 1.$$

$$n(f(x))^{n-1}f'(x) = 1,$$

$$f'(x) = \frac{1}{n}x^{-(1-1/n)}$$

$$= \frac{1}{n}x^{1/n-1}.$$

Example Quotient rule.

$$(f/g)' = \left[f\frac{1}{g}\right]$$

$$= \left[f'\frac{1}{g}\right] + \left[f\frac{-1}{g^2}g'\right]$$

$$= \frac{f'g - fg'}{g^2}.$$

14 The Derivative Of An Inverse Function

Let g be the inverse of f. That is

$$g(f(x) = f^{-1}(f(x)) = x.$$

Let $y = f(x)$, so that $x = g(y)$. Then

$$(g(f(x)))' = 1$$

$$
\begin{aligned}
(g'(f(x)))f'(x) &= 1 \\
g'(f(x)) &= \frac{1}{f'(x)} \\
g'(y) &= \frac{1}{f'(g(y))}.
\end{aligned}
$$

15 The Binomial Theorem

We have

$$(a+b)^2 = a^2 + 2ab + b^2,$$

and

$$(a+b)^3 = a^3 + 3a^2b + 3a^1b^2 + b^3.$$

The general result is called the Binomial Theorem.

Binomial Theorem For each positive integer n we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)...(n-k+1)}{k!},$$

is the number of ways of choosing k things from n things. We read

$$\binom{n}{k},$$

as n choose k. So

$$\binom{n}{0} = 1$$

and

$$\binom{n}{n} = 1.$$

Proof. Suppose the theorem holds for n, then we have

$$(a+b)^{n+1} = a(a+b)^n + (a+b)^n b$$

$$= \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{(n+1)-k} b^k + \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{n-k} b^{k+1}$$

$$= \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{(n+1)-k} b^k + \sum_{k=1}^{n+1} \left(\begin{array}{c} n \\ k-1 \end{array} \right) a^{(n+1)-k} b^k$$

$$= \left(\begin{array}{c} n \\ 0 \end{array} \right) a^{n+1} b^0 + \sum_{k=1}^{n} \left[\left(\begin{array}{c} n \\ k \end{array} \right) + \left(\begin{array}{c} n \\ k-1 \end{array} \right) \right] a^{(n+1)-k} b^k + \left(\begin{array}{c} n \\ n \end{array} \right) a^0 b^{n+1}$$

$$= a^{n+1} b^0 + \sum_{k=1}^{n} \left[\left(\begin{array}{c} n \\ k \end{array} \right) + \left(\begin{array}{c} n \\ k-1 \end{array} \right) \right] a^{(n+1)-k} b^k + a^0 b^{n+1}.$$

The expression

$$\left[\left(\begin{array}{c} n \\ k \end{array} \right) + \left(\begin{array}{c} n \\ k-1 \end{array} \right) \right]$$

is

$$\frac{n!}{k!(n-k)!)} + \frac{n!}{(k-1)!(n-(k-1))!)}$$

$$= \frac{n!}{k!(n-k)!)} + \frac{n!}{(k-1)!((n+1)-k))!)}$$

$$= \frac{n!((n+1)-k))}{k!((n+1)-k))!)} + \frac{kn!}{k!((n+1)-k))!)}$$

$$= \frac{(n+1)!}{k!((n+1)-k))!)}$$

$$= \left(\begin{array}{c} n+1 \\ k \end{array} \right).$$

So the sum above becomes

$$(a+b)^{n+1} = \sum_{k=0}^{n+1} \left(\begin{array}{c} n+1 \\ k \end{array} \right) a^{(n+1)-k} b^k.$$

Therefore by induction, the binomial theorem holds for all integers n.

16 Leibnitz Formula for the nth Derivative of a Product of Two Functions

Let us introduce the notation $f^{(k)}$ for the kth derivative of a function $f(x)$. That is

$$f^{(k)}(x) = \frac{d^k f(x)}{dx^k}.$$

The formula for the derivative of a product is written in this notation as

$$(fg)^{(1)} = f^{(1)}g + fg^{(1)}.$$

Continuing we have

$$\begin{aligned}(fg)^{(2)} &= (f^2g + f^1g^1) + (f^1g^1 + fg^2)\\ &= f^2g + 2f^1g^1 + fg^2,\end{aligned}$$

and

$$\begin{aligned}(fg)^{(3)} &= (f^3g + f^2g^1) + 2(f^2g^1 + f^1g^2) + (f^1g^2 + fg^3)\\ &= f^3g + 3f^2g^1 + 3f^1g^2 + fg^3.\end{aligned}$$

This leads to the general formula for the nth derivative of a product.

Leibnitz Formula. For each positive integer n we have

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}g^{(k)}.$$

Proof. The proof can be carried out by induction in a manner very similar to the proof by induction of the Binomial Theorem.

17 The Binomial Series

A binomial series is an infinite series of the form

$$(1+x)^r = \sum_{k=0}^{\infty} \binom{a}{k} x^k,$$

where r is any real number, and

$$\left(\begin{array}{c} r \\ k \end{array} \right) = \frac{r(r-1)(r-2)...(r-k+1)}{k!}.$$

So for example

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1024}x^6 + ...$$

It is clear that if $0 < x < 1$ that this series converges because it is an alternating series, and the terms are decreasing in magnitude.

Consider

$$(1+x)^{-5/3} = 1 - \frac{5}{3}x + \frac{20}{9}x^2 - \frac{220}{81}x^3 + \frac{770}{243}x^4 - \frac{2618}{729}x^5 + \frac{26180}{6561}x^6 - ...$$

For x outside the interval $(-1, 1)$ this series diverges because the terms do not go to zero.

And it is not quite so obvious that this series converges for $-1 < x < 1$.

Theorem.

$$(1+x)^r = \sum_{k=0}^{\infty} \left(\begin{array}{c} a \\ k \end{array} \right) x^k,$$

for $-1 < x < 1$.

Proof. $(1+x)^r$ is defined to be $\exp(r \ln(1+x))$. Let $f(x) = (1+x)^r$, then the derivatives of f are

$$f^{(1)}(x) = r(1+x)^{r-1},$$

$$f^{(2)}(x) = r(r-1)(1+x)^{r-2},$$

$$f^{(3)}(x) = r(r-1)(r-2)(1+x)^{r-3},$$

and so on. So for any positive integer k

$$f^{(k)}(x) = [r(r-1)(r-2)...(r-k+1)](1+x)^{r-k},$$

and

$$f^{(k)}(0) = r(r-1)(r-2)...(r-k+1).$$

So the Taylor series for $(1+x)^r$ is

$$(1+x)^r = \sum_{k=0}^{\infty} \frac{r(r-1)(r-2)...(r-k+1)}{k!} x^k.$$

The convergence of the binomial series for $-1 < x < 1$ and r any real number can be proven in various ways, so for example, as a consequence of Bernstein's convergence theorem (Sergei Natanovich Bernstein 1880-1968). See p244 of **Mathematical Analysis**, 2nd edition, 1975, by Tom M. Apostol.

18 The Multiplication of Power Series

Let

$$f(x) = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...) = \sum_{i=0}^{\infty} a_i x^i.$$

$$g(x) = (b_0 + b_1 x + b_2 x^2 + b_3 x^3 + ...) = \sum_{i=0}^{\infty} b_i x^i.$$

Then collecting together terms of like degree in the product we have

$$f(x)g(x) = (a_0 b_0) + (a_0 b_1 + a_1 b_0)x + ... + (a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + .. + a_k b_0)x^k + ...$$

$$= \sum_{k=0}^{\infty} (\sum_{j=0}^{k} a_j b_{k-j}) x^k.$$

19 The Number e

The number e is the base of the natural logarithms. In the next section we define the exponential function $\exp(x)$. The number e is the value of the exponential function at 1,

$$e = \exp(1).$$

The exponential function is often written as

$$exp(x) = e^x,$$

which would define the exponential function were exponentiation defined for all real numbers. Exponentiation is defined for integers, and for rational numbers using nth roots. However it is not yet defined for irrational numbers. In fact a definition of a real number a raised to the b power when b is irrational, requires the definition of the exponential function itself.

The number e can be defined as a power series

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

It can also be defined as

$$\lim_{n \to \infty} (1 + 1/n)^n.$$

Approximately

$$e = 2.718281828459045$$

We can show that the two definitions are equal. One can do this by using the binomial theorem to expand the second definition and then showing that if s_n is the partial sum of the first series and t_n is the second defining sequence then they converge to say S and T, and we have both

$$S \leq T$$

and

$$T \leq S,$$

which shows that the two limits are equal.
Proposition. $2 < e < 3$.

Proof. To show that $2 < e$, we write

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + 1/2! + 1/3! + ... > 1 + 1 = 2.$$

To show that $e < 3$, we write

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + 1/2! + 1/3! + ...$$

$$< 1 + (1 + 1/2 + (1/2)^2 + (1/2)^3 + ... = 1 + \frac{1}{1 - 1/2} = 3.$$

We can also show that e is an irrational number, and it is a trancendental number, not an algebraic number. An algebraic number is a number which occurs as a root of some polynomial equation with integer coefficients. So for example $\sqrt{2}$ is an algebraic number, because it is a root of the polynomial equation

$$x^2 - 2 = 0.$$

There is no polynomial equation with integer coefficiets that has e as a root. Such numbers are called trancendental numbers. The number of polynomials with integer coefficients is countable, and each such polynomial of degree n, has n roots. Therefore the number of algebraic numbers is countable. On the other hand the real numbers are uncountable. Thus the transcendental numbers must be uncountable. So the number of trancendental numbers is of a higher order of infinity than the algebraic numbers. However, paradoxically the number of known transcendental numbers is quite small. It is very difficult to prove that a number is trancendental.

Proposition. The number e is irrational.

Proof. Assume that e is rational so that

$$e = \frac{p}{q},$$

where p and q are integers with $q > 1$. We know that q must be greater than 1, because we have shown that $2 < e < 3$, so that e is not an integer. So we start with the assumption that

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = p/q.$$

Multiplying by the integer $q!$ we have

$$eq! = q! \sum_{k=0}^{q} \frac{1}{k!} + q! \sum_{k=q+1}^{\infty} \frac{1}{k!}.$$

The left hand side is an integer, and the first term of the right hand side is also clearly an integer. The second term of the right hand side is greater than zero. If we can show than the second term of the right hand side is not an integer, then we will have a contradiction, namely that the left hand side of the assumed equation does not equal the right hand side.

So we have

$$0 < q! \sum_{k=q+1}^{\infty} \frac{1}{k!} = \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + ...$$

$$= \frac{1}{q+1}\left[1 + \frac{1}{(q+2)} + \frac{1}{(q+2)(q+3)} + ...\right].$$

$$< \frac{1}{3}\left[1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ...\right]$$

$$= \frac{1}{3}\left[\frac{1}{1-1/3}\right] = \frac{1}{2}.$$

So we have a contradiction, and our assumption that e is a rational number is false.

References

[1] Baker, Alan (1975), Transcendental Number Theory, Cambridge University Press, ISBN 0-521-39791-X

[2] A.O.Gelfond, Transcendental and Algebraic Numbers, translated by Leo F. Boron, Dover Publications, 1960.

[3] Lindemann, F. " Über die Zahl π", Mathematische Annalen 20 (1882): pp. 213225.

Johann Heinrich Lambert proved that π is irrational, in the 19th century.

Carl Louis Ferdinand von Lindemann (April 12, 1852 March 6, 1939) was a German mathematician, noted for his proof, published in 1882, that π is a transcendental number, meaning it is not a root of any polynomial with rational coefficients.

20 The Exponential Function and the Natural Logarithm

Define the exponential function as the power series

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

Differentiating the series term by term

$$\frac{d(\exp(x)}{dx} = 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \exp(x).$$

The exponential function has the property

$$\exp(a + b) = \exp(a)\exp(b).$$

This follows by finding the product of power series.

$$\exp(a)\exp(b) = (1 + a + \frac{a^2}{2!} + ..)(1 + b + \frac{b^2}{2!} + ..) =$$

$$= \sum_{k=0}^{\infty} c_k,$$

where

$$c_k = \frac{a^0 b^k}{k!} + \frac{a^1 b^{k-1}}{1!(k-1)!} + \frac{a^2 b^{k-2}}{2!(k-2)!} + .. + \frac{a^k b^0}{k!}$$

$$= \frac{1}{k!}(c(k,0)a^0 b^k + c(k,1)a^1 b^{k-1} + .. + c(k,k)a^k b^0),$$

$$= \frac{(a+b)^k}{k!}.$$

We have used the binomial theorem, and binomial coefficients

$$c(k,j) = \frac{k!}{j!(k-j)!}.$$

We have shown that

$$\exp(a+b) = \exp(a)\exp(b).$$

Define a number e by

$$e = \exp(1).$$

One can prove that e is an irrational and transcendental number. We have

$$e^m = \exp(1)^m = \prod_{i=1}^{m} \exp(1) = \exp(m).$$

Also

$$\exp(1/n)^n = \exp(1) = e.$$

Thus

$$e^{1/n} = \exp(1/n).$$

Thus for any rational number r

$$e^r = \exp(r).$$

If x is irrational, then e^x is not yet defined. However, if e^x is to be a continuous function we must define, for all real x

$$e^x = \exp(x).$$

From the power series definition of $\exp(x)$, we see that it is a monotone increasing function, so it has an inverse. Define the natural logarithm as the inverse of the exponential function

$$\ln(x) = \exp^{-1}(x).$$

Since $\exp(0) = 1$, we have $\ln(1) = 0$, and since $\exp(1) = e$, we have $\ln(e) = 1$. If $\ln(x_1) = y_1$ and $\ln(x_2) = y_2$, then $x_1 = e^{y_1}$ and $x_2 = e^{y_2}$ Then

$$x_1 x_2 = e^{y_1 + y_2},$$

hence

$$\ln(x_1 x_2) = y_1 + y_2 = \ln(x_1) + \ln(x_2).$$

Similarly, if r is a rational number, then

$$\ln(x^r) = r \ln(x).$$

Then

$$x^r = \exp(r \ln(x)).$$

If a is an irrational number, we define

$$x^a = \exp(a \ln(x)).$$

Notice that if

$$f(x) = a^x = \exp(x \ln(a)),$$

then

$$f'(x) = \exp(x \ln(a)) \ln(a) = f(x) \ln(a).$$

Hence for any real number a,

$$\ln(x^a) = a \ln(x).$$

Letting $y = \ln(x)$, We have

$$\frac{d(\ln(x))}{dx} = 1/\frac{d(\exp(y))}{dy} = 1/\exp(\ln(x)) = 1/x.$$

Now given a real number a, we define the power function

$$a^x = \exp(\ln(a^x)) = \exp(x \ln(a)).$$

The logarithm to base a is the inverse of the power function. We write

$$\log_a(x) = y,$$

when

$$y = a^x.$$

Example We shall show that

$$\lim_{n\to\infty}(1 + 1/n)^n = e.$$

By taking the logarithm we have

$$\lim_{n\to\infty} \ln((1 + 1/n)^n)) = \lim_{n\to\infty} \frac{\ln(1 + 1/n)}{1/n}$$

Let $x = 1/n$. We shall use the mean value theorem. The numerator and the denominator each go to zero, so we may replace the numerator and the denominator by their derivatives (L'Hospital's Rule).

$$\lim_{x\to 0} \frac{\ln(1 + x)}{x}$$

$$= \lim_{x\to 0} \frac{1/(1 + x))}{1} = 1.$$

So

$$\lim_{n\to\infty} \ln((1 + 1/n)^n) = 1.$$

Therefore

$$\lim_{n\to\infty}(1 + 1/n)^n = e.$$

21 The Logarithm to the Base b

Given a positive real number b, we define the logarithm $y = \log_b x$ of x to the base b, by

$$x = b^y.$$

So we have

$$\ln(x) = y\ln(b) = log_b x \ln(b).$$

Therefore the logarithm to the base b may be defined in terms of the natural logarithm as

$$\log_b x = \frac{\ln(x)}{\ln(b)}.$$

So $\log_b(x)$ has the same properties as $\ln(x)$, such as

$$\log_b(x_1 x_2) = \log_b(x_1) + \log_b(x_2)$$

and

$$\log_b(x^a) = a \log_b(x).$$

Common logarithms use base $b = 10$. We have approximately

$$\ln(10) = 2.30258509.$$

So we have very approximately,

$$\log_{10}(x) = \frac{\ln(x)}{2.30}.$$

Now consider the relation between $\log_b x$ and $\log_c x$, for two different bases b and c. Notice that $\log_b x \ln(b)$ and $\log_b x \ln(b)$ are equal, because they are both equal to $\ln(x)$. Therefore

$$\log_b(x) = \frac{\log_c(x)\ln(c)}{\ln(b)}.$$

$$= \frac{\log_c(x)}{\ln(b)/\ln(c)}$$

$$= \frac{\log_c(x)}{\log_c(b)}.$$

22 Angle

Two rays emanating from point A define an angle. Let a circle of radius R and center A intersect the rays at points B and C, thereby defining an arc. The measure of the angle, which is written θ, is defined to be the ratio of the arc length of the circle, s, to the radius R:

$$\theta = \frac{s}{R}.$$

The definition is independent of the particular circle chosen. This can be seen by decomposing the angle into many very small angles, so that each very small arc length is nearly equal to the small side of a triangle formed by the small angle. The independence of the circle radius on the definition of angle measure follows from properties of similar triangles, and by taking limits. The number π is by definition the ratio of the arclength of a circle to the diameter $D = 2R$. The arclength of a circle of radius R is $2\pi R$. The complete circle angle formed by a ray with itself has measure

$$\theta = \frac{2\pi R}{R} = 2\pi.$$

A straight angle, which is one half of a full circle angle, has measure $\theta = \pi$, and a right angle, which is one fourth of a full angle, has measure $\theta = \pi/2$.

23 Trigonometric Functions

Let (x, y) be a point on a circle centered at the origin of radius 1. A ray from the origin passing through (x, y) defines an angle with the x-axis. Let the measure of this angle be θ. Then define

$$\cos(\theta) = x,$$

$$\sin(\theta) = y.$$

If $(x, y) = (1, 0)$ then $\theta = 0$. Therefore

$$\cos(0) = 1,$$

$$\sin(0) = 0.$$

If $(x, y) = (0, 1)$ then $\theta = \pi/2$. Therefore

$$\cos(\pi/2) = 0,$$

$$\sin(\pi/2) = 1.$$

If (x, y) lies on the unit circle then

$$x^2 + y^2 = 1.$$

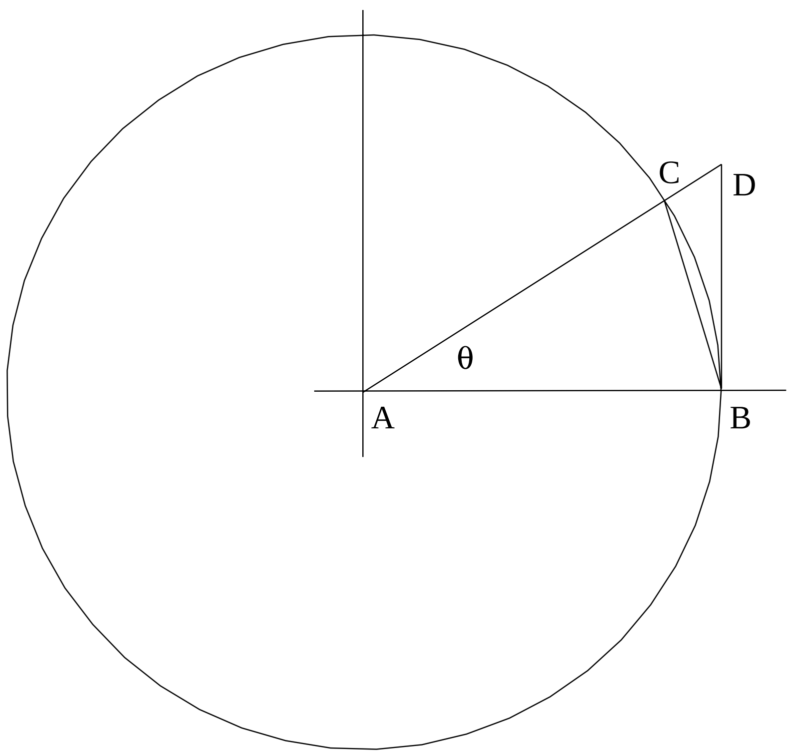

Figure 3: Proof that the Limit of $\sin(\theta)/\theta$ as $\theta \to 0$ is 1. Let the radius of the circle be 1. So the area of triangle ABC is $\sin(\theta)/2$, the area of circular sector ABC is $\theta/2$, and the area of triangle ABD is $\tan(\theta)/2$. As θ goes to zero, $\sin(\theta)/\theta$ goes to one.

Hence
$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

Example 2. We shall show that if
$$f(\theta) = \frac{\sin(\theta)}{\theta},$$
then
$$\lim_{\theta \to 0} f(\theta) = 1.$$

Proof. Construct a circle of radius $r = 1$ as shown in the $\sin(\theta)/\theta$ figure. The area of the inner triangle, ABC is $\sin(\theta)/2$, because the height of the triangle is $\sin(\theta)$, and the base has length 1. Clearly the circular sector ABC has area area $\theta/2$, and the outer triangle ABD has area $\tan(\theta)/2$. Intuitively we can see that as θ goes to zero, the ratio of these areas approaches 1. We can prove this. The area of the inner triangle is less than the area of the circular sector, which in turn is less than the area of the outer triangle. So we have
$$\sin(\theta)/2 < \theta/2 < \tan(\theta)/2,$$
$$1 < \theta/\sin(\theta) < 1/\cos(\theta),$$
$$1 > \sin(\theta)/\theta > \cos(\theta),$$
$$-1 < -\sin(\theta)/\theta < -\cos(\theta).$$

It follows that
$$-(1 - \cos(\theta)) < 0 < 1 - \sin(\theta)/\theta < 1 - \cos(\theta).$$

Therefore
$$|1 - \sin(\theta)/\theta| < 1 - \cos(\theta).$$

Choose $\epsilon > 0$. We have
$$\lim_{\theta \to 0} \cos(\theta) = 1.$$

So we may choose a number δ so that if $|\theta| < \delta$ then $1 - \cos(\theta) < \epsilon$. Then if $|\theta| < \delta$ then
$$|1 - \sin(\theta)/\theta| < \epsilon.$$

We have proved that
$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$$

24 Angle Sum Formula

We shall prove an angle sum formula for the case

$$\theta_2 > 0,$$

$$\theta_1 > 0,$$

$$\theta_1 + \theta_2 < \pi/2.$$

We shall show that

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2).$$

Consider points lying on the first quadrant of the unit circle $E = (x, y)$, and $F = (x_1, y_1)$. Refer to the angle sum formula figure. Let A be the origin. Let $x < x_1$. Drop a perpendicular from E meeting the x-axis at B. Then the angle BAE is the sum of angle BAF and angle FAE. Let θ be the measure of BAE, θ_1 the measure of BAF and θ_2 the measure of FAE. Then

$$\theta = \theta_1 + \theta_2.$$

Construct a line perpendicular to AF through E. Let this line meet AF at D. DAE is a right triangle with unit hypotenuse. Define

$$a = AD,$$

and

$$b = DE.$$

Then

$$a = \cos(\theta_2),$$

and

$$b = \sin(\theta_2).$$

Let AD meet EB at C. Let $h = EC$ and $k = BC$. Then

$$y = h + k.$$

Also we have

$$x = \cos(\theta),$$

$$y = \sin(\theta),$$

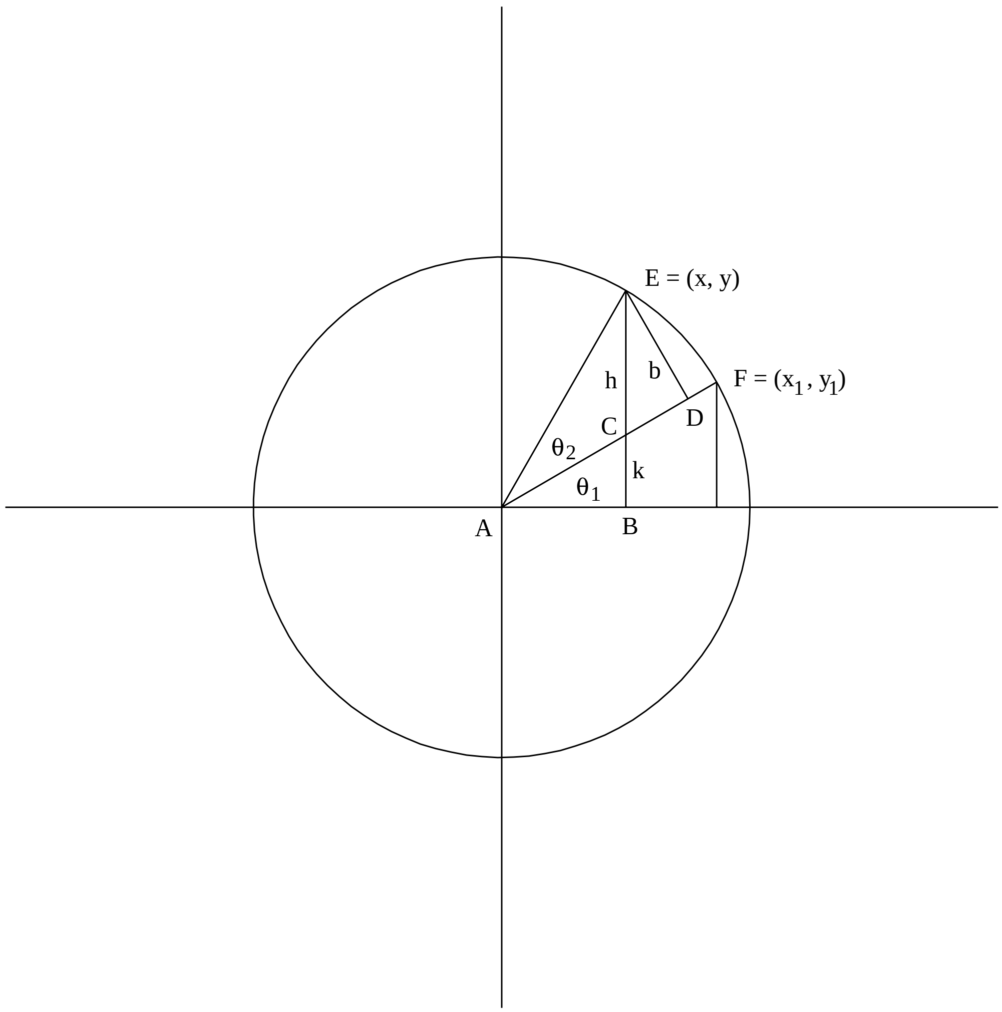

Figure 4: **Angle Sum Formula.** The length of line segment AD is a, the length of ED is b, the length of EC is h and the length of CB is k. The circle has unit radius. EDA is a right angle. The figure shows how similar triangles can be used to prove the formula $\sin(\theta_1+\theta_2) = \sin(\theta_1)\cos(\theta_2)+\cos(\theta_1)\sin(\theta_2)$.

$$x_1 = \cos(\theta_1),$$

and

$$y_1 = \sin(\theta_1).$$

By properties of similar triangles we have

$$\frac{k}{x} = \frac{y_1}{x_1},$$

and

$$\frac{h}{b} = \frac{1}{x_1}.$$

Then

$$k = \frac{xy_1}{x_1},$$

$$h = \frac{b}{x_1}.$$

$$y = h + k = \frac{b}{x_1} + \frac{\sqrt{1-y^2}y_1}{x_1}.$$

Then

$$y_1\sqrt{1-y^2} = x_1 y - b,$$

$$y_1^2(1-y^2) = x_1^2 y^2 - 2x_1 y b + b^2$$

$$y_1^2 = (x_1^2 + y_1^2)y^2 - 2x_1 y b + b^2$$

$$y_1^2 = y^2 - 2y(x_1 b) + (x_1 b)^2 + b^2(1 - x_1^2)$$

$$y_1^2 = (y - x_1 b)^2 + b^2 y_1^2$$

$$(1-b^2)y_1^2 = (y - x_1 b)^2$$

$$a^2 y_1^2 = (y - x_1 b)^2$$

$$y = \pm y_1 a + x_1 b.$$

Then

$$\sin(\theta_1 + \theta_2) = \pm \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2).$$

Because

$$\sin(\theta_1 + \theta_2) > \sin(\theta_2),$$

the plus sign is correct, so

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2).$$

A similar proof shows that if

$$0 < \theta_1 < \pi/2,$$

and

$$\theta_2 < \theta_1,$$

then

$$\sin(\theta_1 - \theta_2) = \sin(\theta_1)\cos(\theta_2) - \cos(\theta_1)\sin(\theta_2).$$

25 The Derivative of the Sine Function

These results allow us to compute the derivative of $\sin(\theta)$.

We shall show that the derivative is

$$\frac{d\sin(\theta)}{d\theta} = \cos(\theta).$$

We have

$$\frac{d\sin(x)}{dx} = \lim_{h\to 0}\frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin((x+h/2)+h/2) - \sin((x+h/2)-h/2)}{h}$$

$$= \lim_{h\to 0}\frac{2\cos(x+h/2)\sin(h/2)}{h}$$

$$= \lim_{h/2\to 0}\cos(x+h/2)\lim_{h/2\to 0}\frac{\sin(h/2)}{h/2}$$

$$= \cos(x)\cdot 1 = \cos(x).$$

To find the derivative of $\cos(\theta)$, we differentiate both sides of

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

We find that

$$2\cos(\theta)\frac{d\cos(\theta)}{d\theta} + 2\sin(\theta)\cos(\theta) = 0.$$

If $2\cos(\theta) \neq 0$, then dividing by $2\cos(\theta)$, we find

$$\frac{d\cos(\theta)}{d\theta} = -\sin(\theta).$$

This is a general result, but our proof is only valid provided $\cos(\theta)$ is not zero. We realize though by invoking continuity at such exceptional points that the result is valid everywhere.

Notice that all the higher derivatives of sin and cos at zero take values $0, 1, -1$. We can define sin and cos as everywhere convergent power series about zero. Expanding each of $\sin(x)$, $\cos(x)$ and $\exp(x)$ about zero in a Taylor series, we find

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...,$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...,$$

and

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + ...$$

Hence we have found Euler's famous formula

$$\exp(i\theta) = \cos(\theta) + i\sin(\theta),$$

where

$$i = \sqrt{-1}.$$

Then

$$\sin(\theta) = \frac{\exp(i\theta) - \exp(-i\theta)}{2i},$$

and

$$\cos(\theta) = \frac{\exp(i\theta) + \exp(-i\theta)}{2}.$$

Now we may prove the angle sum formula for the sine, which was proved above for a special case of the arguments by trigonomitry, for all values of the arguments.

Proposition For all θ_1 and θ_2,

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2).$$

Proof We may use the exponential definitions of $\sin(x)$ and $\cos(x)$, and show that the two sides of the equation are equal.

The exponential definitions of the trigonometric functions may be used to establish general trigonometric identities.

Exercise. Using

$$\cos(x) = \sin(x + \pi/2),$$

show that

$$\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2).$$

26 Some Famous Formulas

26.1 Euler's Formula

Euler's formula

$$e^{\theta i} = \cos(\theta) + i\sin(\theta),$$

may be derived from the Taylor series for $e^{\theta i}$, $\cos(\theta)$, and $\sin(\theta)$.

26.2 de Moivre's Formula

Show that de Moivre's formula

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

is easily derived from Euler's formula.

26.3 The $1, 0, \pi, e$, and i Formula

In the beginning there was one. To relieve the tedium, we began to compare and count, and we got the counting numbers by adding one, again and again. Then by eating, we found we were doing subtraction, and pearing into our empty bag, we realized that it was all about nothing, so we constructed the abstract number zero. Eventually we drew circles in the sand and discovered geometry, and thought of the idea of π, as half the angle around the circle. From multiplication we had realized the idea of exponentiation, and when the calculus arrived, we naturally formed the idea of the exponential function, $\exp(x)$, and the number $e = exp(1)$. With a lot of ego, and a little imaginary thinking, while encountering the quadratic formula, we created i.

Some find it amazing and even magical, that all of these abstractions occur in one single formula

$$e^{\pi i} + 1 = 0.$$

So be so kind as to show that this follows from Euler's formula, through an encounter with -1.

27 The Indefinite Integral

$F(x)$ is called the indefinite integral, or the antiderivative of $f(x)$, when

$$\frac{dF(x)}{dx} = f(x).$$

$F(x)$ is written as

$$F(x) = \int f(x)dx.$$

28 The Riemann Integral

Suppose we have a partition of the interval $[a, b]$

$$a = x_1 < \chi_1 < x_2 < \chi_2 < ... < \chi_{n-1} < x_n = b.$$

We define the definite integral to be

$$\int_a^b f(x)dx = \lim_{|x_{i+1}-x_i| \to 0} \sum_{i=1}^{n} f(\chi_i)(x_{i+1} - x_i).$$

The limit is taken as the distance between the mesh points goes to zero.
Proposition If

$$G(x) = \int_a^x f(x)dx,$$

then

$$G'(x) = f(x).$$

Proof

$$G'(x) = \lim_{h \to 0} \frac{G(x+h) - G(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(x)dx - \int_a^x f(x)dx\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_x^{x+h} f(x)dx\right]$$

Let m be the minimum of f on the interval $[x, x+h]$, and M the maximum of $F(x)$ on $[x, x+h]$. Then

$$m \le \frac{1}{h}\int_x^{x+h} f(x)dx \le M.$$

The limit of both m and M is $f(x)$. Therefore

$$G'(x) = f(x).$$

Proposition If $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Let

$$G(x) = \int_a^x f(t)dt.$$

Then $G'(x) = f(x)$ so

$$G(x) = F(x) + c,$$

for some constant c. Then

$$F(a) + c = G(a) = \int_a^a f(t)dt = 0.$$

Thus

$$c = -F(a).$$

Therefore

$$\int_a^b f(t)dt = G(b) = F(b) + c = F(b) - F(a).$$

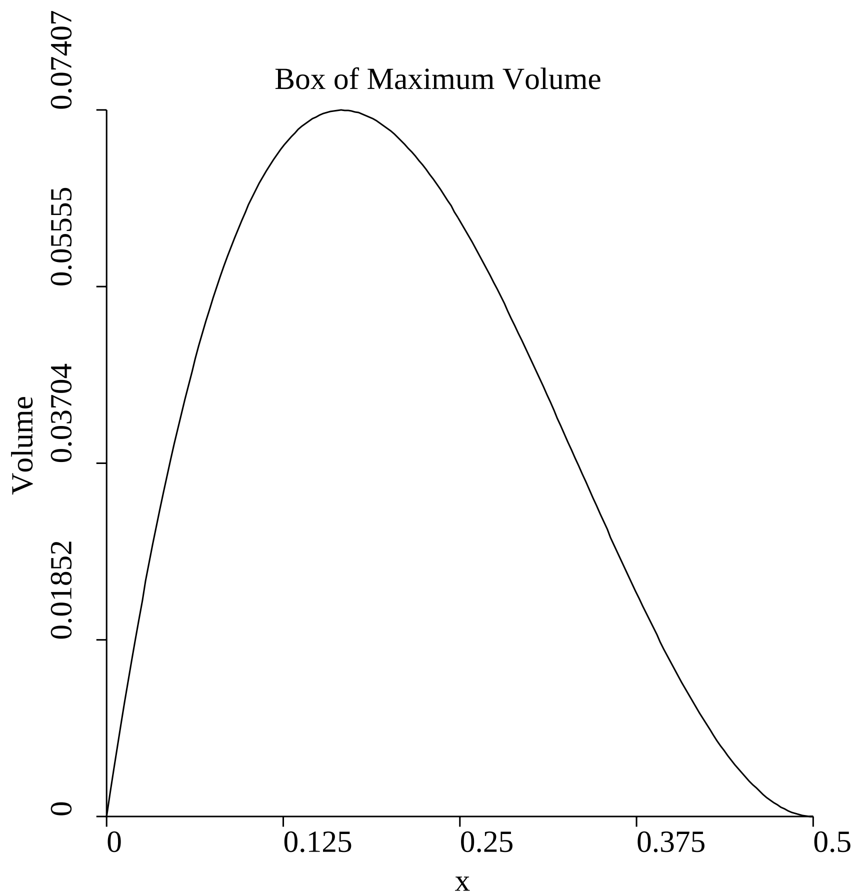

Figure 5: **Maximum Box Volume.** This shows how the volume of the box varies as x varies from 0 to 1/2. x is the side length of the squares that are clipped from the four corners of a unit square.

29 Some Maxima and Minima Examples

Example 1 Given a 1 by 1 square, find the box of maximum volume obtained by clipping squares of side x from each corner and folding up the edges to make a box.
Solution. The volume of the folded box is

$$V = (1 - 2x)^2 x = 4x^3 - 4x^2 + x.$$

We set the volume derivative to 0,

$$\frac{dV}{dx} = 12x^2 - 8x + 1 = 0.$$

The roots of this equation are

$$x = \frac{1}{6}, \frac{1}{2}.$$

So the maximum occurs for $x = 1/6$. See the figure showing the variation of volume with x.
Example 2 *Snell's Law.* Suppose that we can travel in one medium at velocity v_1 and in a second medium at a slower velocity v_2. Suppose we are to travel from point P in the first medium to point R in the second medium. What path results in the minimum travel time?
Solution

Referring to the **Snell's Law** figure, let v_1 be the velocity in the upper plane and v_2 the velocity in the lower plane, with $v_2 < v_1$. We are to find the position of the point $Q = (x, 0)$ to minimize the travel time from point P to point R. The length of the path in the upper plane is

$$\ell_1 = \sqrt{d^2 + x^2},$$

and the length of the path in the lower plane is

$$\ell_2 = \sqrt{e^2 + (c - x)^2}$$

The travel time as a function of x is

$$t = \frac{\ell_2}{v_1} + \frac{\ell_2}{v_2}.$$

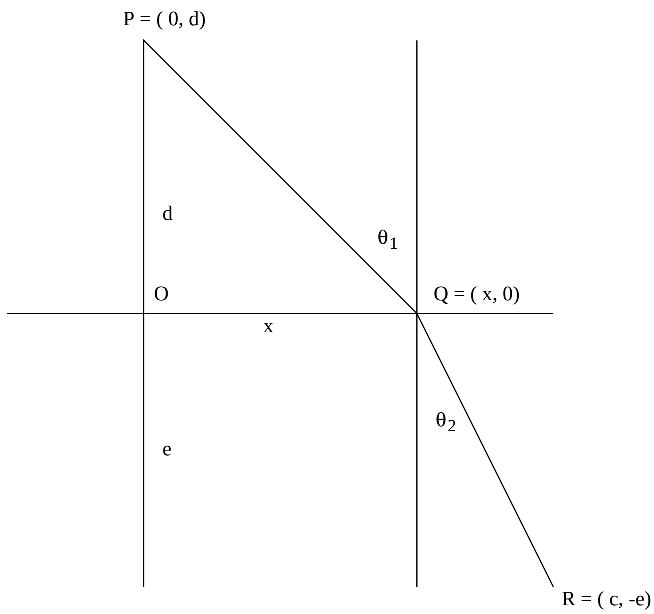

Figure 6: **Snell's Law.** Let two media be separated by the horizontal line. A particle in the upper media travels at velocity v_1, in the lower at velocity v_2, with $v_2 < v_1$. The travel time from P to R is minimized when x the coordinate of Q is selected to satisfy Snell's law.

The derivative of the time is

$$\frac{dt}{dx} = \frac{(x/\sqrt{d^2+x^2})}{v_1} - \frac{((c-x)/\sqrt{e^2+(x-c)^2})}{v_2}$$

$$= \frac{\sin(\theta_1)}{v_1} - \frac{\sin(\theta_2)}{v_2}.$$

Setting this to zero, we find that the condition for a minimum is

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2}.$$

In the case of optics we have the indices of refraction

$$n_1 = \frac{c}{v_1}, n_2 = \frac{c}{v_2},$$

where c is the velocity of light in a vacuum. So we obtain Snell's law of optical refraction.

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2).$$

Example 3 *The Range of a Projectile.* Suppose an object is projected upward at angle θ with velocity v, where air resistance is neglected. Let the acceleration of gravity be g. Then the motion in the vertical y direction is

$$\frac{d^2y}{dt^2} = -g,$$

$$\frac{dy}{dt} = -gt + v\sin(\theta),$$

$$y = -\frac{-gt^2}{2} + v\sin(\theta)t.$$

The motion in the horizontal x direction is

$$x = v\cos(\theta)t.$$

The projectile returns to the ground when

$$0 = t(v\sin(\theta) - \frac{gt}{2}).$$

So

$$t = \frac{2v\sin(\theta)}{g}$$

Then

$$x = \frac{2v^2\sin(\theta)\cos(\theta)}{g} = \frac{v^2\sin(2\theta)}{g}.$$

The range is maximum where $\sin(2\theta)$ is maximum, where $\theta = \pi/4$.

Given a desired range x, the projectile should be launched at angle

$$\theta = \frac{1}{2}\sin^{-1}(\frac{gx}{v^2}).$$

If there is air resistance, say a retarding force proportional to the square of the velocity, then we have to solve a more complicated differential equation, with an approximate numerical technique. This was the problem that led to the invention of electronic computers.

30 Methods of Integration

30.1 Integration by Substitution

By making a variable substitution it may be possible to find an integral for a function. So suppose we have a function $F(x)$ and we want to find an integral for the function, that is a function $f(x)$ such that

$$\frac{df(x)}{dx} = F(x)$$

$$f(x) = \int F(x)dx.$$

Suppose we make the substitution $x = k(u)$ for some function k. We assume that k has an inverse. We have $u = k^{-1}(x)$. Let us write $h = k^{-1}$. We have

$$\frac{du}{dx} = k'(x),$$

and

$$\frac{dx}{du} = h'(u)$$

Notice that

$$k'(x)h'(u) = k'(x)h'(u(x)) = 1.$$

We make the substitution in the original integral

$$dx = h'(u)du,$$

getting

$$\int F(h(u))h'(u)du.$$

Letting

$$G(u) = F(h(u))h'(u),$$

this is

$$\int G(u)du.$$

Suppose we find g so that

$$g(u) = \int G(u)du,$$

that is

$$\frac{dg}{du} = G(u).$$

Now we substitute back defining an f

$$f(x) = g(k(x)).$$

Then

$$\begin{aligned}\frac{df}{dx} &= \frac{dg}{du}\frac{dk}{dx}\\ &= G(u)\frac{dk}{dx}\\ &= G(k(x))k'(x)\\ &= F(h(k(x)))h'(k(x))k'(x)\\ &= F(x).\end{aligned}$$

So

$$f(x) = \int F(x)dx.$$

Example 1. Find

$$f(x) = \int \sqrt{1-x^2}dx.$$

Let

$$x = \sin(\theta)$$

$$dx = \cos(\theta)d\theta.$$

So substituting

$$\int \sqrt{1 - \sin^2(\theta)} \cos(\theta)d\theta$$

$$= \int \cos^2(\theta)d\theta$$

$$= \int \frac{1 + \cos(2\theta)}{2} d\theta$$

$$= \frac{\theta}{2} + \frac{\sin(2\theta)}{4}$$

$$= \frac{\theta}{2} + \frac{\sin(\theta)\cos(\theta)}{2}$$

Substituting back

$$f(x) = \frac{\sin^{-1}(x)}{2} + \frac{x\sqrt{1 - x^2}}{2}.$$

Example 2. Find

$$f(x) = \int \sec(x)dx.$$

Multiplying by

$$u = \sec(x) + \tan(x)$$

we have

$$\int \sec(x)dx = \int \frac{\sec(x)(\sec(x) + \tan(x))}{\sec(x) + \tan(x)} dx.$$

$$= \int \frac{\sec^2(x) + \sec(x)\tan(x))}{\sec(x) + \tan(x)} dx.$$

$$= \int \frac{1}{u} du$$

$$= \ln(|u|)$$

$$= \ln(|\sec(x) + \tan(x)|).$$

Example 3. Find

$$f(x) = \int \sec(x)dx,$$

using the integral

$$\int \frac{1}{1-x^2}dx.$$

We have

$$\frac{1}{1-x^2} = (1/2)(\frac{1}{1-x} + \frac{1}{1+x}),$$

so

$$\int \frac{1}{1-x^2}dx = (1/2)(-\ln(|1-x)|) + \ln(|1+x)|))$$
$$= \ln\sqrt{\frac{1+x}{1-x}}.$$

Now

$$\int \sec(x)dx = \int \frac{\cos(x)}{\cos^2(x)}dx$$
$$= \int \frac{\cos(x)}{1-\sin^2(x)}dx.$$

Let

$$u = \sin(x),$$

then

$$du = \cos(x)dx.$$

So

$$\int \sec(x)dx = \int \frac{1}{1-u^2}du$$
$$= \ln\sqrt{\frac{1+u}{1-u}}$$
$$= \ln\sqrt{\frac{1+\sin(x)}{1-\sin(x}}$$
$$= \ln\sqrt{\frac{(1+\sin(x))^2}{1-\sin^2(x)}}$$
$$= \ln\left|\frac{1+\sin(x)}{\cos(x)}\right|$$
$$= \ln|\sec(x)+\tan(x)|.$$

30.2 Integration by Parts

Integration by parts comes from the formula for the derivative of a product.

$$d(uv) = udv + vdu$$

Integrating

$$uv = \int udv + \int vdu$$

or

$$\int udv = uv - \int vdu.$$

Example 1. Suppose we wish to calculate

$$\int x\sin(x)dx.$$

Let

$$u = x, dv = \sin(x)dx$$

Then

$$du = dx, v = -\cos(x)$$

Then

$$\int x\sin(x)dx = -x\cos(x) - \int -\cos(x)dx$$
$$= -x\cos(x) + \sin(x).$$

Example 2. Suppose we wish to calculate

$$\int x^2\cos(x)dx.$$

We can write a little table and avoid introducing u and v explicitly

x^2	$\cos(x)dx$
$2xdx$	$\sin(x)$

Here we have written what we want u to be in the top left box, and what we want dv to be in the top right box. We differentiate the top left box to get the bottom left box. We integrate the top right box to get the lower right box. Then the integral of the top product, namely

$$\int x^2\cos(x)dx,$$

is equal to the product of the diagonal elements

$$x^2 \sin(x),$$

minus the integral of the product of the lower elements. Thus we have

$$\int x^2 \cos(x) dx = x^2 \sin(x) - \int 2x \sin(x) dx$$

$$= x^2 \sin(x) - 2(-x \cos(x) + \sin(x))$$
$$= x^2 \sin(x) + 2x \cos(x) - 2 \sin(x)).$$

Example 3. Suppose we wish to calculate

$$\int e^x \sin(x) dx.$$

Thus requires a bit of a trick.

If we integrate by parts we end up with a last term

$$\int e^x \cos(x) dx,$$

so we don't seem to be making any progress. However if we now integrate the term involving

$$\int e^x \cos(x) dx,$$

by parts, we get a term involving our original integral

$$\int e^x \sin(x) dx.$$

So we may rearrange this to get

$$\int e^x \sin(x) dx = \frac{1}{2}(e^x \sin(x) - e^x \cos(x)).$$

Example 4. Let us calculate

$$\int \cos^n(x) dx.$$

We use

$\cos^{n-1}(x)$	$\cos(x)dx$
$(n-1)\cos^{n-1}(-\sin(x))(x)dx$	$\sin(x)$

to get.

$$\int \cos^n(x)dx = \sin(x)\cos^{n-1}(x) - \int (n-1)(-\sin^2(x))\cos^{n-2}(x)dx$$

$$= \sin(x)\cos^{n-1}(x) + (n-1)\int (1-\cos^2(x))\cos^{n-2}(x)dx$$

$$= \sin(x)\cos^{n-1}(x) + (n-1)[\int \cos^{n-2}(x)dx - \int \cos^n(x)dx].$$

$$n\int \cos^n(x)dx = \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)dx.$$

Then

$$\int \cos^n(x)dx = \frac{\sin(x)\cos^{n-1}(x)}{n} + \frac{(n-1)}{n}\int \cos^{n-2}(x)dx.$$

For example to compute

$$\int \cos^2(x)dx,$$

we can either use the preceding formula, or

$$\cos^2(x) = \frac{\cos(2x)+1}{2}$$

to compute

$$\int \cos^2(x)dx = \frac{1}{2}(\sin(x)\cos(x) + x).$$

Example 5. Let us calculate

$$\int \frac{1}{(u^2+\alpha^2)^n}du.$$

We can use trigonometric substitution and the result of the preceding example.

We let

$$u = \alpha\tan(\theta).$$

then

$$du = \alpha\sec^2(\theta)d\theta,$$

$$u^2 + \alpha^2 = \alpha^2 \sec^2(\theta).$$

So

$$\int \frac{1}{(u^2 + \alpha^2)^n} du$$

$$= \int \frac{\alpha \sec^2(\theta)}{(\alpha^2 \sec^2(\theta))^n} d\theta$$

$$= \frac{1}{\alpha^{2n-1}} \int \frac{1}{\sec^{2n-2}(\theta)} d\theta$$

$$= \frac{1}{\alpha^{2n-1}} \int \cos^{2n-2}(\theta) d\theta$$

Using the recurrence relation

$$\int \cos^n(x) dx = \frac{\sin(x)\cos^{n-1}(x)}{n} + \frac{(n-1)}{n} \int \cos^{n-2}(x) dx,$$

we can derive a recurrence relation for

$$\int \frac{1}{(u^2 + \alpha^2)^n} du.$$

30.3 The Fundamental Theorem of Algebra

This theorem says that every polynomial $p(z)$ of degree n with real or complex coefficients has a root. If z_1 is a root, then $z - z_1$ divides $p(z)$, so

$$p(z) = (z - z_1) q(z),$$

where $q(z)$ is a polynomial of degree $n - 1$. By the fundamental theorem $q(z)$ has a root. Continuing in this way every complex polynomial of degree n can be factored into products of the form

$$c_0 + c_1 z + c_2 z^2 + c_3 z^3 + ... + c_n z^n =$$

$$c_n (z - z_1)(z - z_2)(z - z_3)(z - z_4)...(z - z_n).$$

Thus every complex polynomial of degree n has exactly n roots. If

$$z = a + bi,$$

its conjugate is

$$\bar{z} = a - bi.$$

By conjugating the whole polynomial, we see that if z is a root, then $\bar{z}$ is also a root. Roots occur in conjugate pairs. If the polynomial has real coefficients and a complex root $z_k = a + bi$, then it also has a root $\bar{z_k}$, so a real quadratic factor

$$(x - z_k)(x - \bar{z}_k) = x^2 - (z_k + \bar{z}_k)x + |z_k|^2.$$

So any real polynomial can be factored into a product of linear and quadratic factors. This will be used in the next section. The fundamental theorem of algebra was first proved by Gauss.

Johann Carl Friedrich Gauss (30 April 1777 23 February 1855) was a German mathematician and scientist who contributed significantly to many fields, including number theory, statistics, analysis, differential geometry, geodesy, geophysics, electrostatics, astronomy and optics. Sometimes known as the Princeps mathematicorum (Latin, "the Prince of Mathematicians" or "the foremost of mathematicians") and "greatest mathematician since antiquity", Gauss had a remarkable influence in many fields of mathematics and science and is ranked as one of history's most influential mathematicians. He referred to mathematics as "the queen of sciences."

There is a simple proof of the Fundamental Theorem of Algebra using elementary facts from complex analysis. We give that proof in a later section. Elementary proofs that do not use complex analysis are long and involved.

30.4 Partial Fractions: Integrating Rational Functions

A rational function is of the form

$$r(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomials. If the degree of $p(x)$ is not less than the degree of $q(x)$ we may divide and get a polynomial plus a new rational function, where the degree of the numerator is less than the degree of the denominator. So we will consider only rational functions where the latter condition holds. If the denominator can be factored into first degree non-repeating factors then the rational function can be expanded in a partial fraction of the form

$$r(x) = \sum_{i=1}^{n} \frac{A_i}{x - x_i)},$$

where $\{x_1, x_2, ..., x_n\}$ are the roots of $q(x)$, and where the $a_1, A_2, ..., A_n$ are constants. Then the integral

$$\int r(x),$$

is equal to a sum of logarithms. If there are repeating roots, say $q(x$ has k roots x_1, then we must include terms of the form

$$\frac{A_1}{(x - x_1)}, \frac{A_2}{(x - x_1)^2}, ..., \frac{A_k}{(x - x_1)^k}.$$

If there are complex roots of $q(x)$ then they occur in pairs, and so we need fractions of the form

$$\frac{A_j + xB_j}{x^2 + b_j x + c_j}.$$

By completing the square and doing a substitution the denominator can be put in the form

$$u^2 + \alpha^2.$$

If we have repeated complex roots we must include powers in the denominator as for non-complex roots. Thus we get integrals of the form

$$\int \frac{A_j + uB_j}{(u^2 + \alpha^2)^m} du = \int \frac{A_j}{(u^2 + \alpha^2)^m} du + \int \frac{uB_j}{(u^2 + \alpha^2)^m} du.$$

The first can be evaluated using a recurance formula evaluating integrals of the form

$$\int \frac{1}{(u^2 + \alpha^2)^m} du.$$

The recurance formula comes from doing a substitution on the result of Example 5 in the section on substituion.

The recurance formula is

$$\int \frac{du}{(u^2 + \alpha^2)^m} = \frac{1}{2\alpha^2(m-1)} \frac{u}{(u^2 + \alpha^2)^{m-1}} + \frac{2m-3}{2\alpha^2(m-1)} \int \frac{du}{(u^2 + \alpha^2)^{m-1}}.$$

30.5 Rational Functions of Sines and Cosines

A rational function involving Sines and Cosines can be integrated by using the special substitution

$$z = \tan(x/2).$$

We find that

$$\cos(x) = \frac{1 - z^2}{1 + z^2}$$
$$\sin(x) = \frac{2z}{1 + z^2}$$
$$dx = \frac{2dz}{1 + z^2}.$$

So a rational function of $\sin(x)$ and $\cos(x)$ can be converted to a rational function in z. This can then be integrated by partial fractions.

30.6 Products of Sines and Cosines

The integral of the products of sines and cosines can be handled by using the identities

$$\sin(mx)\sin(nx) = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$
$$\sin(mx)\cos(nx) = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$$
$$\cos(mx)\cos(nx) = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

31 Hyperbolic Functions

Hyperbolic Functions

The hyperbolic sine is defined as

$$\sinh(z) = \frac{\exp(z) - \exp(-z)}{2}.$$

The hyperbolic cosine is

$$\cosh(z) = \frac{\exp(z) + \exp(-z)}{2}.$$

We have

$$\sinh^2(z) - \cosh^2(z) = 1.$$

The functions $\tanh(z), \coth(z), sech(z), csch(z)$ are defined in the obvious way.

By the usual Tayor series:

$$\sin(z) = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - ...$$

$$\cos(z) = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - ...$$

$$\sinh(z) = z + \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + ...$$

$$\cosh(z) = 1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + ...$$

$$\exp(z) = 1 + \frac{1}{1!}z^1 + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + ...$$

Then

$$\exp(iz) = 1 + i\frac{1}{1!}z - \frac{1}{2!}z^2 - i\frac{1}{3!}z^3 + ...$$
$$= \cos(z) + i\sin(z).$$

We also have

$$\sin(iz) = i\sinh(z)$$

$$\sinh(iz) = i\sin(z)$$

$$\cos(iz) = \cosh(z)$$

$$\cosh(iz) = \cos(z).$$

If $z = x + iy$ then

$$\sin(z) = \sin(x + iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy)$$

$$= \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

$$\cos(z) = (\cos(x + iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy)$$

$$= \cos(x)\cosh(y) - i\sin(x)\sin(y).$$

32 The Inverse Hyperbolic Functions

The hyperbolic functions are defined using exponential functions. Because the inverse of the exponential function is the natural logrithm, one might think that the inverse of hyperbolic functions can be expressed as logarithmic functions. This is indeed the case. As an example consider the inverse of $\sinh(x)$.

We have

$$x = \sinh(u) = \frac{e^u - e^{-u}}{2}$$
$$2x = e^u - 1/e^u$$

Letting

$$e^u = v,$$

we have

$$2xv = v^2 - 1$$
$$v^2 - 2xv - 1 = 0.$$

Thus, because $v \geq 0$ we find

$$v = x + \sqrt{x^2 + 1}.$$

So

$$e^u = x + \sqrt{x^2 + 1},$$

then

$$u = \ln(x + \sqrt{x^2 + 1}).$$

Because $x = \sinh(u)$,

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1}).$$

See a mathematical handbook, such as **Schaum's Mathematical Handbook** By Murray Spiegel, or the **The CRC Mathematical Handbook**, for expressions of the other inverse hyperbolic functions.

33 A Table of Elementary Derivatives

$f(x)$	Domain	Range	df/dx
$\sin(x)$	$(-\infty,\infty)$	$[-1,1]$	$\cos(x)$
$\cos(x)$	$(-\infty,\infty)$	$[-1,1]$	$-\sin(x)$
$\tan(x)$	x not $n\pi/2$	$(-\infty,\infty)$	$\sec^2(x)$
$\cot(x)$	x not $n\pi$	$(-\infty,\infty)$	$-\csc^2(x)$
$\sec(x)$	x not $n\pi/2$	$(-\infty,-1]\cup[1,\infty)$	$\sec(x)\tan(x)$
$\csc(x)$	x not $n\pi$	$(-\infty,-1]\cup[1,\infty)$	$-\csc(x)\cot(x)$
$\sin^{-1}(x)$	$[-1,1]$	$(-\pi/2,\pi/2)$	$1/\sqrt{1-x^2}$
$\cos^{-1}(x)$	$[-1,1]$	$(0,\pi)$	$-1/\sqrt{1-x^2}$
$\tan^{-1}(x)$	$(-\infty,\infty)$	$(-\pi/2,\pi/2)$	$1/(1+x^2)$
$\cot^{-1}(x)$	$(-\infty,\infty)$	$(0,\pi)$	$-1/(1+x^2)$
$\sec^{-1}(x)$	$(-\infty,-1]$	$(\pi/2,\pi)$	$1/(x\sqrt{x^2-1})$
$\sec^{-1}(x)$	$[1,\infty)$	$[0,\pi/2)$	$-1/(x\sqrt{x^2-1})$
$\csc^{-1}(x)$	$(-\infty,-1]$	$[-\pi/2,0)$	$-1/(x\sqrt{x^2-1})$
$\csc^{-1}(x)$	$[1,\infty)$	$(0,\pi/2]$	$1/(x\sqrt{x^2-1})$
$\ln(x)$	$(0,\infty)$	$(-\infty,\infty)$	$1/x$
$\log_a(x)=\log_a(e)\ln(x)$	$(0,\infty)$	$(-\infty,\infty)$	$\log_a(e)/x$
$\exp(x)$	$(-\infty,\infty)$	$(0,\infty)$	$\exp(x)$
$a^x=\exp(x\ln(a))$	$(-\infty,\infty)$	$(0,\infty)$	$a^x\ln(a)$
$sinh(x)=(e^x-e^{-x})/2$	$(-\infty,\infty)$	$(-\infty,\infty)$	$cosh(x)$
$cosh(x)=(e^x+e^{-x})/2$	$(-\infty,\infty)$	$[1,\infty)$	$sinh(x)$
$tanh(x)$	$(-\infty,\infty)$	$(-1,1)$	$sech^2(x)$
$coth(x)$	x not 0	$(-\infty,-1)\cup(1,\infty)$	$-csch^2(x)$
$sech(x)$	$(-\infty,\infty)$	$(0,1]$	$-sech(x)\tanh(x)$
$csch(x)$	x not 0	$(-\infty,0)\cup(0,\infty)$	$-csch(x)\coth(x)$
$sinh^{-1}(x)$	$(-\infty,\infty)$	$(-\infty,\infty)$	$1/\sqrt{x^2+1}$
$cosh^{-1}(x)$	$[1,\infty)$	$[0,\infty)$	$1/\sqrt{x^2-1}$
$cosh^{-1}(x)$	$[1,\infty)$	$(-\infty,0]$	$-1/\sqrt{x^2-1}$
$tanh^{-1}(x)$	$(-1,1)$	$(-\infty,\infty)$	$1/(1-x^2)$
$coth^{-1}(x)$	$(-\infty,-1)\cup(1,\infty)$	$(-\infty,0)\cup(0,\infty)$	$1/(1-x^2)$
$sech_1^{-1}(x)$	$(0,1]$	$(-\infty,0]$	$1/(x\sqrt{1-x^2})$
$sech_2^{-1}(x)$	$(0,1]$	$[0,\infty)$	$-1/(x\sqrt{1-x^2})$
$csch^{-1}(x)$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$	$-1/(\lvert x\rvert\sqrt{1+x^2})$

34 Inequalities

By the triangle inequality

$$|a+b| \leq |a| + |b|.$$

So if

$$a = b + c,$$

then

$$|a| \leq |b| + |c|.$$

And so

$$|b| \geq |a| - |c|.$$

We have

$$|a+b+c| \leq |a| + |b+c| \leq |a| + |b| + |c|$$

and so on. If

$$w = \sum_{k=1}^{n} z_k,$$

then

$$w - \sum_{k=2}^{n} z_k = z_1,$$

so

$$|w| \geq |z_1| - \sum_{k=2}^{n} |z_k|.$$

35 Convergence of Sequences

A sequence is a set indexed by the positive integers written

$$\{s_n\}_0^\infty.$$

The elements of the set might be numbers or functions. For example

$$s_n = 1/n.$$

As n goes to infinity, s_n goes to 0. A sequence of numbers $\{s_n\}_0^\infty$ converges to a if for each $\epsilon > 0$, there exists an integer N so that for all $n > N$,

$$|s_n - a| < \epsilon.$$

So the sequence $\{s_n\}_0^\infty$ given by

$$s_n = \frac{2+n+3n^3}{3+n^2+5n^3}$$

converges to 3/5, as can be seen by writing

$$s_n = \frac{2+n+3n^3}{3+n^2+5n^3} = \frac{2/n^3+1/n^2+3}{3/n^3+1/n+5}.$$

A sequence of functions $\{f_n\}_0^\infty$ defined on a domain D converges to a function f defined on D if every pointwise sequence $\{f_n(x)\}_0^\infty$ converges to $f(x)$. So for example if for $x \in (0,1)$

$$f_n(x) = x^n,$$

then f_n converges to the zero function on $(0,1)$.

A sequence with the property that for every $\epsilon > 0$ there exists an integer N so that

$$|s_n - s_m| < \epsilon$$

for all $n, m > N$ is called a Cauchy sequence.

proposition A convergent sequence is a Cauchy sequence.

Proof. So given an $\epsilon/2 > 0$ there exist an integer so that for all $n > N$ and $m > N$

$$|s_n - s_m| = |s_n - a - (s_m - a)| \leq |s_n - a| + |s_m - a| < \epsilon/2 + \epsilon/2 = \epsilon.$$

But a Cauchy sequence need not converge. A space in which every Cauchy sequence converges is called a complete space. There are Cauchy sequences of rational numbers that converge to $\sqrt{2}$, which is not a rational number. So the rational numbers are not a complete space. The real numbers are complete.

36 Infinite Series and Power Series

The nth partial sum of the infinite series

$$\sum_{k=0}^{\infty} a_k,$$

is

$$s_n = \sum_{k=0}^{n} a_k.$$

These partial sums form a sequence

$$\{s_n\}_0^\infty$$

The infinite series converges if the partial sums converge. If an infinite sequence converges then the sequence $\{|a_n|\}_0^\infty$ must converge to zero. This follows from the fact that the partial sums of the sequence, being a convergent sequence, are a Cauchy sequence. An alternating series (terms alternating in sign), where each term decreases in magnitude always converges.

A power series has the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

In general a power series has a radius of convergence R so that the series converges for all z such that $|z| < R$. Every power series converges for $z = 0$.

36.1 The Geometric Series

If

$$S = \sum_{k=0}^{n} x^k$$

then

$$Sx = S - 1 + x^{n+1}$$
$$S(1 - x) = 1 - x^{n+1}$$

or

$$S = \frac{1 - x^{n+1}}{x - 1}$$

If

$$0 < x < 1$$

then

$$\frac{x^{n+1}}{x - 1} \to 0$$

as $n \to \infty$. So

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}.$$

36.2 Comparison Test

If there exists an $0 < x < 1$ so that

$$0 < a_n < x^n,$$

then

$$\sum_{n=0}^{\infty} a_n$$

converges because the geometric series converges.

36.3 The Root Test

If

$$0 < a_n$$

and

$$\lim_{n\to\infty} (a_n)^{1/n} < 1,$$

then

$$\sum_{n=0}^{\infty} a_n$$

converges.

Proof. If the limit is less than 1, then there exists an $0 < x < 1$ and an integer N so that for all $n > N$

$$a_n^{1/n} < x$$

or

$$a_n < x^n.$$

and so the series converges by comparison with the geometric series.

Similarly if the limit is greater than 1, then the series diverges, because there exists an $x > 1$, and an N so that for all $n > N$

$$a_n > x^n,$$

and the geometric series diverges for $x > 1$.

36.4 The Ratio Test

If

$$0 < a_n$$

and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1,$$

then

$$\sum_{n=0}^{\infty} a_n$$

converges.

Proof. If the limit is less than 1, then there exists an $0 < x < 1$ and an integer N so that for all $n \geq N$

$$a_{n+1} < x a_n,$$

which implies that

$$a_n < x^{n-N} c,$$

for some constant c. So the series is eventually dominated by a convergent geometric series, and so must converge. And again if

$$\lim_{n\to\infty} \frac{a^{n+1}}{a^n} > 1,$$

then the series eventually dominates a divergent geometric series, so diverges.

37 Polar Coordinates

If $p = (x, y)$ is a point in the plane, let

$$\theta = \tan^{-1}(y/x),$$

and

$$r = \sqrt{x^2 + y^2}.$$

Then θ and r are called the polar coordinates of the point p. Given θ and r we have

$$x = r\cos(\theta),$$

and

$$y = r\sin(\theta).$$

A curve can be defined as a function $r(\theta)$. Example the spiral of Achimedies is defined as

$$r(\theta) = \alpha\theta.$$

38 Areas, Volumes, Moments of Inertia

Example Calculate the center of mass of a semicircular disk of unit thickness, unit density, and radius r. Let the semicircular disk lie above the x-axis. Moment about the x axis is

$$M_x = \int_A y dA = \int_0^r y 2x dy$$
$$= \int_0^r 2y\sqrt{r^2 - y^2} dy$$
$$= \frac{2r^3}{3}.$$

The y coordinate of the center of mass is

$$c_y = \frac{M_x}{(\pi/2)r^2} = \frac{4r}{3\pi}$$

39 Complex Analysis

`www.stem2.org/je/complex.pdf`

40 The Elements of Complex Analysis

Let

$$w = f(z)$$

be a complex function of a complex variable $z = x + yi$. Write

$$w = u + vi.$$

The derivative of f at z_0 is defined as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

If this limit is to exist, it must exist for z approaching z_0 from any direction. So it must exist in the special case where

$$z = x + y_0 i, z_0 = x_0 + y_o i.$$

Then since $w = u + vi$ and u and v are both functions of x and y, we see that

$$f'(z_0) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} i.$$

Similarly letting z approach z_0 in the y direction, we have

$$f'(z_0) = \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} i.$$

So a necessary condition for the existence of the derivative is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are called the Cauchy-Riemann equations. A function is called analytic or regular at a point z_0 if f has a derivative in an open neighborhood of z_0. We have shown that the Cauchy-Riemann equations are a necessary condition for f to be differentiable. This condition is also sufficient for a function to be analytic. That is if the four partial derivatives exist and are continuous in a neighborhood of z_0, and if the Cauchy-Riemann equations hold, then f is analytic at z_0.

An analytic function is given by a power series (Taylor Series) in a neighborhood of the point z_0

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

where the kth coefficient is defined by the kth derivative at z_0

$$a_k = \frac{f^{(k)}(z_0)}{k!}.$$

Cauchy's Theorem. The integral of an analytic function around a closed path is zero.

From this result we may deduce that the value of an analytic function at a point z_0 is given by **Cauchy's integral formula,**

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz,$$

where C is a closed path containing z_0 in its interior. Similarly all derivatives are defined by similar integrals.

To prove the integral formula we start with the result

$$2\pi i = \int_C \frac{1}{z - z_0}dz.$$

Then

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z_0)}{z - z_0}dz + \frac{1}{2\pi i}\int_C \frac{f(z) - f(z_0)}{z - z_0}dz.$$

The first integral is equal to

$$\frac{2\pi i f(z_0)}{2\pi i} = f(z_0).$$

According to Cauchy's theorem we may replace the path C by a circle K of arbitrarily small radius ρ. Then the second integral becomes

$$\frac{1}{2\pi i\rho}\int_K (f(z) - f(z_0))dz.$$

Given an $\epsilon > 0$, we can find a radius ρ so that

$$|f(z) - f(z_0)| < \epsilon.$$

So the second interval is arbitrarily small and hence zero. This proves the integral formula.

We have an integral formula for the nth derivative

$$f^{(n)}(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}dz.$$

Cauchy's Inequality Given a powseries representation for an analytic function $f(z)$, the nth coefficient is given by

$$a_n = \frac{f^{(n)}(z_o)}{n!} = \frac{1}{n!2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where C is a circle about Z_0 contained in the region of regularity of f. Let ρ be the radius of the circle, and let M be the maximum value of $|f(z)|$ on C. Then

$$|a_n| \leq \frac{M2\pi}{n!2\pi\rho^{n+1}} \leq \frac{M}{n!\rho^{n+1}}.$$

Theorem. A bounded entire function is a constant.
Proof. Let the entire function have a power series representation

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

Suppose $f(z)$ is bounded by a number $M > 0$. Then using Cauchy's inequality

$$|a_k| \leq \frac{M}{k!\rho^{n+1}}.$$

But because f is an entire function, the radius ρ may be taken arbitrarily large, so the right side can be made arbitrarily small. Therefore a_k is zero for $k > 0$, so

$$f(z) = a_0,$$

a constant.

41 A Polynomial is Unbounded

Given a polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n,$$

we have

$$p(z) - (a_0 + a_1 z + a_2 z^2 + ... + a_{n-1} z^{n-1} = a_n z^n).$$

Thus

$$|p(z)| + |a_0| + |a_1||z| + ... + |a_{n-1}||z^{n-1}| \geq |a_n||z^n|.$$

Let $r = |z|$, then

$$|p(z)| \geq |a_n|r^n - (|a_0| + |a_1|r^n + ... + |a_{n-1}|r^{n-1}).$$

So

$$|p(z)| \geq r^n(|a_n| - (\frac{|a_0|}{r^n} + \frac{|a_1|}{r^{n-1}} + ... + \frac{|a_{n-1}|}{r})).$$

Clearly

$$\frac{|a_0|}{r^n} + \frac{|a_1|}{r^{n-1}} + ... + \frac{|a_{n-1}|}{r}$$

goes to zero as r goes to infinity. So there is some $R > 1$ so that if $r > R$ then

$$|a_n| - (\frac{|a_0|}{r^n} + \frac{|a_1|}{r^{n-1}} + ... + \frac{|a_{n-1}|}{r}) > |a_n|/2.$$

So if $r > R$, then

$$|p(z)| \geq r^n(|a_n|/2.$$

Then given an arbitrarily large M, an $r > R$ can be chosen so that

$$r^n|a_n|/2 > M.$$

Hence given any $M > 0$, there exists a circle with center at the origin with radius r so that for all z outside of this circle.

$$p(z) > M.$$

Theorem. Given a polynomial $p(z)$ and a number $M > 0$ there exists a circle about the origin so that $\forall z$ outside of this circle.

$$|p(z)| > M.$$

42 A Proof of the Fundamental Theorem of Algebra

A bounded entire function is a constant. Given a non-constant polynomial $p(z)$. Suppose $p(z)$ does not have a root. Then

$$\frac{1}{p(z)}$$

is an entire function. But because $p(z)$ is a polynomial, $1/|p(z)|$ is say less than 1 for all points outside of some circle. That is it is bounded, and so a bounded entire function, and so a constant. This is a contradiction. Therefore $p(z)$ has a root.

43 Laurent Series

If f is analytic in $r_1 < |z - z_0| < r_2$ then

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z - z_0)^n,$$

where

$$A_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

44 The Residue Theorem

At an isolated singular point

$$A_{-1} = \frac{1}{2\pi i} \int_C f(z) dz.$$

So that

$$\int_C f(z) dz = 2\pi i A_{-1}$$

A_{-1} is called the residue of $f(z)$ at the isolated singular point z_0.

45 Calculating Residues

Let $f(z)$ have a simple pole of order n at z_0. Then

$$\phi(z) = (z - z_0)^n f(z),$$

is analytic in a neighborhood of z_0. Then the $n - 1$ coefficient of the Taylor expansion of $\phi(z)$ is

$$A_{-1} = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}\phi(z)}{dz^{n-1}}.$$

46 The Inversion of the Laplace Transform

We define the Fourier transform as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt.$$

The Fourier inversion theorem is

$$f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega.$$

The double sided Laplace transform is

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt.$$

Let $s = \phi + i\omega$. Then $F(s)$ is the Fourier transform of $g_\phi(t) = f(t)e^{-\phi t}$, that is

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} f(t) e^{-\phi t} e^{-i\omega t} dt \\ &= \hat{g}_\phi(\omega). \end{aligned}$$

Formally applying the Fourier inversion theorem, we have

$$\begin{aligned} f(t)e^{-\phi t} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}_\phi(\omega) e^{i\omega t} d\omega. \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{i\omega t} d\omega. \end{aligned}$$

Then

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{\phi t} e^{i\omega t} d\omega. \\ &= \frac{1}{2\pi i} \int_{C_\phi} F(s) e^{st} ds, \end{aligned}$$

where C_ϕ is the Bromwich contour defined by

$$\{\phi + i\omega : -\infty < \omega < \infty\}.$$

Note that i appears in the expression $2\pi i$ because

$$ds = i d\omega.$$

In general we will find that if we define a closed curve consisting of a finite line of length $2R$ on the bromwich contour, and a semicircle of radius R to the left, then as R goes to infinity, the integral over the semicircle goes to zero, so that the total integral over the curve is equal to the integral on the Bromwich line, which is thus equal to $2\pi i$ times the residues of $F(s)e^{st}$ in the left halfspace bounded by the contour. Our inversion expression is therefore

equal to the sum of the residues themselves. We get the single sided Laplace transform from the double when $f(t)$ is equal to zero for $t \le 0$.
Example: Consider

$$F(s) = \frac{1}{s-1},$$

for $\Re(s) > 1$. The residue of $F(s)e^{st}$ is

$$\lim_{s\to 1}(s-1)F(s)e^{st} = e^t.$$

Therefore

$$f(t) = e^t.$$

Example: Consider

$$F(s) = \frac{1}{s^2+1} = \frac{1}{(s-i)(s+i)},$$

for $\Re(s) > 0$. The residues of $F(s)e^{st}$ are

$$\lim_{s\to i}(s-i)F(s)e^{st} = \frac{e^{it}}{2i},$$

and

$$\lim_{s\to -i}(s+i)F(s)e^{st} = \frac{e^{-it}}{-2i},$$

Therefore

$$f(t) = \frac{e^{it} - e^{-it}}{2i} = \sin(t).$$

47 Parametric Curves

A parametric curve is a vector function where each component is a function of some parameter t. For example

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

$$= a\cos(t)\mathbf{i} + b\sin(t)\mathbf{j} + t^2\mathbf{k},$$

is some kind of elliptical spiral.

48 Arc Length

Let a parametric curve be defined by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

The velocity is

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

The velocity is is tangent to the curve. Define the arclength as

$$s(t) = \int_0^t \|\mathbf{v}\| dt$$

$$= \int_0^t \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} dt.$$

If

$$x(t) = r\cos(t),$$
$$y(t) = r\sin(t).$$
$$z(t) = 0,$$

then

$$dx/dt = -r\sin(t),$$
$$dy/dt = r\cos(t).$$

So

$$s(t) = \int_0^t r dt = rt.$$

On the other hand if the curve is

$$x(t) = a\cos(t),$$
$$y(t) = b\sin(t).$$
$$z(t) = 0,$$

which is an ellipse, then

$$dx/dt = -a\sin(t),$$
$$dy/dt = b\cos(t).$$

So

$$s(t) = \int_0^t \sqrt{a^2\sin(t)^2 + a^2\cos(t)^2} dt.$$

This is not an elementary integral, but is a type of elliptic integral, whose values can be expressed as a standard elliptic integral, whose values in turn are computed numerically and tabulated in mathematical handbooks.

49 Curvature and Elementary Differential Geometry

Curvature is the ratio of the change in turning to the distance traveled. Consider the circular path. As a point on the circle moves through an angle change $\Delta\theta$, it moves a distance $\Delta s = r\Delta\theta$. The ratio is a measure of the curvature

$$\frac{\Delta\theta}{\Delta s} = \frac{\Delta\theta}{r\Delta\theta} = \frac{1}{r}.$$

The angle change of the tangent $\Delta\phi$ is here equal to the angle change $\Delta\theta$, so we can use the tangent angle in our definition of the curvature. So suppose we are given a general curve in the plane

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}.$$

Suppose $dx/dt \neq 0$, then the angle of the tangent is

$$\phi = \tan^{-1}\left[\frac{dy/dt}{dx/dt}\right].$$

Let us write

$$\frac{dx}{dt} = \dot{x},$$
$$\frac{dy}{dt} = \dot{y},$$
$$\frac{d^2x}{dt^2} = \ddot{x},$$
$$\frac{d^2y}{dt^2} = \ddot{y}.$$

Then we have

$$\frac{d\phi}{dt} = \frac{1}{1+(\dot{y}/\dot{x})^2}\frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2} = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}.$$

We have

$$\frac{ds}{dt} = \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{\dot{x}^2 + \dot{x}^2}$$

So the curvature κ is

$$\begin{aligned}\kappa = \frac{d\phi}{ds} &= \frac{d\phi/dt}{ds/dt} \\ &= \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2} \frac{1}{\sqrt{\dot{x}^2 + \dot{x}^2}}. \\ &= \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.\end{aligned}$$

Now we excluded the case where $dx/dt = 0$. However we can just as well define the tangent angle as

$$\phi = \cot^{-1}\left[\frac{dx/dt}{dy/dt}\right].$$

If we carry out the derivation above we shall find that we get the same formula for the curvature. Hence as long as at least one of dx/dt or dy/dt is not zero, the formula holds.

Suppose we have a function $y = f(x)$. This is a curve with $t = x$ as the parameter. Then $\dot{x} = 1$, $\ddot{x} = 0$, and the curvature formula reduces to

$$\kappa = \frac{d\phi}{ds} = \frac{\ddot{y}}{(1 + \dot{y}^2)^{3/2}},$$

where

$$\ddot{y} = \frac{d^2 y}{dx^2},$$

and

$$\dot{y} = \frac{dy}{dx}.$$

In three dimensional space there is no obvious tsngent angle. So we must define curvature using another approch. Let

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

The velocity is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = dx/dt(t)\mathbf{i} + dy/dt(t)\mathbf{j} + dz/dt(t)\mathbf{k}.$$

The magnitude of the velocity is

$$v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \frac{ds}{dt}.$$

The unit tangent vector $\mathbf{T}$ is defined as

$$\mathbf{T} = \frac{\mathbf{v}}{v} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{d\mathbf{r}}{ds}.$$

$\mathbf{T}$ is a unit vector so

$$\mathbf{T} \cdot \mathbf{T} = 1.$$

We have

$$\frac{d(\mathbf{T} \cdot \mathbf{T})}{ds} = d\mathbf{T}/ds \cdot \mathbf{T} + \mathbf{T} \cdot d\mathbf{T}/ds = 0,$$

which implies that

$$2\mathbf{T} \cdot d\mathbf{T}/ds = 0.$$

So $\mathbf{T}$ and it derivative are orthogonal. Thus $d\mathbf{T}/ds$ is a vector normal to the curve. The unit normal vector $\mathbf{N}$ is defined as

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{\|d\mathbf{T}/ds\|}.$$

Thus

$$d\mathbf{T}/ds = \|d\mathbf{T}/ds\|\mathbf{N}.$$

We can define the curvature κ as

$$\kappa = \|d\mathbf{T}/ds\|,$$

because we can show that for a two dimensional curve this agrees with the two dimensional curvature.

Indeed, in two dimension the unit tangent vector $\mathbf{T}$ can be written as

$$\mathbf{T} = \cos(\phi)\mathbf{i} + \sin(\phi)\mathbf{j},$$

where the tangent angle ϕ is a function of the arc length s. Then

$$d\mathbf{T}/ds = -\sin(\phi)(d\phi/ds)\mathbf{i} + \cos(\phi)(d\phi/ds)\mathbf{j}.$$

So

$$\|d\mathbf{T}/ds\| = |d\phi/ds|\sqrt{\sin^2(\phi) + \cos^2(\phi)} = |d\phi/ds| = \kappa.$$

Notice that here the curvature is always nonnegative. We can therefore write

$$d\mathbf{T}/ds = \kappa\mathbf{N}.$$

Let the curve $r(s)$ in Euclidean 3-space, be parameterized by arc length. Define the tangent vector

$$t = \frac{dr}{ds},$$

the curvature

$$\kappa = \left|\frac{dt}{ds}\right|,$$

the normal vector

$$n = \frac{1}{\kappa}\frac{dt}{ds},$$

and the binormal vector

$$b = t \times n.$$

The vectors t, n, and b, are called the frame vectors. The derivatives of these frame vectors with respect to arc length s are equal to linear combinations of the frame vectors themselves. These are called the Serret-Frenet formulas. The Serret-Frenet formulas are derived from the facts that the frame vectors are mutually perpendicular, and that they have unit length. The dot product of any pair of frame vectors is zero. So the derivative of their dot product is also zero. Unit vectors are perpendicular to their derivatives, and n is a unit vector. So dn/ds is perpendicular to n. Consequently dn/ds can be written as a linear combination of t and b only. Thus

$$\frac{dn}{ds} = a_1 t + a_3 b.$$

Because t is perpendicular to n,

$$a_1 = \frac{dn}{ds} \cdot t = -n \cdot \frac{dt}{ds}.$$

By definition the right hand expression is equal to $-\kappa$. So we conclude that a_1 is equal to the curvature κ.

Define the torsion τ to be a_3. Thus

$$\frac{dn}{ds} = -\kappa t + \tau b.$$

Let

$$\frac{db}{ds} = b_1 t + b_2 n.$$

$$b_1 = \frac{db}{ds} \cdot t = -b \cdot \frac{dt}{ds} = -b \cdot n = 0.$$

$$\frac{db}{ds} \cdot n = -b \cdot \frac{dn}{ds} = -b \cdot (-\kappa t + \tau b) = -\tau.$$

Thus

$$\frac{db}{ds} = -\tau n.$$

Therefore we have the Serret-Frenet transformation,

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

Notice that the matrix of the transformation is antisymmetric, and that the top right element of the matrix is zero. An antisymmetric matrix has a zero diagonal. The lower triangular part has negative elements. These facts might aid one in remembering the formula.

50 Curves and Surfaces

There are parametric curves and algebraic curves. The same is true for surfaces. There are reasons to convert from one representation to the other. Curves and surfacces may be defined piecewise. See the document called **Geometric Calculations**:

`www.stem2.org/je/geometry.pdf`

51 Interpolation, Piecewise Curves, Splines

`www.stem2.org/je/numanal.pdf`
`www.stem2.org/je/cs.pdf`

52 Vector Spaces

53 Linear Transformations and Matrices

See linearal.tex

54 Multivariable Calculus

55 Elementary Differential Equations

`www.stem2.org/je/diffeq.pdf`

56 Elements of Vector Analysis

`www.stem2.org/je/vecana.pdf`

See vecana.tex

Vector analysis is a classical subject dealing with those aspects of vectors which have application in applied mathematics and specifically to physics. Vector analysis deals largely with vector calculus. Linear Algebra also deals with vectors and vector spaces, but confines itself to algebra.

57 The Law of Cosines and the Inner Product

We shall prove the law of cosines. Suppose we have three points

$$p_0 = (0,0), p_1 = (b,0), p_2 = (x,y) = (a\cos(\theta), a\sin(\theta)),$$

where we take $y \geq 0$. These points form a triangle with sides p_0p_2, p_0p_1, p_2p_1. These sides have lengths a, b, c. The angle between side p_0p_1 and side p_0p_2 is θ. We have

$$\begin{aligned} c^2 &= (x-b)^2 + y^2 \\ &= (x-b)^2 + a^2 - x^2 \\ &= x^2 - 2xb + b^2 + a^2 - x^2 \\ &= a^2 + b^2 - 2xb = a^2 + b^2 - 2ab\cos(\theta). \end{aligned}$$

Thus we have the law of cosines, namely the square of the side opposite an angle of a triangle, is equal to the sum of the squares of the adjacent sides, minus two times the product of the sides and the cosine of the angle. That is,

$$c^2 = a^2 + b^2 - 2ab\cos(\theta).$$

The inner product (dot product) of two vectors, A and B, is defined as

$$A \cdot B = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Then the dot product of a vector with itself is the square of its length. That is,

$$A \cdot A = a_1 a_1 + a_2 a_2 + a_3 a_3 = \|A\|^2.$$

Let

$$C = B - A.$$

Then

$$\|C\|^2 = (B - A) \cdot (B - A)$$

$$= B \cdot B - B \cdot A - A \cdot B + A \cdot A$$

$$= \|B\|^2 - 2A \cdot B + \|A\|^2.$$

From which it follows that

$$2A \cdot B = \|A\| + \|B\|^2 - \|C\|^2.$$

But the right hand side is, by the law of cosines,

$$2\|A\|\|B\|\cos(\theta),$$

where θ is the angle between vectors A and B. Hence

$$A \cdot B = \|A\|\|B\|\cos(\theta).$$

Thus if the the dot product is zero, then the cosine is zero, and so the angle between the vectors is plus or minus $\pi/2$, and the vectors are perpendicular.

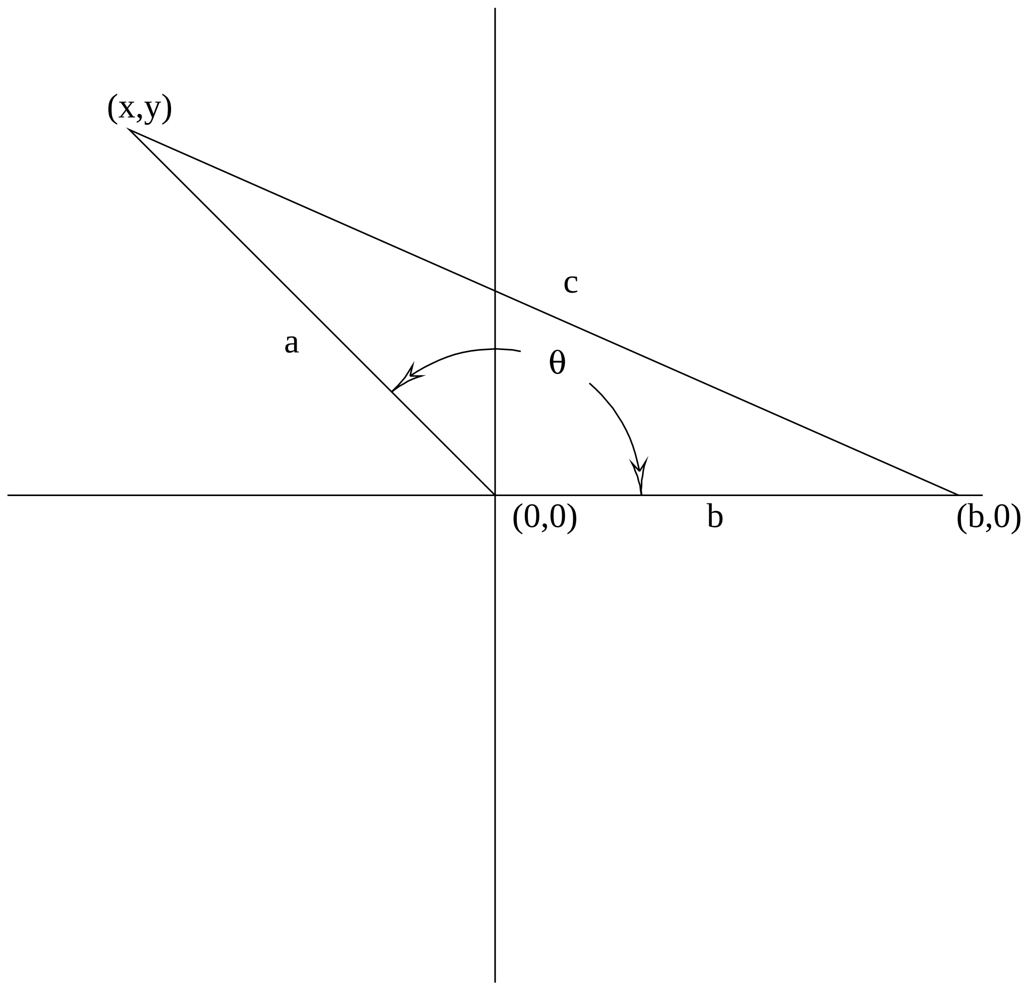

Figure 7: Derivation of the law of cosines. $x = a\cos(\theta)$, $y = a\sin(\theta)$. Computing c^2, we find that $c^2 = a^2 + b^2 - 2ab\cos(\theta)$.

58 The Vector Product

The vector product of two vectors A and B, (the cross product), is defined to be

$$A \times B = (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k,$$

where i, j, k are the unit coordinate vectors. This may be written as a determinant with i, j, k in the first row, the components of A in the second, and the components of B in the third row.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

When the rows of a determinant are interchanged, the sign of the determinant changes, hence

$$A \times B = -B \times A.$$

Then

$$A \times A = -A \times A.$$

But this can be true only if

$$A \times A = 0.$$

We have shown that the vector product of any two parallel vectors is zero.

Given three vectors A, B, C, we see that

$$A \cdot (B \times C),$$

is given as the determinant that has rows A, B, and C. By interchanging these rows twice, we see that

$$A \cdot (B \times C) = (A \times B) \cdot C.$$

That is, in the scalar triple product, the dot and the cross may be interchanged. Now using this result, we see that

$$(A \times B) \cdot B = A \cdot (B \times B) = A \cdot 0 = 0.$$

Then $A \times B$ is perpendicular to B. Similarly it is perpendicular to A. Therefore we have shown that the vector product of two vectors is perpendicular to each of them. This establishes the direction of the vector product, except possibly for sign. One may further establish the right hand rule. The direction of $A \times B$ is given by the right hand rule: Curl the fingers of your right hand from A to B, then $A \times B$ is in the direction of your thumb. One may verify directly that if V is a vector in the upper xy half plane that

$$i \times V$$

points in the positive z direction. This verifies the right hand rule in this case. One may also show the invariance of the cross product to a rigid motion, which establishes the right hand rule in general.

We have established the direction of the cross product, now we shall find its magnitude. We shall do this with the aid of the "Back Minus Cab Rule"

By direct computation one may verify that the vector triple product satisfies

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B).$$

This is the "Back Minus Cab Rule" (Note that each term is a single vector, followed by an expression in parentheses).

Exercise. Evaluate $i \times (k \times i)$ in two ways

(1) By the "Back Minus Cab Rule," and

(2) by using $k \times i = j$

To find the magnitude of $A \times B$, let

$$C = A \times B.$$

Then

$$\begin{aligned} \|C\|^2 &= C \cdot C \\ &= (A \times B) \cdot C \\ &= A \cdot (B \times C) \\ &= A \cdot (B \times (A \times B)) \\ &= A \cdot (A(B \cdot B) - B(B \cdot A)) \\ &= (A \cdot A)(B \cdot B) - (A \cdot B)^2 \\ &= \|A\|^2\|B\|^2(1 - \cos^2(\theta)) = \|A\|^2\|B\|^2 \sin^2(\theta). \end{aligned}$$

The magnitude of the cross product is the product of the lengths of the vectors, times the sine of the angle between them,

$$\|A \times B\| = \|A\|\|B\| \sin(\theta).$$

Example The equation of a plane. Let the plane have a unit normal vector N. Let $P = (x, y, z)$ be a point on the plane. Let d be the distance from the origin to the plane. Then d is equal to the length of P times the cosine of the angle between P and the normal N. Hence

$$d = P \cdot N.$$

Therefore the equation of the plane is

$$P \cdot N - d = xn_1 + yn_2 + zn_3 - d = 0.$$

Suppose we are given three points P_1, P_2, P_3 and we wish to find the equation of the plane passing through these points. The normal to the plane is perpendicular to each of $P_2 - P_1$ and $P_3 - P_1$. Therefore

$$N = \frac{(P_2 - P_1) \times (P_3 - P_1)}{\|(P_2 - P_1) \times (P_3 - P_1)\|}$$

Also d is equal to the inner product of N with any one of the three points. For example

$$d = P_1 \cdot N.$$

Then the equation of the plane is

$$P \cdot N - P_1 \cdot N = xn_1 + yn_2 + zn_3 - d = 0.$$

59 Heron's Formula for the Area of a Triangle

Let T be the area of a triangle with sides given by vectors A, B, and C, and corresponding side lengths a, b and c. The area is one half of the magnitude of the cross product of the vectors A and B. That is,

$$2T = \|A \times B\|.$$

So

$$4T^2 = a^2b^2 \sin^2(\theta) = a^2b^2(1 - \cos^2(\theta)) = a^2b^2 - \|A \cdot B\|^2.$$

Also

$$c^2 = \|C\|^2 = \|A - B\|^2 = (A - B) \cdot (A - B) = a^2 - 2A \cdot B + b^2.$$

Then

$$\|A \cdot B\|^2 = \frac{(c^2 - (a^2 + b^2))^2}{4}.$$

Substituting this into the equation that we found above, namely

$$4T^2 = a^2b^2 - \|A \cdot B\|,$$

we get

$$\begin{aligned}
16T^2 &= 4a^2b^2 - (a^2 + b^2 - c^2)^2 \\
&= [2ab - (a^2 + b^2 - c^2)][2ab + (a^2 + b^2 - c^2)] \\
&= [c^2 - (a - b)^2][(a + b)^2 - c^2] \\
&= [c - (a - b)][c + (a + b)][a + b - c][a + b + c] \\
&= [c + b - a][c + a - b][a + b - c][a + b + c] \\
&= [a + b + c - 2a][a + b + c + -2b][a + b + c - 2c][a + b + c].
\end{aligned}$$

Dividing each product on the right by 2, we get

$$T^2 = (s - a)(s - b)(s - c)s,$$

where

$$s = \frac{a + b + c}{2},$$

is the half perimeter of the triangle. Taking the square root, we get Heron's formula,

$$T = \sqrt{(s - a)(s - b)(s - c)s}.$$

This derivation is suggested in a problem in Apostol's Calculus.

60 Line Integrals

A line integral is the integration of a vector function along a curve. So let C be a curve given by

$$\mathbf{r}(t) = x(r)\mathbf{i} + y(r)\mathbf{j} + z(r)\mathbf{k},$$

for $a \leq t \leq b$. Let $\mathbf{A}$ be a vector field. Then the line integral of $\mathbf{A}$ along curve C is

$$\int_C \mathbf{A} \cdot d\mathbf{r} = \int_a^b \mathbf{A}(\mathbf{r}) \cdot (d\mathbf{r}/dt)dt.$$

A vector field is a vector function defined in a region of space.

61 Curl, Divergence, Gradient

The curl of a vector field $\mathbf{A}$ in cartesian coordinates is

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{i}$$

$$- \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \mathbf{j}$$

$$\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{k}.$$

Divergence thereom. Stokes theorem. Directional derivative.

The divergence of a vector field $\mathbf{A}$ in cartesian coordinates is

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

If a surface S has bounding curve ∂S, Stokes theorem is

$$\int_S \nabla \times \mathbf{A} \cdot \mathbf{n} ds = \int_{\partial S} \mathbf{A} \cdot d\mathbf{r},$$

which allows a surface integral to be evaluated as a line integral around the boundary of the surface. The surface normal is $\mathbf{n}$.

The divergence theorem allows a volume integral to be evaluated as a surface integral. Let V be a volume and ∂V be it enclosing surface. Then

$$\int_V \nabla \cdot \mathbf{A} dv = \int_{\partial V} \mathbf{A} \cdot \mathbf{n} ds.$$

If S is an area in the x, y plane then a special case of Stokes Theorem gives

$$\int_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) ds$$

$$= \int_{\partial S} \mathbf{A} \cdot d\mathbf{r}$$

$$= \int_{\partial S} (A_x dx + A_y dy).$$

As an application of this formula we can find the area enclosed by a curve by evaluating a line integral around the curve. So let $A_x = -y/2$ and $A_y = x/2$, then

$$\int_S ds =$$

$$\int_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) ds$$

$$= \int_{\partial S} (A_x dx + A_y dy).$$

$$= (1/2) \int_{\partial S} (ydx - xdy).$$

62 Optimization: Lagrange Multipliers

See cnstrop.tex.

63 Numerical Analysis

See

`www.stem2.org/je/numanal.pdf`

64 Exercises

Exercise [1]
By carrying out the multiplication show that

$$(a-b)(a+b) = a^2 - b^2.$$

Exercise [2] Use the fact that

$$(a+b)^2 = a^2 + 2ab + b^2$$

to show that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Exercise [3]

Given that the equation of a standard ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

show that the parametric equations

$$x = a\cos(\theta),$$

and

$$y = b\sin(\theta),$$

are parametric equations of an ellipse.

Exercise [4]

Prove that the derivative of

$$f(x) = \frac{1}{x}$$

is

$$-\frac{1}{x^2}$$

from the definition of the derivative. Try to do this without looking at the solution below.

Solution.

We have

$$\frac{df}{dx} = \lim_{h\to 0}(\frac{1}{x+h} - \frac{1}{x})/h$$

$$= \lim_{h\to 0}(\frac{x-(x+h)}{(x+h)x})/h$$

$$= \lim_{h\to 0}\frac{-1}{(x+h)x}$$

$$= -\frac{1}{x^2}$$

The LaTeXcode for this little section is:

```
Prove that the derivative of
\[ f(x) = \frac{1}{x} \]
is
\[ -\frac{1}{x^2} \]
from the definition of the derivative.

We have
\[ \frac{df}{dx} = \lim_{h \rightarrow 0}
 ( \frac{1}{x+h} - \frac{1}{x})/h \]
\[ = \lim_{h \rightarrow 0}( \frac{x -(x+h)}{(x+h)x})/h \]
\[ = \lim_{h \rightarrow 0} \frac{ -1}{(x+h)x} \]
\[ = - \frac{1}{x^2} \]
```

65 $\tau\epsilon\chi$nical Writing

There is joy in TeX, and ecstacy in LaTeX.

66 Appendix A, Computer Programs

66.1 Listing sine.ftn

The Fortran program **sine.ftn** computes a Taylor approximation to function **sin**(**x**). The program asks for the order n of the Taylor polynomial approximation.

```
c     test of taylor series for sine
c     10-29-2014
      implicit real*8(a-h,o-z)

c     dimension x(100),y(100)
      dimension a(2)
       nfile1=1
       open(nfile1,file='p1.eg',status='unknown')
       nfile2=2
       open(nfile2,file='p2.eg',status='unknown')
```

```
      one=1.
      zero=0.
      pi=4.*atan(one)
      nt=37
      write(*,*)' Enter number of Taylor Series terms'
      call readr(nf, a, nr)
      if(nr .gt. 0)nt=a(1)
      write(*,*)' Highest order Taylor term is ',nt
      nz=nt/2
      if(2*(nt/2) .ne. nt)nz=nz+1
      write(*,*)' Number of nonzero terms is', nz
      n=25
      do i=0,n
       j=md(i,4)
c write(*,*)' i=',i,' i mod 4= ',j
      end do
      x=.1
      y=x-x**3/6. + x**5/120.
c     write(*,'(3(1x,g21.14))')x,sin(x),y
      call taylor(x,nt,y)
c     write(*,'(3(1x,g21.14))')x,sin(x),y
      write(*,*)

      xmn=0.
      xmx=2.*pi
c
      do i=1,n
       x=(i-1)*(xmx-xmn)/(n-1) + xmn
call taylor(x,nt,y)
write(*,'(3(1x,g21.14))')x,sin(x),y
      end do

      write(nfile1,'(a,3(1x,g12.5))')'m',0.,0.
      write(nfile1,'(a,3(1x,g12.5))')'d',6.28,0.

c     write plot points for sine curve
      n=100
      do i=1,n
```

```
        x=(i-1)*(xmx-xmn)/(n-1) + xmn
        if(i .eq. 1)then
 write(nfile1,'(a,3(1x,g12.5))')'m',x,sin(x)
else
         write(nfile1,'(a,3(1x,g12.5))')'d',x,sin(x)
        end if
       end do

c      write plot points for Taylor series approximation to sine
       n=100
       do i=1,n
        x=(i-1)*(xmx-xmn)/(n-1) + xmn
call taylor(x,nt,y)
        if(abs(y) .lt. 2.0d0)then
 if(i .eq. 1)then
  write(nfile2,'(a,3(1x,g12.5))')'m',x,y
 else
          write(nfile2,'(a,3(1x,g12.5))')'d',x,y
         end if
        endif
       end do

      end
c
       subroutine taylor(x,k,y)
        implicit real*8(a-h,o-z)
zero=0.
        f=1
        y=0.
        do i=1,k
         f=f*i
c        calculate i mod 4
         j=md(i,4)
         if(j .eq. 0)t=zero
         if(j .eq. 1)t=x**i/f
         if(j .eq. 2)t=zero
         if(j .eq. 3)t=-x**i/f
         y=y+t
```

```
         enddo
        return
        end
c
        function md(i,m)
         integer d,r
d=i/m
r=i-m*d
md=r
return
end
c
c+ valsub converts string to floating point number (r*8)
      subroutine valsub(s,v,ier)
      implicit real*8(a-h,o-z)
c     examples of valid strings are: 12.13 34 45e4 4.78e-6 4E2
c     the string is checked for valid characters,
c     but the string can still be invalid.
c      s-string
c      v-returned value
c      ier- 0 normal
c           1 if invalid character found, v returned 0
c
      logical p
      character s*(*),c*50,t*50,ch*15
      character z*1
      data ch/'1234567890+-.eE'/
      v=0.
      ier=1
      l=lenstr(s)
      if(l.eq.0)return
      p=.true.
      do 10 i=1,l
      z=s(i:i)
      if((z.eq.'D').or.(z.eq.'d'))then
          s(i:i)='e'
      endif
      p=p.and.(index(ch,s(i:i)).ne.0)
```

```
 10   continue
      if(.not.p)return
      n=index(s,'.')
      if(n.eq.0)then
          n=index(s,'e')
          if(n.eq.0)n=index(s,'E')
          if(n.eq.0)n=index(s,'d')
          if(n.eq.0)n=index(s,'D')
          if(n.eq.0)then
             s=s(1:l)//'.'
          else
            t=s(n:l)
            s=s(1:(n-1))//'.'//t
          endif
          l=l+1
      endif
      write(c,'(a30)')s(1:l)
      read(c,'(g30.23)')v
      ier=0
      return
      end

c+ readr read a row of numbers and return in double precision array
      subroutine readr(nf, a, nr)
      implicit real*8(a-h,o-z)
c Input:
c nf    unit number of file to read
c       nf=0 is the standard input file (keyboard)
c Output:
c a     array containing double precision numbers found
c nr    number of values in returned array,
c     or 0 for empty or blank line,
c       or -1 for end of file on unit nf.
c   Numbers are separated by spaces.
c   Examples of valid numbers are:
c   12.13 34 45e4 4.78e-6 4e2,5.6D-23,10000.d015
c requires subroutine valsub and function lenstr
c a semicolon and all characters following are ignored.
```

```
c This can be used for comments.
c modified 6/16/97 added semicolon feature
      dimension a(*)
      character*200  b
      character*200  c
      character*1 d
        c=' '
      if(nf.eq.0)then
         read(*,'(a)',end=99)b
      else
         read(nf,'(a)',end=99)b
      endif
      nr=0
      lsemi=index(b,';')
      if(lsemi .gt. 0)then
       if(lsemi .gt. 1)then
        b=b(1:(lsemi-1))
       else
        return
       endif
      endif
      l=lenstr(b)
      if(l.ge.200)then
        write(*,*)' error in readr subroutine '
        write(*,*)' record is too long '
      endif
      do 1 i=1,l
       d=b(i:i)
       if (d.ne.' ') then
        k=lenstr(c)
        if (k.gt.0)then
           c=c(1:k)//d
        else
           c=d
        endif
       endif
       if( (d.eq.' ').or.(i.eq.l)) then
        if (c.ne.' ') then
```

```
          nr=nr+1
           call valsub(c,a(nr),ier)
          c=' '
        endif
       endif
1     continue
      return
99    nr=-1
      return
      end
c+ lenstr   nonblank length of string
      function lenstr(s)
c  length of the substring of s obtained by deleting all
c  trailing blanks from s.  thus the length of a string
c  containing only blanks will be 0.
      character   s*(*)
      lenstr=0
      n=len(s)
      do 10 i=n,1,-1
      if(s(i:i) .ne. ' ')then
         lenstr=i
         return
      endif
10    continue
      return
      end
```

67 Appendix B, What is Calculus?

68 The Parts of Calculus

Traditionally Calculus was divided into two parts, Differential Calculus and Integral Calculus, and taught successively. Now this division is rather old fashioned and not made often. Rather the material for a course in Calculus is normally mixed and integrated, (no pun intended). Calculus used to be called "The Infinitesimal Calculus," which stressed its concern with the infinitely small.

Differential Calculus deals with the derivative, which is a limiting ratio of small changes. The most common example is instantaneous velocity, which is a ratio of a small distance change to a small time change.

Integral Calculus concerns itself with taking small bits and adding them together to get a whole. Thus for example, the determination of area or volume is, except for simple cases, a problem in Integral Calculus.

The word integral means "whole" or "oneness." Politicians often talk about "having integrity," which according to them means being truthful. The real meaning of integrity is a bit different. One has integrity, if he behave just "one" way to all people. He does not say one thing to one person while saying a different thing to a second person, behind the back. That is, one with integrity is "whole." In fact, one could have integrity by lying uniformly to everyone. One fact is indubitably clear, to have true integrity, one must do Calculus.

One good reason for mixing the two parts of calculus is that differentiation and integration are inverses of one another, one splits apart and analyzes, the other puts together and synthesizes. Calculus is also known as Analysis, especially in its advanced treatment and ramifications.

Often one might hear someone say, "What is Calculus used for?" as in the age old statement of students, "What do I have to learn this for? I will never use it! I am going to be a politician, a football player, an exotic dancer, or whatever." But calculus is about everything and used for everything. It is about how everything in the world works.

69 The Derivative

A derivative of a function $y = f(x)$ is the limit of ratio of a change in y, written Δy to a change in x written Δx as Δx goes to zero. Put another way, the derivative of f, at coordinate x, is the slope of the tangent line to the curve at the point (x, y). Refer to figure 1, where we see the slope of the chord joining the points (x, y) and (x_1, y_1), and we see that as $x_1 \to x$, this chord becomes the tangent line. The derivative is often given as

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

where $\lim_{h\to 0}$ means the limit as h goes to 0, the limiting value as h becomes very small. Here h plays the role of $\Delta x = x_1 - x$ and $f(x + h) - f(x)$ plays the role of Δy.

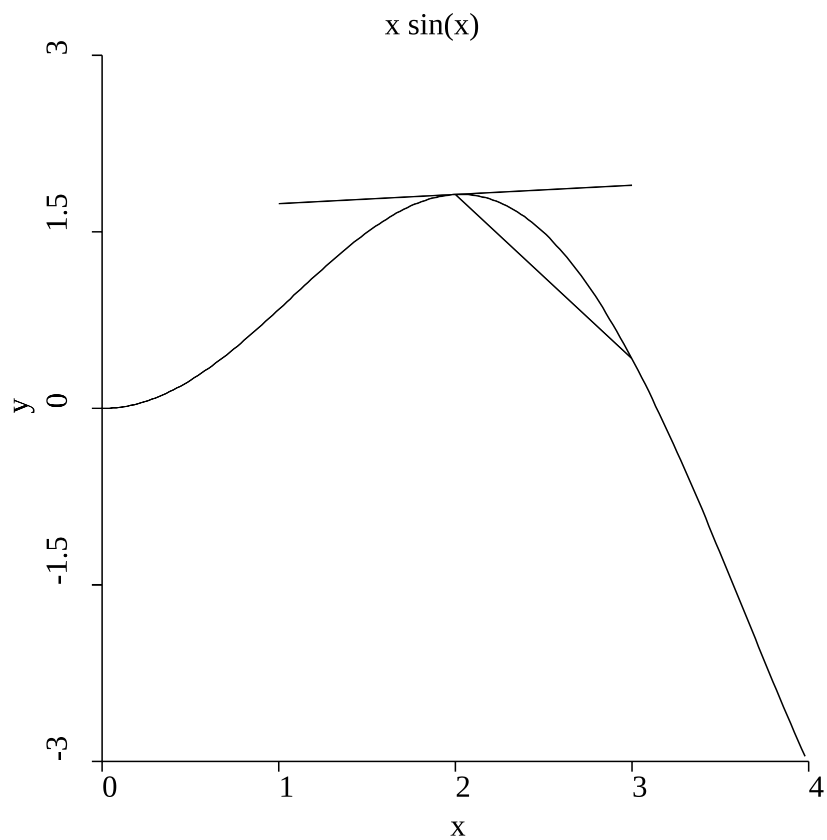

Figure 8: **The derivative is the slope of the tangent line**. This figure shows a plot of the function $y = f(x) = x\sin(x)$. The tangent line at the point on the curve where $x = 2$ is shown. A chord of the curve is a line joining two points of the curve, such as the line joining the points $(x, f(x))$ and $(x+h, f(x+h))$. Shown here is such a chord, where $x = 2$, and $h = 1$. The limit of the slope of this chord line as h goes to zero, is the derivative of f at x, which is the ratio of the change in y to the change in x, as h goes to zero.

Let us attempt to calculate a derivative approximately with a computer program. We use the function $f(x) = \sin(x)$, and calculate the derivative at $x = 1$.

The Program

```
// derfun.c, compute the derivative of sin(x) at 1.
#include <stdio.h>
#include <math.h>
double f(double);
int main (){
 double x;
 double y;
 double h;
 double dydx,dydx2;
 int n;
 int i;
 n=45;
 h=.5;
 x=1.;
  printf(" x = %21.15g \n",x);
  printf(" cos(x) = %21.15g \n",cos(x));
 for(i=0;i<n;i++){
  dydx=(f(x+h)-f(x))/h;
  dydx2=(f(x+h)-f(x-h))/(2.*h);
  printf(" h = %21.15g dydx = %21.15g dydx_2 = %21.15g\n",h,dydx,dydx2);
  h=h/2.;
 }

 return(1);
}
double f(double x){
 double y;
 y=sin(x);
 return (y);
}
```

The Output of the Program

```
x =                     1
 cos(x) =      0.54030230586814
 h =                   0.5 dydx =     0.312048003592316 dydx_2 =     0.518069447999851
 h =                  0.25 dydx =     0.430054538190759 dydx_2 =     0.534691718664504
 h =                 0.125 dydx =     0.486372874329589 dydx_2 =     0.538896367452272
 h =                0.0625 dydx =     0.513663205746793 dydx_2 =     0.539950615251025
 h =               0.03125 dydx =     0.527067456146781 dydx_2 =     0.540214370333548
 h =              0.015625 dydx =     0.533706462857715 dydx_2 =     0.540280321179402
 h =             0.0078125 dydx =     0.537009830329723 dydx_2 =     0.540296809645639
 h =            0.00390625 dydx =     0.538657435881987 dydx_2 =     0.540300931809369
 h =           0.001953125 dydx =     0.539480213605884 dydx_2 =     0.540301962353254
 h =          0.0009765625 dydx =     0.539891345517731 dydx_2 =     0.540302219989371
 h =         0.00048828125 dydx =      0.54009684715038 dydx_2 =     0.540302284398422
```

```
h =        0.000244140625 dydx =     0.540199581875186 dydx_2 =     0.540302300500798
h =       0.0001220703125 dydx =     0.540250945213302 dydx_2 =     0.540302304526449
h =      6.103515625e-05 dydx =     0.540276625875777 dydx_2 =      0.54030230553235
h =     3.0517578125e-05 dydx =     0.540289465956448 dydx_2 =      0.54030230578428
h =    1.52587890625e-05 dydx =     0.540295885934029 dydx_2 =     0.540302305846126
h =    7.62939453125e-06 dydx =     0.540299095911905 dydx_2 =     0.540302305867954
h =   3.814697265625e-06 dydx =     0.540300700900843 dydx_2 =      0.54030230587523
h =  1.9073486328125e-06 dydx =     0.540301503380761 dydx_2 =     0.540302305860678
h =  9.5367431640625e-07 dydx =     0.540301904664375 dydx_2 =     0.540302305889782
h = 4.76837158203125e-07 dydx =     0.540302105247974 dydx_2 =     0.540302305831574
h = 2.38418579101562e-07 dydx =     0.540302205365151 dydx_2 =     0.540302305715159
h = 1.19209289550781e-07 dydx =     0.540302256122231 dydx_2 =     0.540302305947989
h = 5.96046447753906e-08 dydx =     0.540302280336618 dydx_2 =     0.540302305482328
h = 2.98023223876953e-08 dydx =      0.54030229523778 dydx_2 =     0.540302306413651
h = 1.49011611938477e-08 dydx =      0.54030229896307 dydx_2 =     0.540302306413651
h = 7.45058059692383e-09 dydx =     0.540302306413651 dydx_2 =     0.540302306413651
h = 3.72529029846191e-09 dydx =     0.540302306413651 dydx_2 =     0.540302306413651
h = 1.86264514923096e-09 dydx =     0.540302276611328 dydx_2 =     0.540302276611328
h = 9.31322574615479e-10 dydx =     0.540302276611328 dydx_2 =     0.540302276611328
h = 4.65661287307739e-10 dydx =     0.540302276611328 dydx_2 =     0.540302276611328
h =  2.3283064365387e-10 dydx =     0.540302276611328 dydx_2 =     0.540302276611328
h = 1.16415321826935e-10 dydx =     0.540302276611328 dydx_2 =     0.540302276611328
h = 5.82076609134674e-11 dydx =     0.540302276611328 dydx_2 =     0.540302276611328
h = 2.91038304567337e-11 dydx =     0.540302276611328 dydx_2 =     0.540302276611328
h = 1.45519152283669e-11 dydx =     0.540306091308594 dydx_2 =     0.540302276611328
h = 7.27595761418343e-12 dydx =     0.540298461914062 dydx_2 =     0.540298461914062
h = 3.63797880709171e-12 dydx =     0.540313720703125 dydx_2 =     0.540313720703125
h = 1.81898940354586e-12 dydx =        0.540283203125 dydx_2 =        0.540283203125
h = 9.09494701772928e-13 dydx =        0.540283203125 dydx_2 =        0.540283203125
h = 4.54747350886464e-13 dydx =        0.540283203125 dydx_2 =        0.540283203125
h = 2.27373675443232e-13 dydx =         0.54052734375 dydx_2 =         0.54052734375
h = 1.13686837721616e-13 dydx =          0.5400390625 dydx_2 =          0.5400390625
h =  5.6843418860808e-14 dydx =           0.541015625 dydx_2 =           0.541015625
h =  2.8421709430404e-14 dydx =             0.5390625 dydx_2 =             0.5390625
```

The actual derivative of $\sin(x)$ is $\cos(x)$ and

$$\cos(1) = 0.54030230586814,$$

and our best approximation to this occurs where h is about 10^{-9}. As h decreases more the error actually increases. This is because of roundoff error, the fact that we are using numbers of only about 15 digits in the computer. Notice that the $dydx_2$ result is a bit more accurate. This is because we use a central difference approximation where the truncation error is of the order h^2, rather than of order h in the one sided difference.

70 The Derivative of $f(x) = x^n$

Consider the derivative of the function $f(x) = x$. From the definition

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = 1.$$

Consider the derivative of $f(x) = x^3$. We have

$$f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3.$$

So

$$\begin{aligned} \frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \\ = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ = \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2. \end{aligned}$$

As $h \to 0$, we are left with $3x^2$. In the same way we can compute the derivative of x^n using the binomial theorem

$$(x+h)^n = x^n + nx^{n-1}h + ... + h^n.$$

So to compute the derivative of x^n, we multiply by n and decrease the exponent by 1, getting nx^{n-1} . This also holds for $n < 0$.

We can use the derivative to find the maximum or minimum of a function. This is because at the minimum or maximum point of a function, it has a tangent line of zero slope. So we only to need find where the derivative of a function is zero.

Example. Suppose we want to build a cylindrical tank to hold a specified volume of liquid. The surface area is to be minimized. How should the radius r and the height h of the tank be chosen.

Let the constant volume be

$$V = \pi r^2 h$$

then

$$h = \frac{V}{\pi r^2}.$$

The area is

$$\begin{aligned} A &= 2\pi r^2 + 2r\pi h \\ &= 2\pi r^2 + 2r\pi \frac{V}{\pi r^2} \end{aligned}$$

$$= 2\pi r^2 + \frac{2V}{r}$$

The derivative is

$$\frac{dA}{dr} = 4\pi r - \frac{2V}{r^2}$$

Setting the derivative to zero, we have

$$4\pi r^3 - 2V = 0.$$

So

$$r^3 = \frac{V}{2\pi}$$

and so

$$r = \left[\frac{V}{2\pi}\right]^{1/3}.$$

From above

$$h = \frac{V}{\pi r^2} = \frac{V}{\pi \left[\frac{V}{2\pi}\right]^{2/3}}$$

So

$$h/2 = \frac{\frac{V}{2\pi}}{\left[\frac{V}{2\pi}\right]^{2/3}} = \left[\frac{V}{2\pi}\right]^{1/3} = r.$$

The diameter of the tank is

$$d = 2r = h.$$

The tank of minimum surface area has a diameter equal to the height.

71 A Table of Elementary Derivatives

$f(x)$	Domain	Range	df/dx
$\sin(x)$	$(-\infty, \infty)$	$[-1, 1]$	$\cos(x)$
$\cos(x)$	$(-\infty, \infty)$	$[-1, 1]$	$-\sin(x)$
$\tan(x)$	x not $n\pi/2$	$(-\infty, \infty)$	$\sec^2(x)$
$\cot(x)$	x not $n\pi$	$(-\infty, \infty)$	$-\csc^2(x)$
$\sec(x)$	x not $n\pi/2$	$(-\infty, -1] \cup [1, \infty)$	$\sec(x)\tan(x)$
$\csc(x)$	x not $n\pi$	$(-\infty, -1] \cup [1, \infty)$	$-\csc(x)\cot(x)$
$\sin^{-1}(x)$	$[-1, 1]$	$(-\pi/2, \pi/2)$	$1/\sqrt{1-x^2}$
$\cos^{-1}(x)$	$[-1, 1]$	$(0, \pi)$	$-1/\sqrt{1-x^2}$
$\tan^{-1}(x)$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$	$1/(1+x^2)$
$\cot^{-1}(x)$	$(-\infty, \infty)$	$(0, \pi)$	$-1/(1+x^2)$
$\sec^{-1}(x)$	$(-\infty, -1]$	$(\pi/2, \pi)$	$1/(x\sqrt{x^2-1})$
$\sec^{-1}(x)$	$[1, \infty)$	$[0, \pi/2)$	$-1/(x\sqrt{x^2-1})$
$\csc^{-1}(x)$	$(-\infty, -1]$	$[-\pi/2, 0)$	$-1/(x\sqrt{x^2-1})$
$\csc^{-1}(x)$	$[1, \infty)$	$(0, \pi/2]$	$1/(x\sqrt{x^2-1})$
$\ln(x)$	$(0, \infty)$	$(-\infty, \infty)$	$1/x$
$\log_a(x) = \log_a(e)\ln(x)$	$(0, \infty)$	$(-\infty, \infty)$	$\log_a(e)/x$
$\exp(x)$	$(-\infty, \infty)$	$(0, \infty)$	$\exp(x)$
$a^x = \exp(x\ln(a))$	$(-\infty, \infty)$	$(0, \infty)$	$a^x\ln(a)$
$sinh(x) = (e^x - e^{-x})/2$	$(-\infty, \infty)$	$(-\infty, \infty)$	$cosh(x)$
$cosh(x) = (e^x + e^{-x})/2$	$(-\infty, \infty)$	$[1, \infty)$	$sinh(x)$
$tanh(x)$	$(-\infty, \infty)$	$(-1, 1)$	$sech^2(x)$
$coth(x)$	x not 0	$(-\infty, -1) \cup (1, \infty)$	$-csch^2(x)$
$sech(x)$	$(-\infty, \infty)$	$(0, 1]$	$-sech(x)\tanh(x)$
$csch(x)$	x not 0	$(-\infty, 0) \cup (0, \infty)$	$-csch(x)\coth(x)$
$sinh^{-1}(x)$	$(-\infty, \infty)$	$(-\infty, \infty)$	$1/\sqrt{x^2+1}$
$cosh^{-1}(x)$	$[1, \infty)$	$[0, \infty)$	$1/\sqrt{x^2-1}$
$cosh^{-1}(x)$	$[1, \infty)$	$(-\infty, 0]$	$-1/\sqrt{x^2-1}$
$tanh^{-1}(x)$	$(-1, 1)$	$(-\infty, \infty)$	$1/(1-x^2)$
$coth^{-1}(x)$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$	$1/(1-x^2)$
$sech_1^{-1}(x)$	$(0, 1]$	$(-\infty, 0]$	$1/(x\sqrt{1-x^2})$
$sech_2^{-1}(x)$	$(0, 1]$	$[0, \infty)$	$-1/(x\sqrt{1-x^2})$
$csch^{-1}(x)$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$	$-1/(\|x\|\sqrt{1+x^2})$

72 The Integral

The integral of a function

$$I = \int_a^b f(x)dx$$

is the area under the curve of the function $f(x)$ for x varying from a to b. Suppose we have a partition of the interval $[a, b]$

$$a = x_1 < \chi_1 < x_2 < \chi_2 < ... < \chi_{n-1} < x_n = b.$$

We define the definite integral to be

$$\int_a^b f(x)dx = \lim_{|x_{i+1}-x_i|\to 0} \sum_{i=1}^{n} f(\chi_i)(x_{i+1} - x_i).$$

The limit is taken as the distance between the mesh points goes to zero.
The First Fundamental Theorem of Integral Calculus. Define

$$G(x) = \int_A^x f(x)dx.$$

Then

$$\frac{dG(x)}{dx} = f(x).$$

Proof. We have, recalling that the integral is the area under the curve, that

$$G(x+h) - G(x) = \int_A^{x+h} f(x)dx - \int_A^x f(x)dx = \int_x^{x+h} f(x)dx = f(c)h,$$

for some c between x and $x + h$. Dividing by h and taking the limit we get the result. This says that $G(x)$ is an antiderivative of $f(x)$, which means that the derivative of $G(x)$ is $f(x)$.
The Second Fundamental Theorem of Integral Calculus. If $F(x)$ is any antiderivative of $f(x)$, then

$$\int_A^B f(x)dx = F(B) - F(A).$$

Proof. If $F(x)$ is any antiderivative of $f(x)$, and $G(x)$ is the antiderivative

$$G(x) = \int_A^x f(x)dx,$$

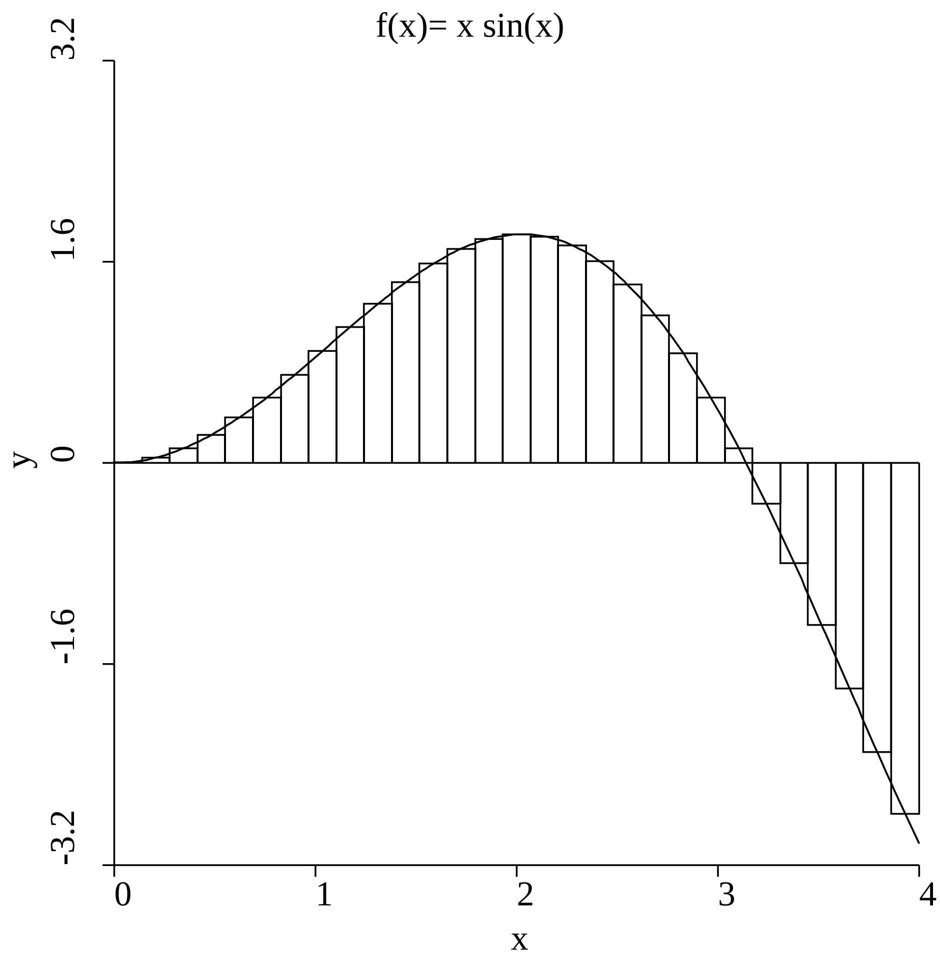

Figure 9: **The definite integral is the area under the curve.** . This figure shows a plot of the function $y = f(x) = x\sin(x)$. The integral of this function $\int_0^4 f(x)dx$ is approximated by the sum of the areas of the rectangles, where the areas of the rectangles under the $y = 0$ axis are negative. The integral is defined as the limit of such sums as the width of the rectangles goes to zero.

then $F(x)$ and $G(x)$ differ by some constant C. So

$$F(x) = G(x) + C.$$

$$G(B) = \int_A^B f(x)dx = F(B) - C.$$

$$F(A) = G(A) + C = \int_A^A f(x)dx + C = 0 + C = C.$$

So

$$\int_A^B f(x)dx = F(B) - F(A).$$

Example Let us calculate the volume of a sphere. A sphere has equation

$$x^2 + y^2 + z^2 = R^2,$$

where R is the radius of the sphere. Let us evaluate one half of the volume. We take a cylindrical slice of the sphere at some x value. The area of this slice is $\pi r^2(x)$, where $r(x)$ is the radius of the plane slice at x. If the thickness of the slice is dx, then the volume of a thin section is $\pi r^2(x)dx$. To compute the volume we add up an infinite number of infinitesimally small such sections. Then the volume of the half sphere is

$$\int_0^R \pi r^2(x)dx$$

We have

$$r^2(x) = R^2 - x^2.$$

So half the volume of the sphere is

$$V/2 = \int_0^R \pi(R^2 - x^2)dx = \int_0^R f(x)dx.$$

We need an antiderivative. Consider

$$F(x) = \pi(xR^2 - x^3/3)$$

Differentiating this we get

$$\pi(R^2 - \frac{3x^2}{3}) = \pi(R^2 - \frac{3x^2}{3}) = f(x)$$

So $F(x)$ is an antiderivative, so we have

$$V/2 = F(R) - F(0) = \pi((R^3 - R^3/3) - (0 - 0)) = \frac{2}{3}\pi R^3.$$

So the volume of the sphere is

$$V = \frac{4}{3}\pi R^3.$$

73 Snell's Law, the Early Bird Gets the Worm

Suppose that we can travel in one medium at velocity v_1 and in a second medium at a slower velocity v_2. Suppose we are to travel from point P in the first medium to point R in the second medium. What path results in the minimum travel time?

Solution

Referring to the **Snell's Law** figure, let v_1 be the velocity in the upper plane and v_2 the velocity in the lower plane, with $v_2 < v_1$. We are to find the position of the point $Q = (x, 0)$ to minimize the travel time from point P to point R. The length of the path in the upper plane is

$$\ell_1 = \sqrt{d^2 + x^2},$$

and the length of the path in the lower plane is

$$\ell_2 = \sqrt{e^2 + (c - x)^2}$$

The travel time as a function of x is

$$t = \frac{\ell_2}{v_1} + \frac{\ell_2}{v_2}.$$

The derivative of the time is

$$\begin{aligned} \frac{dt}{dx} &= \frac{(x/\sqrt{d^2 + x^2})}{v_1} - \frac{((c - x)/\sqrt{e^2 + (x - c)^2})}{v_2} \\ &= \frac{\sin(\theta_1)}{v_1} - \frac{\sin(\theta_2)}{v_2}. \end{aligned}$$

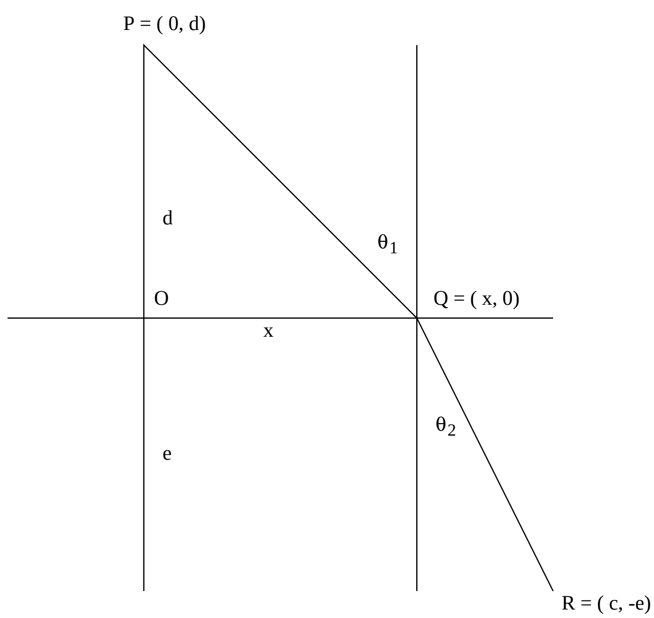

Figure 10: **Snell's Law.** Let two media be separated by the horizontal line. A particle in the upper media travels at velocity v_1, in the lower at velocity v_2, with $v_2 < v_1$. The travel time from P to R is minimized when x the coordinate of Q is selected to satisfy Snell's law.

Setting this to zero, we find that the condition for a minimum is

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2}.$$

In the case of optics we have the indices of refraction

$$n_1 = \frac{c}{v_1}, n_2 = \frac{c}{v_2},$$

where c is the velocity of light in a vacuum. So we obtain Snell's law of optical refraction.

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2).$$

74 Calculus On Vectors

Vectors in two or three space are objects that have x, y, and in the case of 3 dimensional vectors, z components. We write vectors in boldface. So suppose $\mathbf{R}$ is some vector

$$\mathbf{R} = (t^2 + 1, \sin(t), \ln(t))$$

Then the derivative of the vector is obtained by calculating the derivatives of the x, y, z components. So

$$\frac{d\mathbf{R}}{dt} = (2t, \cos(t), 1/t)$$

Vectors are commonly written using the unit coordinate basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

$$\mathbf{R} = (t^2 + 1, \sin(t), \ln(t)) = (t^2 + 1)\mathbf{i} + \sin(t)\mathbf{j} + \ln(t)\mathbf{k}$$

75 A Mass Point in Circular Motion

Let a mass point be specified in polar coordinates (θ, r). Let the point be constrained to lie on a circle of radius r. Let the polar coordinate unit vectors be

$$\mathbf{u}_r = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j},$$

$$\mathbf{u}_\theta = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}.$$

The first vector is perpendicular to the circle and the second is tangent to it. Let the position vector of the point be

$$\mathbf{p} = r\mathbf{u}_r.$$

The velocity is

$$\mathbf{v} = \frac{d\mathbf{p}}{dt} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\mathbf{u}_r}{dt} = r\frac{d\mathbf{u}_r}{dt},$$

because here r is constant. We have

$$\frac{d\mathbf{u}_r}{dt} = \frac{d\mathbf{u}_r}{d\theta}\frac{d\theta}{dt}$$

$$= \mathbf{u}_\theta \frac{d\theta}{dt}.$$

So

$$\mathbf{v} = r\frac{d\theta}{dt}\mathbf{u}_\theta = r\omega\mathbf{u}_\theta,$$

where ω is the angular velocity. The acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = r\frac{d\omega}{dt}\mathbf{u}_\theta + r\omega\frac{d\mathbf{u}_\theta}{dt}$$

$$= r\frac{d\omega}{dt}\mathbf{u}_\theta - r\omega^2\mathbf{u}_r$$

$$= r\frac{d\omega}{dt}\mathbf{u}_\theta - \frac{v^2}{r}\mathbf{u}_r,$$

where $d\omega/dt$ is the angular acceleration, and $v = r\omega$ is the tangential velocity. If the angular acceleration is zero then v^2/r is the magnitude of the centripetal acceleration directed toward the center of the circle.

76 The Period of an Earth Satellite

From the preceding section if a satellite is rotating around the earth, the magnitude of its centripetal acceleration is

$$r\omega^2,$$

where ω is the angular velocity in radians.

If its mass is m then the force keeping the mass in orbit is

$$F = r\omega^2 m.$$

This force is supplied by the earth's gravity and this force is given by

$$f = G\frac{mM}{r^2} = mG\frac{M}{r^2}$$

So the acceleration is

$$a = G\frac{M}{r^2}$$

Then

$$r\omega^2 = G\frac{M}{r^2},$$

or

$$\omega^2 = GM\frac{1}{r^3}.$$

The period of the orbit is the time it takes to rotate 2π radians

$$T = \frac{2\pi}{\omega} = \frac{2\pi r^{3/2}}{\sqrt{GM}}.$$

The period of the moon is about

$$T_m = (28)24) = 672$$

hours. The radius of the earth is about

$$r_e = 6371km$$

the distance to the moon is about

$$r_m = 384403km$$

So if a satellite could orbit right at the surface of the earth we would have

$$\frac{T_e}{T_m} = \left(\frac{r_e}{r_m}\right)^{3/2}.$$

So the satellite would orbit in about 1.43 hours. At a distance d from the earth the satellite period would be

$$T = T_m((r_e + d)/r_m)^{3/2}$$

A geosynchronous orbit where the period is about 24 hours is about $re + d =$ 42164 km.

77 Moment, Center of Gravity, Moment of Inertia

A moment is the twisting force due to a force acting on a lever arm. Given a plane area there is an area moment acting about the origin of the coordinate system. The x moment is given by an integral

$$M_x = \int x dm,$$

where dm is an element of mass. In the case of a plane object we have an area density σ, so that the element of mass $dm = \sigma dA$, where dA is an element of area.

Let us calculate the center of gravity of a triangle of base b and height a. We suppose the triangle base is on the x axis, with vertices $(0,0), (b,0)$ and (b,a). So let

$$y = (a/b)x$$

Consider the area under this curve on the interval $[0,b]$. We let the density $\sigma = 1$, so that $dm = dA$. We have $dA = ydx$, So the x moment is

$$M_x = \int_0^b xydx$$

$$= \int_0^b \frac{a}{b} x^2 dx$$
$$= \frac{a}{b}[x^3/3]_0^b$$
$$= ab^2/3.$$

The x coordinate of the center of gravity is obtained by dividing by the area $A = ab/2$ of the triangle,

$$\bar{x} = \frac{m_X}{A} = \frac{ab^2/3}{ab/2} = (2/3)b.$$

The y moment is

$$M_y = \int_0^a y(b-x)dy$$

$$= \int_0^a y(b - \frac{b}{a}y)dy$$

$$= \int_0^a (yb - \frac{b}{a}y^2)dy$$

$$= [by^2/2 - \frac{b}{a}y^3/3]_0^a$$

$$= ba^2/2 - ba^2/3$$

$$= ba^2(1/6)$$

So dividing by the area of the triangle $ab/2$, the y coordinate of the center of gravity is

$$\bar{y} = a/3.$$

So the center of gravity is at $(2/3b, 1/3a)$.

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be the vertices of any triangle. The midpoint of the side opposite A is

$$\frac{\mathbf{B} + \mathbf{C}}{2}$$

The line from $\mathbf{A}$ to the midpoint $\frac{\mathbf{B}+\mathbf{C}}{2}$ is called a median of the triangle. A point on the median 2/3 of the way from $\mathbf{A}$ to $\frac{\mathbf{B}+\mathbf{C}}{2}$ is

$$\mathbf{A} + \frac{2}{3}\left(\frac{\mathbf{B} + \mathbf{C}}{2} - \mathbf{A}\right)$$

$$= \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3}.$$

We get the same result starting from the other two vertices. Hence this point is the intersection of the three medians of a triangle. A calculation like the center of gravity argument above for the triangle shows that the intersection of the medians is the center of gravity of any triangle. So cut out a triangle draw the intersecting medians and try it.

The rotational moment of inertia of a body rotating about say the z axis is

$$I_z = \int (x^2 + y^2)dm,$$

where dm is an element of mass of the body. One can show that the rotational kinetic energy is

$$E_{rot} = \frac{1}{2}I_z\omega^2,$$

where ω is the angular velocity. Here ω plays the roll of velocity, and I_z plays the roll of the mass in the usual kinetic energy formula.

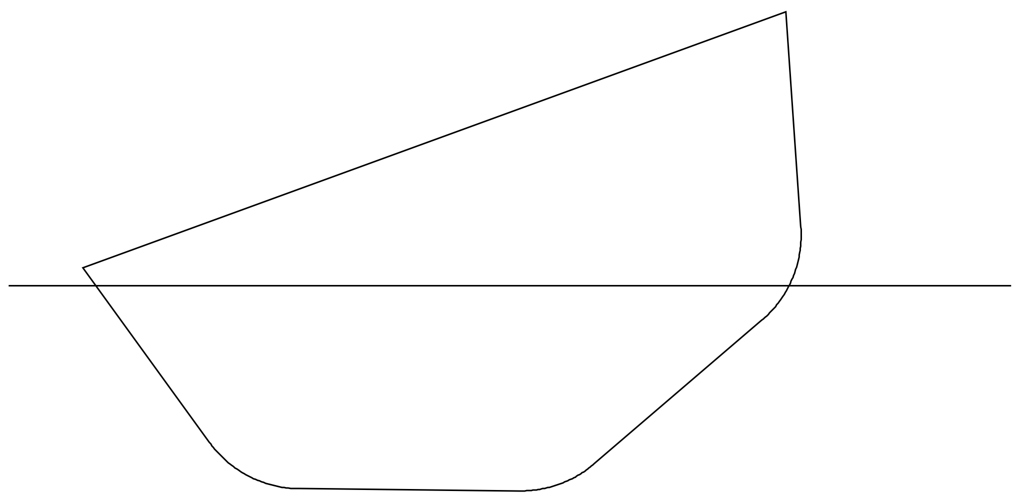

Figure 11: **Boat Stability**. As the boat is tipped to the left, the center of buoyancy lies to the left of the vertical center line of the boat. Assuming that the center of gravity lies on the center line, if the center of gravity is low enough, there will be a torque to level the boat. If the center of gravity is too high, it will lie to the left of the center of buoyancy in this tipped position, and the boat will capsize. This is because the force at the center of buoyancy acts upward and the force at the center of gravity acts downward.

78 Boat Stability

When a boat hull displaces water it produces antigravity. This antigravity has a center of antigravity called the center of buoyancy just like the mass and cargo of the boat has a center of mass. The antigravity force pushes upward, the gravity force pushes downward. If the boat is inclined at an angle, usually the antigravity force with the gravity force create a couple to right the boat. The relation of the center of gravity and the center of buoyancy determines how much side force, say on a sail can be tolerated without the boat capsizing.

79 Infinite Compound Interest

Let P be the principal, r the yearly interest, and A the current value. Then after one year, for simple interest

$$A = (1 + r)P$$

Suppose the interest is compounded twice a year. Then the half year interest rate is $r/2$, so at the end of the year we have

$$A = (1 + r/2)^2 P$$

Likewise if the amount is compounded n times per year we have

$$A = (1 + r/n)^n P$$

at the end of the year. What happens if we compound infinitely often? This is given by

$$A = \lim_{n\to\infty} (1 + r/n)^n P.$$

So let us compute

$$\lim_{n\to\infty} (1 + r/n)^n.$$

It is convenient to set $x = 1/n$ then the limit becomes

$$\lim_{x\to 0} (1 + rx)^{1/x}.$$

Let us consider the logarithm of our expression

$$f(x) = (1 + rx)^{1/x}$$

. Let

$$g(x) = \ln(f(x) = \ln((1 + rx)^{1/x}) = \frac{\ln(1 + rx)}{x}$$

As $x \to 0$ both numerator and denominator go to 0, and we get the indeterminant form 0/0. By L' Hospital's rule, in this case we may compute the limit as the limit of the derivative of the numerator divided by the derivative of the denominator. Thus

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{r}{1 + rx} = r$$

So the limit of $g(x)$ is

$$\ln(f(x)) = r.$$

Thus the limit of $f(x)$ as $x \to 0$ is

$$e^r,$$

where $e = exp(1) = 2.7182818284590\,45$. Most accounts are compounded daily, so $n = 365$. This is a large number, so the yearly amount is quite close to

$$A = e^r P.$$

The Annual Percentage Yield (APY) is then

$$e^r - 1$$

which is the effective yearly interest rate. So for example if the interest rate is $r = 5$ percent, then the APY is

$$e^{.05} - 1 = 0.05127109637602$$

or approximately 5.127 percent.

80 The Numerical Calculation of Integrals

One of the simplest ways of approximating an integral is to approximate the integral with many trapezoids. This is called the trapezoid method. It is not very accurate. But when combined with Richardson extrapolation, it becomes the Romberg method and is very accurate.

Trapezoid Method Program.

```
// trapezoid.c
#include <stdio.h>
#include <math.h>
double f(double);
double trapez(double (*fp)(double),double,double,int);
void main(){
 double a,b,v;
 int n,i;
 a=0.;
 b=4.;
 for(n=3;n < 1000000;n=2*n+1 ){
  v=trapez(f,a,b,n);
  printf(" v = %21.15g n= %d \n",v,n);
 }
 v= sin(b)-b*cos(b);
 printf(" sin(b)-b*cos(b) = %21.15g \n",v);
}
//c+ trapez trapezoid integration
double trapez(double (*fp)(double) ,double a,double b,int n){
//parameters
//    f-external function to be integrated
//  a,b-integration interval
//    n-interval divided into n-1 pieces
//    v-value returned for integral
 double x,y,v,h;
 int i;
 for(i=1,v=0.;i<=n;i++){
  x=(i-1)*(b-a)/(n-1)+a;
  y=(*fp)(x);
  if((i == 1)||(i == n)){
   y=y/2;
  }
  v=v+y;
 }
 h=(b-a)/(n-1);
 v*=h;
 return v;
 }
 //c+ f  function
 double f(double x){
  double v;
  v=x*sin(x);
  return v;
 }
```

Output of the Program.

```
 v =     0.609979726071014 n= 3
 v =      1.73154715222909 n= 7
 v =      1.83479212300388 n= 15
 v =      1.85277521009651 n= 31
 v =      1.85660247170371 n= 63
 v =      1.85748883881977 n= 127
 v =      1.85770231246565 n= 255
```

```
v =      1.85775470565523 n= 511
v =      1.85776768442241 n= 1023
v =      1.85777091431825 n= 2047
v =       1.8577717199517 n= 4095
v =      1.85777192113055 n= 8191
v =      1.85777197139663 n= 16383
v =      1.85777198395957 n= 32767
v =      1.85777198709985 n= 65535
v =      1.85777198788485 n= 131071
v =      1.85777198808109 n= 262143
v =      1.85777198813021 n= 524287
sin(b)-b*cos(b) =      1.85777198814652
```

Notice that to get good accuracy we had to take over 500000 steps with a very small step size. A very small step size often leads to roundoff error.

81 Explosions and Icebergs

There are frequent explosions where the air contains very small particles that can burn, such as at grain elevators. Why is that? When a body is heated or cooled, the amount of heat transferred is proportional to the surface area of the body. The temperature change of a body for a given supply of heat is inversely proportional to the mass of the body, that is, to the volume of the body. So the rate of temperature change of a heated or cooled body is proportional to the ratio of the surface area to the volume. This ratio for a spherical body is

$$\frac{4\pi r^2}{4\pi r^3/3}.$$

Therefore the heating rate is proportional to $1/r$. If a particle is very small, r is small and $1/r$ is large. So the body is very quickly heated to its combustion temperature. An explosion occurs. On the other hand, for a very large object, such as an iceberg, the heating rate is very slow. So an iceberg or a glacier can exist for a very long time.

82 The Rate of Chemical Reactions: We Are Chemical Machines.

Chemical reactions between two molecules occur when they collide and have sufficient energy to react. So in the simple case the reaction rate is proportional to the product of the concentrations of the two reactants multiplied by a rate constant. The rate constant is usually an exponential function of the temperature. Suppose a molecule with concentration A reacts with a molecule having concentration B to produce a new molecule with concentration C. Then the differential equation for the concentration of C might be

$$\frac{dC}{dt} = kAB,$$

where k is a rate constant, that typically depends on the temperature, faster reactions for higher temperature. At higher temperatures there are more molecules that have enough energy to react. So for a chemical system we will have a system of differential equations similar to this one above for each of the molecular concentrations.

A very complex such system is the metabolism in our bodies. Others concern the concentrations of enzymes and drugs in our blood system.

83 The Motion of an Electron in a Cathode Ray Tube

The Lorenz force on a moving charged particle such as an electron is a sum of an electric force and a magnetic force. The electric force is in the direction of the electric field. The magnetic force is perpendicular to the particle velocity and to the magnetic field.

The Lorentz force is the sum of the electric and magnetic forces

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

So to solve for the path of the electron we must solve a system of differential equations, which are obtained by setting the particle acceleration equal to the Lorenz force.

In a television CRT the electrons are accelerated toward the screen by the electric field **E** due to a very high voltage. The deflection of the electrons, vertical and horizontal, are caused by inductive windings on the neck of the CRT, creating a magnetic field **B**. In the case of a uniform magnetic field, the electrons are forced to move in circles. So as the electron passes through the region of the coils they move in short arcs, thereby changing their direction.

84 Effective AC Voltage

In an alternating current circuit, the power given to a resistive load is given by i^2R where i is the instantaneous current

$$i = I\sin(\omega t).$$

The average power is

$$P_{ave} = \frac{1}{T}I^2R\int_0^T \sin^2(\omega t)dt,$$

where the period is $T = \frac{2\pi}{\omega}$. Evaluating

$$\frac{1}{T}\int_0^T \sin^2(\omega t)dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin^2(u)du$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\cos(2u)}{2}du$$

$$= \frac{1}{2},$$

after the change of variable $u = \omega t$. So the average power is

$$P_{ave} = \frac{I^2R}{2}$$

Therefore the effective constant current is

$$I_{eff} = \frac{I}{\sqrt{2}},$$

where I is the peak current. The effective constant voltage is similarly

$$V_{eff} = \frac{V}{\sqrt{2}}.$$

Thus the common AC line voltage of 110 volts corresponds to a peak voltage of

$$110\sqrt{2} = 155.56\text{v}olts.$$

85 Power Factor in Electric Power Transmission

The power taken by a load in an AC circuit is proportional to the average power taken over a cycle, which is the integral of the product of the voltage by the current divided by the period T. If a load is purely inductive, then the voltage and current are 90 degrees out of phase, so the integral will be zero. The amount of power taken by the load is proportional to $\cos(\phi)$, where ϕ is the phase angle difference between the voltage and current. This $\cos(\phi)$ proportionality is easily shown by looking at the average of the instantaneous power. This $\cos(\phi)$ is called the power factor. The instantaneous power is the product of the current and voltage. Large electric motors tend to have a large value of ϕ because the load is very inductive and are said to have a lagging power factor, because the current lags the voltage by ϕ. Capacitors can be added to the load to decrease ϕ, raise the power factor, and increase electrical efficiency. It does this by reducing the power line losses.

86 Spherical Mass

A spherical mass, for gravitational purposes can be treated as a point mass located at the center of the sphere. This can be established using a calculus argument. This is one of the calculations Newton had to do for his theory of gravity applied to the motion of the planets. Newton had to invent calculus in order to complete his theory.

87 Motion of Planets, Kepler's Laws

Kepler's three laws of planetary motion are:

1. **Planets move in elliptical orbits around the Sun with the Sun at a focus.**

2. **A line joining the Sun to the planet sweeps out equal areas in equal times.**

3. **The period of the orbit is proportional to the 3/2 power of the major diameter of the ellipse.**

Kepler found these laws by observation. Newton proved the laws using calculus and his theory of gravitational force.

In the case of a body of mass m orbiting a body of mass M such as the sun, Newton's law is the following differential equation

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GmM\mathbf{r}}{r^3.}$$

Orbits are usually calculated by solving differential equations of this sort numerically with a computer.

88 The Deterministic Motion of All Particles and Things

All the particles of the universe are acted on by forces. These forces in turn are due to particles. Using these known forces there are differential equations for their motion obtained by setting the acceleration to the net force on each particle. Knowing at any one time the position and velocity of every particle in the universe, these differential equations determine all future motions and positions of all the particles in the universe. Thus the future of the universe, and of humans has been determined. Thus calculus determines everything. He who knows calculus is king.

According to Heisenberg's uncertainty principle, there is some uncertainty in measuring both position and velocity of a particle. So philosophers debate whether free will has been restored.

89 Bibliography

There are a huge number of books on Calculus. Modern university textbooks in general are scandalous, extremmely overpriced, produced in China for a couple of bucks, but sometimes costing the students hundreds of dollars, greedily updated by the publisher every two years to destroy the used book market, poorly bound with glued pages, heavy, so making the purchase of barbells unnecessary. The latter fact perhaps an ironic savings.

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