Quick Calculus

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1 Introduction

The title of this work is Quick Calculus. It is meant to give much of Calculus theory in a quick and abbreviated form. The ideas are presented in a way to give only the essence of the ideas, techniques, and proofs. This can be further developed and completed by the reader. Careful statements of the conditions under which theorems are true is mostly not present, so must be added for full rigor. Normally A big weighty Calculus book is full of examples, comments, and guidance. It is not the purpose of this document to attempt an immitation of such a book. Rather it is presented mostly for review and recall. We sometimes make an informal requirement that a function is "nice." By a nice function we usually mean that it is a function that has at least a continuous derivative.

2 The Limit of a Function

Let f be a real valued function of a real variable. The limit of $f(x)$ as x goes to x_0 is c, if and only if, for every positive number $\epsilon > 0$ there exists a number $\delta > 0$ such that, if $|x - x_0| < \delta$, then $|f(x) - c| < \epsilon$. This is written as

$$\lim_{x \to x_0} f(x) = c.$$

Example 1. It is intuitive that if $f(x) = x^2$, then

$$\lim_{x \to 3} f(x) = 9.$$

We must prove this fact using the definition. We must show that

$$|x^2 - 9|$$

can be made small, when

$$|x - 3|$$

is sufficiently small. To do this we shall find a relationship between these two expressions. Suppose δ is some positive number, and suppose $|x - 3| < \delta$. Then

$$|x| = |x - 3 + 3| \leq |x - 3| + |3| < \delta + 3.$$

Then

$$|x + 3| \leq |x| + 3 < \delta + 3 + 3 = \delta + 6.$$

Now we can find an inequality for the difference of the squares.

$$|x^2 - 9| = |x - 3||x + 3| \leq \delta(\delta + 6).$$

Now given an arbitrary $\epsilon > 0$, we can find a proper δ. Indeed, choose δ to be less than 1 and less than $\epsilon/7$, then

$$|x^2 - 9| = |x - 3||x + 3| \leq \delta(\delta + 6) < \frac{\epsilon}{7}(1 + 6) = \epsilon.$$

3 Limit Theorems

$$\lim_{x \to a} f(x) + \lim_{x \to a} g(x) = \lim_{x \to a}(f(x) + g(x)).$$

$$\lim_{x \to a} f(x) \lim_{x \to a} g(x) = \lim_{x \to a}(f(x)g(x)).$$

$$\lim_{x \to a}(\alpha f(x)) = \alpha \lim_{x \to a} f(x).$$

If $\lim_{x \to a} f(x)$ is not zero, then

$$\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{\lim_{x \to a} f(x)}.$$

Example. We shall prove the product formula. Let

$$\lim_{x \to a} f(x) = b,$$

and

$$\lim_{x \to a} g(x) = c.$$

Then

$$\begin{aligned} & |f(x)g(x) - bc| \\ = & |f(x)g(x) - f(x)c + cf(x) - bc| \\ \leq & |f(x)||g(x) - c| + |c||f(x) - b|. \end{aligned}$$

Given $\epsilon_1 > 0 \ \exists \ \delta_1$, such that if $|x - a| < \delta_1$, then $|f(x) - b| < \epsilon_1$, and given $\epsilon_2 > 0 \ \exists \ \delta_2$, such that if $|x - a| < \delta_2$, then $|g(x) - c| < \epsilon_2$. We have

$$|f(x)| = |f(x) - b + b| \leq \epsilon_1 + |b|,$$

so

$$|f(x)g(x) - bc| \leq (\epsilon_1 + |b|)\epsilon_2 + |c|\epsilon_1.$$

Given ϵ, we may choose ϵ_1 and ϵ_2 so that

$$(\epsilon_1 + |b|)\epsilon_2 + |c|\epsilon_1 < \epsilon.$$

Let δ be the smaller of δ_1 and δ_2. Then if $|x - a| < \delta$, then

$$|f(x)g(x) - bc| < \epsilon.$$

4 Continuity

A function is continuous at a point a iff (if and only if)

$$\lim_{x \to a} f(x) = f(a).$$

Example. Define function f by, $f(x) = 0$, if $x < 0$, and $f(x) = 1$ if $x \geq 0$. Then f is not continuous at 0. To prove this, we only need to find one $\epsilon > 0$ for which there is no $\delta > 0$, so that there is some x whose distance to 0 is less than delta, but for which $|f(x)| > \epsilon$. Let us choose $\epsilon = 1/2$. Let δ be any positive number. Let $x = -\delta/2$. Then $|x - 0| < \delta$, but

$$|f(x) - f(0)| = |0 - 1| = 1 > \epsilon.$$

We can find no δ that works for this $\epsilon = 1/2$. Hence f is not continuous at 0.

5 The Derivative

Given a function $y = f(x)$, the slope of the secant line, which passes through the curve at $(x, f(x))$ and at $(x+h, f(x+h)$, is

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}.$$

As $h \to 0$, this secant line approaches the tangent line at $(x, f(x))$. The slope of this limiting tangent line is called the derivative of f at x. We write the derivative as

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Example Given two differentiable functions f and g, the derivative of the product is

$$\frac{d(fg)}{dx} = \frac{df}{dx}g + f\frac{dg}{dx}.$$

Proof.

$$\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}g(x)$$

$$= f\frac{dg}{dx} + \frac{df}{dx}g.$$

Example. The derivative of a constant is zero. Let $f(x) = c$. Then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0.$$

Example. The derivative of the identity is one. Let $f(x) = x$. Then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x + h - x}{h} = 1.$$

Example. Let $f(x) = x^n$. Then

$$\frac{df}{dx} = nx^{n-1}.$$

Proof. This is true for the identity where $n = 1$. Assume it is true for n. Let $f(x) = x^{n+1}$ and $g(x) = x^n$. Then

$$g'(x) = nx^{n-1}.$$

$$f'(x) = (xg(x))' = 1g(x) + xg'(x) = x^n + xnx^{n-1} = (n+1)x^n.$$

So it is true for $n + 1$. By induction it is true for all positive integers.

6 Maxima and Minima

If a nice function f has a relative maxima, or a relative minima at a point a, then $f'(a) = 0$.
Proof. Suppose there is a relative maxima at a, then for small $h > 0$,

$$f(a+h) - f(a) \leq 0$$

and

$$\frac{f(a+h) - f(a)}{h} \leq 0.$$

Similarly for $h < 0$,

$$\frac{f(a+h) - f(a)}{h} \geq 0.$$

Therefore both

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \leq 0,$$

and

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \geq 0.$$

Therefore $f'(a) = 0$.

7 Rolle's Theorem

Let f be a nice function. Suppose $f(a) = f(b)$. Then there is a number c, $a < c < b$ so that $f'(c) = 0$.
Proof. There must be a relative maximum or a relative minimum of f at some point c between a and b.

8 The Mean Value Theorem

Let f be a nice function. There exists a point c, $a < c < b$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Let function g be formed by subtracting the straight line that passes through the points $(a, f(a))$ and $(b, f(b)$. Then

$$g(x) = f(x) - [\frac{b - x}{b - a} f(a) + \frac{a - x}{a - b} f(b)].$$

Then $g(a) = g(b) = 0$, so by Rolle's Theorem there is a c, $a < c < b$, so that $g'(c) = 0$. Then

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Hence

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

9 Taylor's Formula

If f is a nice function, then there is a number c, $a < c < b$, so that

$$f(b) = f(a) + f^{(1)}(x - a) + f^{(2)}\frac{(x - a)^2}{2!} + f^{(3)}\frac{(x - a)^3}{3!} + ... +$$

$$f^{(n-1)}(a)\frac{(x - a)^{n-1}}{(n - 1)!} + f^{(n)}(c)\frac{(x - a)^n}{n!}$$

Proof. Let p be the polynomial

$$p(x) = f(a) + f^{(1)}(x - a) + f^{(2)}\frac{(x - a)^2}{2!} + ... + f^{(n-1)}(a)\frac{(x - a)^{n-1}}{(n - 1)!}.$$

Let M be defined by

$$f(b) = p(b) + \frac{M(b - a)^n}{n!}.$$

Define

$$g(x) = f(x) - (p(x) + \frac{M(x - a)^n}{n!}).$$

Then $g(a) = f(a) - f(a) + 0 = 0$, and $g(b) = 0$ by the definition of M. By Rolle's' Theorem there is a number b_1, $a < b_1 < b$, so that $g'(b_1) = 0$. Clearly $g'(a) = 0$, so we may apply Rolle's Theorem to g' and obtain a number $b_2, a < b_2 < b_1$, so that $g^{(2)}(b_2) = 0$. Continuing in this way, after n steps, we find that there is a c so that $a < c < b$ and

$$f^{(n)}(c) - M = g^{(n)}(c) = 0.$$

Therefore $M = f^{(n)}(c)$ and

$$f(b) = p(b) + f^{(n)}(c)\frac{(b-a)^n}{n!},$$

which is Taylor's Formula.

10 L'Hospital's Rule

Suppose

$$\lim_{x \to a} f(x) = 0$$

and

$$\lim_{x \to a} g(x) = 0.$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Proof. Certain conditions must be specified, for example suppose the limit is from the right, the derivatives exist in the interval (a, b), and $g'(x)$ is not zero in this interval. It is not required that f and g be defined at a. But extend functions f and g to F and G defined on $[a, b)$, by defining $F(a) = 0$ and $G(a) = 0$. Then F and G are continuous in $[a, b)$. Let

$$h(x) = F(x)(G(b) - G(a)) - G(x)(F(b) - F(a)).$$

Then $h(a) = h(b)$. So by Rolle's Theorem there exists a c, $a < c < b$ so that

$$0 = h'(c) = F'(c)(G(b) - G(a)) - G'(c)(F(b) - F(a)) = f'(c)g(b) - g'(c)f(b).$$

So

$$\frac{f'(c)}{g'(c)} = \frac{f(b)}{f(b)}.$$

Taking limits as b and hence c go to a, we get the result.

L'Hospital's Rule also holds when $x \to \infty$ and when the limits in the numerator and denominator are infinity.

L'Hospital, Guillaume de (1661-1704) was a French mathematician who, at age 15, solved a difficult problem about cycloids posed by Pascal. He published the first book ever on differential calculus, L'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes (1696). In this book, l'Hospital included l'Hospital's rule. l'Hospital's name is commonly seen spelled both "l'Hospital" and "l'Hopital" (e.g., Maurer 1981, p. 426), the two being equivalent in French spelling.

11 The Chain Rule

Suppose $k(x) = g(f(x))$. Then

$$\frac{dk}{dx} = g'(f(x))f'(x).$$

$$\frac{dk}{dx} = \lim_{h\to 0} \frac{k(x+h) - k(x)}{h}$$

$$= \lim_{h\to 0} \frac{g(f(x+h)) - g(f(x))}{h}$$

Let

$$f(x) = y, f(x+h) = y + j(h),$$

where

$$\lim_{h\to 0} j(h) = 0.$$

We have

$$j(h) = f(x+h) - f(x).$$

$$\frac{dk}{dx} = \lim_{h\to 0} \left[\frac{g(f(x+h)) - g(f(x))}{j(h)} \frac{j(h)}{h} \right]$$

$$= \lim_{j\to 0} \frac{g(y+j) - g(y)}{j} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= g'(y)f'(x)$$

$$= g'(f(x))f'(x).$$

This proof is valid provided j is bounded away from zero, in a neighborhood of zero.

We shall present a second proof. By the mean value theorem

$$g(f(x+h)) - g(f(x)) = g'(f(\chi_1))(f(x+h) - f(x)) = g'(f(\chi_1))f'(\chi_2)h.$$

We divide by h, and then let h go to zero. If g' and f' are continuous, then

$$\begin{aligned} k'(x) &= \lim_{h\to 0} \frac{g(f(x+h)) - g(f(x))}{h} \\ &= \lim_{h\to 0} g'(f(\chi_1))f'(\chi_2) \\ &= g'(f(x))f'(x). \end{aligned}$$

Example

Let $f(x) = 1/x$, $1 = xf(x)$ We differentiate this last equation

$$0 = f(x) + xf'(x),$$

so

$$f'(x) = \frac{-1}{x^2}.$$

Example Let $f(x) = x^{1/n}$, let $g(x) = x^n$, then

$$g(f(x)) = x.$$

Thus

$$\begin{aligned} g'(f(x))f'(x) &= 1. \\ n(f(x))^{n-1}f'(x) &= 1, \\ f'(x) &= \frac{1}{n}x^{-(1-1/n)} \\ &= \frac{1}{n}x^{1/n-1}. \end{aligned}$$

Example Quotient rule.

$$\begin{aligned} (f/g)' &= \left[f\frac{1}{g}\right] \\ = \left[f'\frac{1}{g}\right] &+ \left[f\frac{-1}{g^2}g'\right] \\ &= \frac{f'g - fg'}{g^2}. \end{aligned}$$

12 The Derivative Of An Inverse Function

Let g be the inverse of f. That is

$$g(f(x) = f^{-1}(f(x)) = x.$$

Let $y = f(x)$, so that $x = g(y)$. Then

$$(g(f(x)))' = 1$$

$$(g'(f(x)))f'(x) = 1$$

$$g'(f(x)) = \frac{1}{f'(x)}$$

$$g'(y) = \frac{1}{f'(g(y))}.$$

13 The Binomial Theorem

We have results such as

$$(a+b)^2 = a^2 + 2ab + b^2,$$

and

$$(a+b)^3 = a^3 + 3a^2b + 3a^1b^2 + b^3.$$

The general result is called the binomial theorem.
Binomial Theorem For each positive integer n we have

$$(a+b)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{n-k} b^k,$$

where

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)...(n-k+1)}{k!},$$

is the number of ways of choosing k things from n things. We read

$$\left(\begin{array}{c} n \\ k \end{array} \right),$$

as n choose k. So

$$\binom{n}{0} = 1$$

and

$$\binom{n}{n} = 1.$$

Proof. Suppose the theorem holds for n, then we have

$$(a+b)^{n+1} = a(a+b)^n + (a+b)^n b$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{(n+1)-k} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{(n+1)-k} b^k + \sum_{k=1}^{n+1} \binom{n}{k-1} a^{(n+1)-k} b^k$$

$$= \binom{n}{0} a^{n+1} b^0 + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1}\right] a^{(n+1)-k} b^k + \binom{n}{n} a^0 b^{n+1}$$

$$= a^{n+1} b^0 + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1}\right] a^{(n+1)-k} b^k + a^0 b^{n+1}.$$

The expression

$$\left[\binom{n}{k} + \binom{n}{k-1}\right]$$

is

$$\frac{n!}{k!(n-k)!)} + \frac{n!}{(k-1)!(n-(k-1))!)}$$

$$= \frac{n!}{k!(n-k)!)} + \frac{n!}{(k-1)!((n+1)-k))!)}$$

$$= \frac{n!((n+1)-k))}{k!((n+1)-k))!)} + \frac{kn!}{k!((n+1)-k))!)}$$

$$= \frac{(n+1)!}{k!((n+1)-k))!)}$$

$$= \binom{n+1}{k}.$$

So the sum above becomes

$$(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{(n+1)-k} b^k.$$

Therefore by induction, the binomial theorem holds for all integers n.

14 The Binomial Series

A binomial series is an infinite series of the form

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k,$$

where a is any real number, and

$$\binom{a}{k} = \frac{a(a-1)(a-2)...(a-k+1)}{k!}.$$

So for example

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1024}x^6 + ...$$

It is clear that if $-1 < x < 1$ that this series converges because it is an alternating series, and the terms are decreasing in magnitude.

Consider

$$(1+x)^{-5/3} = 1 - \frac{5}{3}x + \frac{20}{9}x^2 - \frac{220}{81}x^3 + \frac{770}{243}x^4 - \frac{2618}{729}x^5 + \frac{26180}{6561}x^6 - ...$$

It is not quite so obvious that this series also converges for $-1 < x < 1$.

The convergence of the binomial series for $-1 < x < 1$ and a any real number can be proven as a consequence of Bernstein's convergence theorem. See for example p244 of **Mathematical Analysis**, 2nd edition , 1975, by Tom M. Apostol.

15 The Multiplication of Power Series

Let

$$f(x) = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...) = \sum_{i=0}^{\infty} a_i x^i.$$

$$g(x) = (b_0 + b_1 x + b_2 x^2 + b_3 x^3 + ...) = \sum_{i=0}^{\infty} b_i x^i.$$

Then collecting together terms of like degree in the product we have

$$f(x)g(x) = (a_0 b_0) + (a_0 b_1 + a_1 b_0)x + ... + (a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + .. + a_k b_0)x^k + ...$$

$$= \sum_{k=0}^{\infty} (\sum_{j=0}^{k} a_j b_{k-j}) x^k.$$

16 The Exponential Function and the Logarithm

Define the exponential function as the power series

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

Differentiating the series term by term

$$\frac{d(\exp(x)}{dx} = 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \exp(x).$$

The exponential function has the property

$$\exp(a + b) = \exp(a)\exp(b).$$

This follows by finding the product of power series.

$$\exp(a)\exp(b) = (1 + a + \frac{a^2}{2!} + ..)(1 + b + \frac{b^2}{2!} + ..) =$$

$$= \sum_{k=0}^{\infty} c_k,$$

where

$$c_k = \frac{a^0 b^k}{k!} + \frac{a^1 b^{k-1}}{1!(k-1)!} + \frac{a^2 b^{k-2}}{2!(k-2)!} + .. + \frac{a^k b^0}{k!}$$

$$= \frac{1}{k!}(c(k,0)a^0 b^k + c(k,1)a^1 b^{k-1} + .. + c(k,k)a^k b^0),$$

$$= \frac{(a+b)^k}{k!}.$$

We have used the binomial theorem, and binomial coefficients

$$c(k,j) = \frac{k!}{j!(k-j)!}.$$

We have shown that

$$\exp(a+b) = \exp(a)\exp(b).$$

Define a number e by

$$e = \exp(1).$$

One can prove that e is an irrational and transcendental number. We have

$$e^m = \exp(1)^m = \prod_{i=1}^{m} \exp(1) = \exp(m).$$

Also

$$\exp(1/n)^n = \exp(1) = e.$$

Thus

$$e^{1/n} = \exp(1/n).$$

Thus for any rational number r

$$e^r = \exp(r).$$

If x is irrational, then e^x is not yet defined. However, if e^x is to be a continuous function we must define, for all real x

$$e^x = \exp(x).$$

From the power series definition of $\exp(x)$, we see that it is a monotone increasing function, so it has an inverse. Define the natural logarithm as the inverse of the exponential function

$$\ln(x) = \exp^{-1}(x).$$

Since $\exp(0) = 1$, we have $\ln(1) = 0$, and since $\exp(1) = e$, we have $\ln(e) = 1$. If $\ln(x_1) = y_1$ and $\ln(x_2) = y_2$, then $x_1 = e^{y_1}$ and $x_2 = e^{y_2}$ Then

$$x_1 x_2 = e^{y_1 + y_2},$$

hence

$$\ln(x_1 x_2) = y_1 + y_2 = \ln(x_1) + \ln(x_2).$$

Similarly, if r is a rational number, then

$$\ln(x^r) = r\ln(x).$$

Then

$$x^r = \exp(r\ln(x)).$$

If a is an irrational number, we define

$$x^a = \exp(a\ln(x)).$$

Hence for any real number a,

$$\ln(x^a) = a\ln(x).$$

Letting $y = \ln(x)$, We have

$$\frac{d(\ln(x))}{dx} = 1/\frac{d(\exp(y))}{dy} = 1/\exp(\ln(x)) = 1/x.$$

Now given a real number a, we define the power function

$$a^x = \exp(\ln(a^x)) = \exp(x\ln(a)).$$

The logarithm to base a is the inverse of the power function. We write

$$\log_a(x) = y,$$

when

$$y = a^x.$$

Example We shall show that

$$\lim_{n\to\infty}(1 + 1/n)^n = e.$$

By taking the logarithm we have

$$\lim_{n\to\infty}\ln((1 + 1/n)^n) = \lim_{n\to\infty}\frac{\ln(1 + 1/n)}{1/n}$$

Let $x = 1/n$. We shall use the mean value theorem. The numerator and the denominator each go to zero, so we may replace the numerator and the denominator by their derivatives (L'Hospital's Rule).

$$\lim_{x \to 0} \frac{\ln(1+x)}{x}$$

$$= \lim_{x \to 0} \frac{1/(1+x))}{1} = 1.$$

So

$$\lim_{n \to \infty} \ln((1+1/n)^n) = 1.$$

Therefore

$$\lim_{n \to \infty} (1+1/n)^n = e.$$

17 Angle

Two rays emanating from point A define an angle. Let a circle of radius R and center A intersect the rays at points B and C, thereby defining an arc. The measure of the angle, which is written θ, is defined to be the ratio of the arc length of the circle, s, to the radius R:

$$\theta = \frac{s}{R}.$$

The definition is independent of the particular circle chosen. This can be seen by decomposing the angle into many very small angles, so that each very small arc length is nearly equal to the small side of a triangle formed by the small angle. The independence of the circle radius on the definition of angle measure follows from properties of similar triangles, and by taking limits. The number π is by definition the ratio of the arclength of a circle to the diameter $D = 2R$. The arclength of a circle of radius R is $2\pi R$. The complete circle angle formed by a ray with itself has measure

$$\theta = \frac{2\pi R}{R} = 2\pi.$$

A straight angle, which is one half of a full circle angle, has measure $\theta = \pi$, and a right angle, which is one fourth of a full angle, has measure $\theta = \pi/2$.

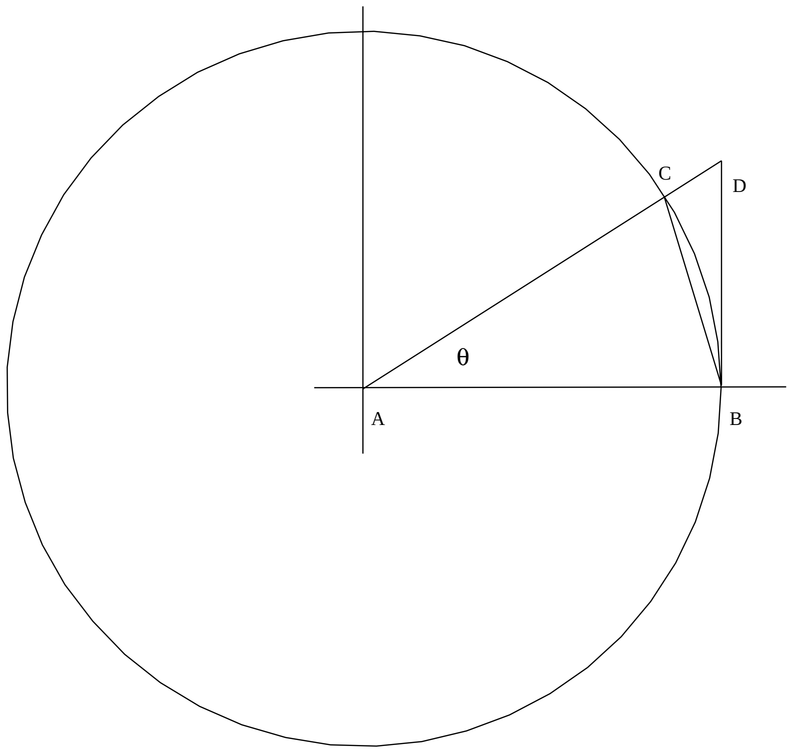

Figure 1: Proof that the Limit of $\sin(\theta)/\theta$ as $\theta \to 0$ is 1. Let the radius of the circle be 1. So the area of triangle ABC is $\sin(\theta)/2$, the area of circular sector ABC is $\theta/2$, and the area of triangle ABD is $\tan(\theta)/2$. As θ goes to zero, $\sin(\theta)/\theta$ goes to one.

18 Trigonometric Functions

Let (x, y) be a point on a circle centered at the origin of radius 1. A ray from the origin passing through (x, y) defines an angle with the x-axis. Let the measure of this angle be θ. Then define

$$\cos(\theta) = x,$$

$$\sin(\theta) = y.$$

If $(x, y) = (1, 0)$ then $\theta = 0$. Therefore

$$\cos(0) = 1,$$

$$\sin(0) = 0.$$

If $(x, y) = (0, 1)$ then $\theta = \pi/2$. Therefore

$$\cos(\pi/2) = 0,$$

$$\sin(\pi/2) = 1.$$

If (x, y) lies on the unit circle then

$$x^2 + y^2 = 1.$$

Hence

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

Example 2. We shall show that if

$$f(\theta) = \frac{\sin(\theta)}{\theta},$$

then

$$\lim_{\theta \to 0} f(\theta) = 1.$$

Proof. Construct a circle of radius $r = 1$ as shown in the $\sin(\theta)/\theta$ figure. The area of the inner triangle, ABC is $\sin(\theta)/2$, because the height of the triangle is $\sin(\theta)$, and the base has length 1. Clearly the circular sector ABC has area area $\theta/2$, and the outer triangle ABD has area $\tan(\theta)/2$. Intuitively we can see that as θ goes to zero, the ratio of these areas approaches 1. We can prove this. The area of the inner triangle is less than the area of the circular sector, which in turn is less than the area of the outer triangle. So we have

$$\sin(\theta)/2 < \theta/2 < \tan(\theta)/2,$$

$$1 < \theta/\sin(\theta) < 1/\cos(\theta),$$

$$1 > \sin(\theta)/\theta > \cos(\theta),$$

$$-1 < -\sin(\theta)/\theta < -\cos(\theta).$$

It follows that

$$-(1 - \cos(\theta)) < 0 < 1 - \sin(\theta)/\theta < 1 - \cos(\theta).$$

Therefore

$$|1 - \sin(\theta)/\theta| < 1 - \cos(\theta).$$

Choose $\epsilon > 0$. We have

$$\lim_{\theta \to 0} \cos(\theta) = 1.$$

So we may choose a number δ so that if $|\theta| < \delta$ then $1 - \cos(\theta) < \epsilon$. Then if $|\theta| < \delta$ then

$$|1 - \sin(\theta)/\theta| < \epsilon.$$

We have proved that

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$$

19 Angle Sum Formula

We shall prove an angle sum formula for the case

$$\theta_2 > 0,$$

$$\theta_1 > 0,$$

$$\theta_1 + \theta_2 < \pi/2.$$

We shall show that

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2).$$

Consider points lying on the first quadrant of the unit circle $E = (x, y)$, and $F = (x_1, y_1)$. Refer to the angle sum formula figure. Let A be the origin. Let $x < x_1$. Drop a perpendicular from E meeting the x-axis at B. Then the angle BAE is the sum of angle BAF and angle FAE. Let θ be the measure of BAE, θ_1 the measure of BAF and θ_2 the measure of FAE. Then

$$\theta = \theta_1 + \theta_2.$$

Construct a line perpendicular to AF through E. Let this line meet AF at D. DAE is a right triangle with unit hypotenuse. Define

$$a = AD,$$

and

$$b = DE.$$

Then

$$a = \cos(\theta_2),$$

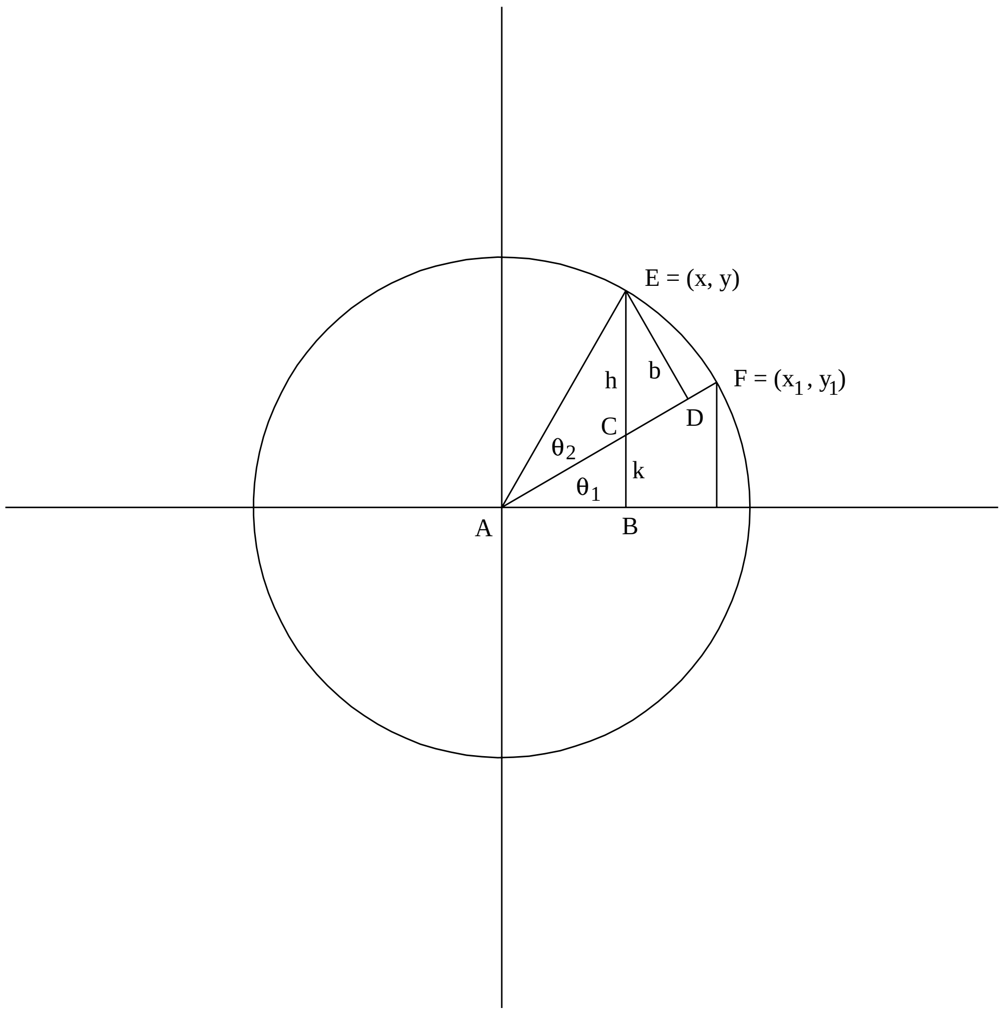

Figure 2: **Angle Sum Formula.** The length of line segment AD is a, the length of ED is b, the length of EC is h and the length of CB is k. The circle has unit radius. EDA is a right angle. The figure shows how similar triangles can be used to prove the formula $\sin(\theta_1+\theta_2) = \sin(\theta_1)\cos(\theta_2)+\cos(\theta_1)\sin(\theta_2)$.

and

$$b = \sin(\theta_2).$$

Let AD meet EB at C. Let $h = EC$ and $k = BC$. Then

$$y = h + k.$$

Also we have

$$x = \cos(\theta),$$

$$y = \sin(\theta),$$

$$x_1 = \cos(\theta_1),$$

and

$$y_1 = \sin(\theta_1).$$

By properties of similar triangles we have

$$\frac{k}{x} = \frac{y_1}{x_1},$$

and

$$\frac{h}{b} = \frac{1}{x_1}.$$

Then

$$k = \frac{xy_1}{x_1},$$

$$h = \frac{b}{x_1}.$$

$$y = h + k = \frac{b}{x_1} + \frac{\sqrt{1-y^2}y_1}{x_1}.$$

Then

$$y_1\sqrt{1-y^2} = x_1 y - b,$$

$$y_1^2(1-y^2) = x_1^2 y^2 - 2x_1 y b + b^2$$

$$y_1^2 = (x_1^2 + y_1^2)y^2 - 2x_1 y b + b^2$$

$$y_1^2 = y^2 - 2y(x_1 b) + (x_1 b)^2 + b^2(1 - x_1^2)$$

$$y_1^2 = (y - x_1 b)^2 + b^2 y_1^2$$

$$(1-b^2)y_1^2 = (y - x_1 b)^2$$

$$a^2 y_1^2 = (y - x_1 b)^2$$
$$y = \pm y_1 a + x_1 b.$$

Then

$$\sin(\theta_1 + \theta_2) = \pm \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2).$$

Because

$$\sin(\theta_1 + \theta_2) > \sin(\theta_2),$$

the plus sign is correct, so

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2).$$

A similar proof shows that if

$$0 < \theta_1 < \pi/2,$$

and

$$\theta_2 < \theta_1,$$

then

$$\sin(\theta_1 - \theta_2) = \sin(\theta_1)\cos(\theta_2) - \cos(\theta_1)\sin(\theta_2).$$

20 The Derivative of the Sin Function

These results allow us to compute the derivative of $\sin(\theta)$.

We shall show that the derivative is

$$\frac{d\sin(\theta)}{d\theta} = \cos(\theta).$$

We have

$$\frac{d\sin(x)}{dx} = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$
$$= \lim_{h\to 0} \frac{\sin((x+h/2)+h/2) - \sin((x+h/2)-h/2)}{h}$$
$$= \lim_{h\to 0} \frac{2\cos(x+h/2)\sin(h/2)}{h}$$
$$= \lim_{h/2\to 0} \cos(x+h/2) \lim_{h/2\to 0} \frac{\sin(h/2)}{h/2}$$

$$= \cos(x) \cdot 1 = \cos(x).$$

To find the derivative of $\cos(\theta)$, we differentiate both sides of

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

We find that

$$2\cos(\theta)\frac{d\cos(\theta)}{d\theta} + 2\sin(\theta)\cos(\theta) = 0.$$

If $2\cos(\theta) \neq 0$, then dividing by $2\cos(\theta)$, we find

$$\frac{d\cos(\theta)}{d\theta} = -\sin(\theta).$$

This is a general result, but our proof is only valid provided $\cos(\theta)$ is not zero. We realize though by invoking continuity at such exceptional points that the result is valid everywhere.

Notice that all the higher derivatives of sin and cos at zero take values $0, 1, -1$. We can define sin and cos as everywhere convergent power series about zero. Expanding each of $\sin(\theta)$ and $\cos(\theta)$ about zero in a Taylor series, we find

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

and

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

Hence

$$\exp(i\theta) = \cos(\theta) + i\sin(\theta),$$

where

$$i = \sqrt{-1}.$$

Then

$$\sin(\theta) = \frac{\exp(i\theta) - \exp(-i\theta)}{2i},$$

and

$$\cos(\theta) = \frac{\exp(i\theta) + \exp(-i\theta)}{2}.$$

Now we may prove our sum formula for all values of the arguments.

Proposition For all θ_1 and θ_2,

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2).$$

Proof We may use the exponential definitions of $\sin(x)$ and $\cos(x)$, and show that the two sides of the equation are equal.

The exponential definitions of the trigonometric functions may be used to establish general trigonometric identities.

21 The Indefinite Integral

$F(x)$ is called the indefinite integral, or the antiderivative of $f(x)$, when

$$\frac{dF(x)}{dx} = f(x).$$

$F(x)$ is written as

$$F(x) = \int f(x)dx.$$

22 The Riemann Integral

Suppose we have a partition of the interval $[a, b]$

$$a = x_1 < \chi_1 < x_2 < \chi_2 < ... < \chi_{n-1} < x_n = b.$$

We define the definite integral to be

$$\int_a^b f(x)dx = \lim_{|x_{i+1}-x_i|\to 0} \sum_{i=1}^{n} f(\chi_i)(x_{i+1} - x_i).$$

The limit is taken as the distance between the mesh points goes to zero.
Proposition If

$$G(x) = \int_a^x f(x)dx,$$

then

$$G'(x) = f(x).$$

Proof

$$G'(x) = \lim_{h \to 0} \frac{G(x+h) - G(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\int_a^{x+h} f(x)dx - \int_a^x f(x)dx \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\int_x^{x+h} f(x)dx \right]$$

Let m be the minimum of f on the interval $[x, x+h]$, and M the maximum of $F(x)$ on $[x, x+h]$. Then

$$m \leq \frac{1}{h} \int_x^{x+h} f(x)dx \leq M.$$

The limit of both m and M is $f(x)$. Therefore

$$G'(x) = f(x).$$

Proposition If $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Let

$$G(x) = \int_a^x f(t)dt.$$

Then $G'(x) = f(x)$ so

$$G(x) = F(x) + c,$$

for some constant c. Then

$$F(a) + c = G(a) = \int_a^a f(t)dt = 0.$$

Thus

$$c = -F(a).$$

Therefore

$$\int_a^b f(t)dt = G(b) = F(b) + c = F(b) - F(a).$$

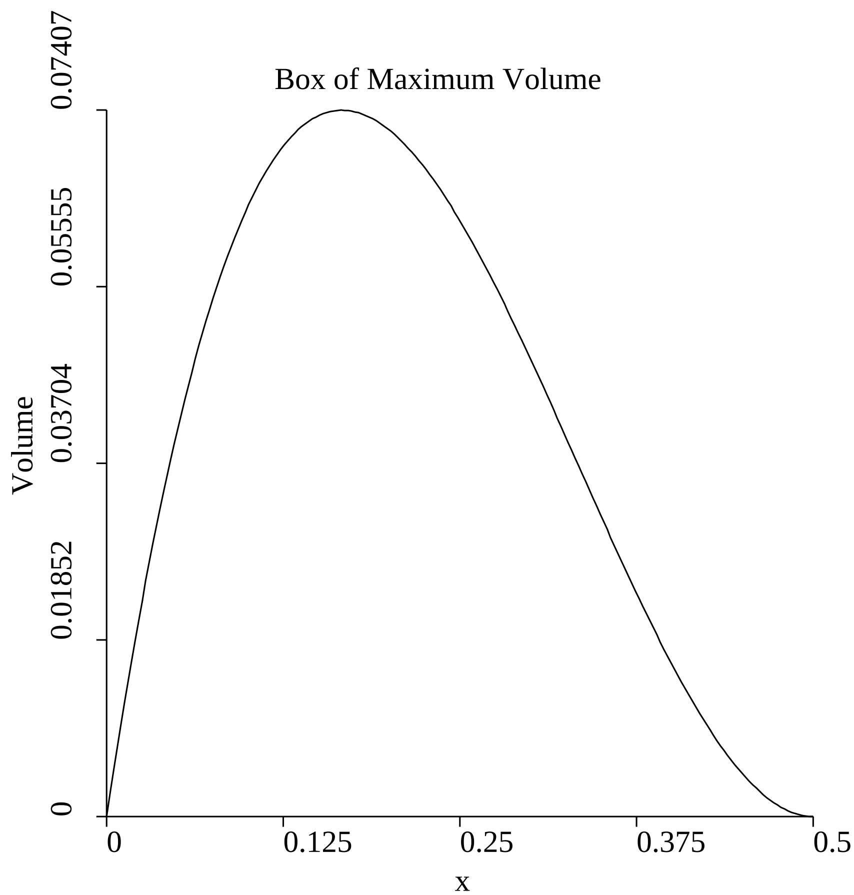

Figure 3: **Maximum Box Volume.** This shows how the volume of the box varies as x varies from 0 to 1/2. x is the side length of the squares that are clipped from the four corners of a unit square.

23 Some Maxima and Minima Examples

Example 1 Given a 1 by 1 square, find the box of maximum volume obtained by clipping squares of side x from each corner and folding up the edges to make a box.
Solution. The volume of the folded box is

$$V = (1-2x)^2 x = 4x^3 - 4x^2 + x.$$

We set the volume derivative to 0,

$$\frac{dV}{dx} = 12x^2 - 8x + 1 = 0.$$

The roots of this equation are

$$x = \frac{1}{6}, \frac{1}{2}.$$

So the maximum occurs for $x = 1/6$. See the figure showing the variation of volume with x.
Example 2 *Snell's Law.* Suppose that we can travel in one medium at velocity v_1 and in a second medium at a slower velocity v_2. Suppose we are to travel from point P in the first medium to point R in the second medium. What path results in the minimum travel time?
Solution

Referring to the **Snell's Law** figure, let v_1 be the velocity in the upper plane and v_2 the velocity in the lower plane, with $v_2 < v_1$. We are to find the position of the point $Q = (x, 0)$ to minimize the travel time from point P to point R. The length of the path in the upper plane is

$$\ell_1 = \sqrt{d^2 + x^2},$$

and the length of the path in the lower plane is

$$\ell_2 = \sqrt{e^2 + (c-x)^2}$$

The travel time as a function of x is

$$t = \frac{\ell_2}{v_1} + \frac{\ell_2}{v_2}.$$

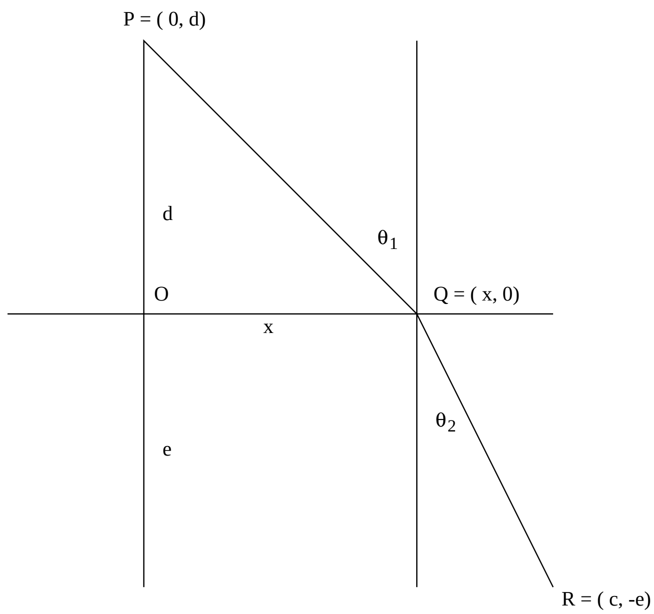

Figure 4: **Snell's Law.** Let two media be separated by the horizontal line. A particle in the upper media travels at velocity v_1, in the lower at velocity v_2, with $v_2 < v_1$. The travel time from P to R is minimized when x the coordinate of Q is selected to satisfy Snell's law.

The derivative of the time is

$$\frac{dt}{dx} = \frac{(x/\sqrt{d^2+x^2})}{v_1} - \frac{((c-x)/\sqrt{e^2+(x-c)^2})}{v_2}$$

$$= \frac{\sin(\theta_1)}{v_1} - \frac{\sin(\theta_2)}{v_2}.$$

Setting this to zero, we find that the condition for a minimum is

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2}.$$

In the case of optics we have the indices of refraction

$$n_1 = \frac{c}{v_1}, n_2 = \frac{c}{v_2},$$

where c is the velocity of light in a vacuum. So we obtain Snell's law of optical refraction.

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2).$$

Example 3 *The Range of a Projectile.* Suppose an object is projected upward at angle θ with velocity v, where air resistance is neglected. Let the acceleration of gravity be g. Then the motion in the vertical y direction is

$$\frac{d^2y}{dt^2} = -g,$$

$$\frac{dy}{dt} = -gt + v\sin(\theta),$$

$$y = -\frac{-gt^2}{2} + v\sin(\theta)t.$$

The motion in the horizontal x direction is

$$x = v\cos(\theta)t.$$

The projectile returns to the ground when

$$0 = t(v\sin(\theta) - \frac{gt}{2}).$$

So

$$t = \frac{2v\sin(\theta)}{g}$$

Then

$$x = \frac{2v^2\sin(\theta)\cos(\theta)}{g} = \frac{v^2\sin(2\theta)}{g}.$$

The range is maximum where $\sin(2\theta)$ is maximum, where $\theta = \pi/4$.

Given a desired range x, the projectile should be launched at angle

$$\theta = \frac{1}{2}\sin^{-1}(\frac{gx}{v^2}).$$

If there is air resistance, say a retarding force proportional to the square of the velocity, then we have to solve a more complicated differential equation, with an approximate numerical technique. This was the problem that led to the invention of electronic computers.

24 Methods of Integration

24.1 Integration by Substitution

By making a variable substitution it may be possible to find an integral for a function. So suppose we have a function $F(x)$ and we want to find an integral for the function, that is a function $f(x)$ such that

$$\frac{df(x)}{dx} = F(x)$$

$$f(x) = \int F(x)dx.$$

Suppose we make the substitution $x = k(u)$ for some function k. We assume that k has an inverse. We have $u = k^{-1}(x)$. Let us write $h = k^{-1}$. We have

$$\frac{du}{dx} = k'(x),$$

and

$$\frac{dx}{du} = h'(u)$$

Notice that
$$k'(x)h'(u) = k'(x)h'(u(x)) = 1.$$
We make the substitution in the original integral
$$dx = h'(u)du,$$
getting
$$\int F(h(u))h'(u)du.$$
Letting
$$G(u) = F(h(u))h'(u),$$
this is
$$\int G(u)du.$$
Suppose we find g so that
$$g(u) = \int G(u)du,$$
that is
$$\frac{dg}{du} = G(u).$$
Now we substitute back defining an f
$$f(x) = g(k(x)).$$
Then
$$\begin{aligned} \frac{df}{dx} &= \frac{dg}{du}\frac{dk}{dx} \\ &= G(u)\frac{dk}{dx} \\ &= G(k(x))k'(x) \\ &= F(h(k(x)))h'(k(x))k'(x) \\ &= F(x). \end{aligned}$$
So
$$f(x) = \int F(x)dx.$$
Example 1. Find
$$f(x) = \int \sqrt{1-x^2}dx.$$

Let

$$x = \sin(\theta)$$

$$dx = \cos(\theta)d\theta.$$

So substituting

$$\int \sqrt{1-\sin^2(\theta)}\cos(\theta)d\theta$$

$$= \int \cos^2(\theta)d\theta$$

$$= \int \frac{1+\cos(2\theta)}{2}d\theta$$

$$= \frac{\theta}{2} + \frac{\sin(2\theta)}{4}$$

$$= \frac{\theta}{2} + \frac{\sin(\theta)\cos(\theta)}{2}$$

Substituting back

$$f(x) = \frac{\sin^{-1}(x)}{2} + \frac{x\sqrt{1-x^2}}{2}.$$

Example 2. Find

$$f(x) = \int \sec(x)dx.$$

Multiplying by

$$u = \sec(x) + \tan(x)$$

we have

$$\int \sec(x)dx = \int \frac{\sec(x)(\sec(x)+\tan(x))}{\sec(x)+\tan(x)}dx.$$

$$= \int \frac{\sec^2(x)+\sec(x)\tan(x))}{\sec(x)+\tan(x)}dx.$$

$$= \int \frac{1}{u}du$$

$$= \ln(|u|)$$

$$= \ln(|\sec(x)+\tan(x)|).$$

Example 3. Find

$$f(x) = \int \sec(x) dx,$$

using the integral

$$\int \frac{1}{1-x^2} dx.$$

We have

$$\frac{1}{1-x^2} = (1/2)(\frac{1}{1-x} + \frac{1}{1+x}),$$

so

$$\int \frac{1}{1-x^2} dx = (1/2)(-\ln(|1-x)|) + \ln(|1+x)|))$$

$$= \ln \sqrt{\frac{1+x}{1-x}}.$$

Now

$$\int \sec(x) dx = \int \frac{\cos(x)}{\cos^2(x)} dx$$

$$= \int \frac{\cos(x)}{1-\sin^2(x)} dx.$$

Let

$$u = \sin(x),$$

then

$$du = \cos(x) dx.$$

So

$$\int \sec(x) dx = \int \frac{1}{1-u^2} du$$

$$= \ln \sqrt{\frac{1+u}{1-u}}$$

$$= \ln \sqrt{\frac{1+\sin(x)}{1-\sin(x}}$$

$$= \ln \sqrt{\frac{(1+\sin(x))^2}{1-\sin^2(x)}}$$

$$= \ln \left|\frac{1+\sin(x)}{\cos(x)}\right|$$

$$= \ln |\sec(x) + \tan(x)|.$$

24.2 Integration by Parts

Integration by parts comes from the formula for the derivative of a product.

$$d(uv) = udv + vdu$$

Integrating

$$uv = \int udv + \int vdu$$

or

$$\int udv = uv - \int vdu.$$

Example 1. Suppose we wish to calculate

$$\int x\sin(x)dx.$$

Let

$$u = x, dv = \sin(x)dx$$

Then

$$du = dx, v = -\cos(x)$$

Then

$$\begin{aligned}\int x\sin(x)dx &= -x\cos(x) - \int -\cos(x)dx \\ &= -x\cos(x) + \sin(x).\end{aligned}$$

Example 2. Suppose we wish to calculate

$$\int x^2\cos(x)dx.$$

We can write a little table and avoid introducing u and v explicitly

x^2	$\cos(x)dx$
$2xdx$	$\sin(x)$

Here we have written what we want u to be in the top left box, and what we want dv to be in the top right box. We differentiate the top left box to get the bottom left box. We integrate the top right box to get the lower right box. Then the integral of the top product, namely

$$\int x^2\cos(x)dx,$$

is equal to the product of the diagonal elements

$$x^2 \sin(x),$$

minus the integral of the product of the lower elements. Thus we have

$$\int x^2 \cos(x) dx = x^2 \sin(x) - \int 2x \sin(x) dx$$

$$= x^2 \sin(x) - 2(-x\cos(x) + \sin(x))$$
$$= x^2 \sin(x) + 2x\cos(x) - 2\sin(x)).$$

Example 3. Suppose we wish to calculate

$$\int e^x \sin(x) dx.$$

Thus requires a bit of a trick.

If we integrate by parts we end up with a last term

$$\int e^x \cos(x) dx,$$

so we don't seem to be making any progress. However if we now integrate the term involving

$$\int e^x \cos(x) dx,$$

by parts, we get a term involving our original integral

$$\int e^x \sin(x) dx.$$

So we may rearrange this to get

$$\int e^x \sin(x) dx = \frac{1}{2}(e^x \sin(x) - e^x \cos(x)).$$

Example 4. Let us calculate

$$\int \cos^n(x) dx.$$

We use

$\cos^{n-1}(x)$	$\cos(x)dx$
$(n-1)\cos^{n-1}(-\sin(x))(x)dx$	$\sin(x)$

to get.

$$\int \cos^n(x)dx = \sin(x)\cos^{n-1}(x) - \int (n-1)(-\sin^2(x))\cos^{n-2}(x)dx$$

$$= \sin(x)\cos^{n-1}(x) + (n-1)\int (1-\cos^2(x))\cos^{n-2}(x)dx$$

$$= \sin(x)\cos^{n-1}(x) + (n-1)[\int \cos^{n-2}(x)dx - \int \cos^n(x)dx].$$

$$n\int \cos^n(x)dx = \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)dx.$$

Then

$$\int \cos^n(x)dx = \frac{\sin(x)\cos^{n-1}(x)}{n} + \frac{(n-1)}{n}\int \cos^{n-2}(x)dx.$$

For example to compute

$$\int \cos^2(x)dx,$$

we can either use the preceding formula, or

$$\cos^2(x) = \frac{\cos(2x)+1}{2}$$

to compute

$$\int \cos^2(x)dx = \frac{1}{2}(\sin(x)\cos(x) + x).$$

Example 5. Let us calculate

$$\int \frac{1}{(u^2+\alpha^2)^n}du.$$

We can use trigonometric substitution and the result of the preceding example.

We let

$$u = \alpha\tan(\theta).$$

then

$$du = \alpha\sec^2(\theta)d\theta,$$

$$u^2 + \alpha^2 = \alpha^2 \sec^2(\theta).$$

So

$$\int \frac{1}{(u^2+\alpha^2)^n} du$$

$$= \int \frac{\alpha \sec^2(\theta)}{(\alpha^2 \sec^2(\theta))^n} d\theta$$

$$= \frac{1}{\alpha^{2n-1}} \int \frac{1}{\sec^{2n-2}(\theta)} d\theta$$

$$= \frac{1}{\alpha^{2n-1}} \int \cos^{2n-2}(\theta) d\theta$$

Using the recurrence relation

$$\int \cos^n(x)dx = \frac{\sin(x)\cos^{n-1}(x)}{n} + \frac{(n-1)}{n} \int \cos^{n-2}(x)dx,$$

we can derive a recurrence relation for

$$\int \frac{1}{(u^2+\alpha^2)^n} du.$$

24.3 The Fundamental Theorem of Algebra

This theorem says that every polynomial $p(z)$ of degree n with real or complex coefficients has a root. If z_1 is a root, then $z - z_1$ divides $p(z)$, so

$$p(z) = (z - z_1)q(z),$$

where $q(z)$ is a polynomial of degree $n-1$. By the fundamental theorem $q(z)$ has a root. Continuing in this way every complex polynomial of degree n can be factored into products of the form

$$c_0 + c_1 z + c_2 z^2 + c_3 z^3 + ... + c_n z^n =$$

$$c_n(z-z_1)(z-z_2)(z-z_3)(z-z_4)...(z-z_n).$$

Thus every complex polynomial of degree n has exactly n roots. If

$$z = a + bi,$$

its conjugate is

$$\bar{z} = a - bi.$$

By conjugating the whole polynomial, we see that if z is a root, then $\bar{z}$ is also a root. Roots occur in conjugate pairs. If the polynomial has real coefficients and a complex root $z_k = a + bi$, then it also has a root $\bar{z_k}$, so a real quadratic factor

$$(x - z_k)(x - \bar{z}_k) = x^2 - (z_k + \bar{z}_k)x + |z_k|^2.$$

So any real polynomial can be factored into a product of linear and quadratic factors. This will be used in the next section. The fundamental theorem of algebra was first proved by Gauss.

Johann Carl Friedrich Gauss (30 April 1777 23 February 1855) was a German mathematician and scientist who contributed significantly to many fields, including number theory, statistics, analysis, differential geometry, geodesy, geophysics, electrostatics, astronomy and optics. Sometimes known as the Princeps mathematicorum (Latin, "the Prince of Mathematicians" or "the foremost of mathematicians") and "greatest mathematician since antiquity", Gauss had a remarkable influence in many fields of mathematics and science and is ranked as one of history's most influential mathematicians. He referred to mathematics as "the queen of sciences."

There is a simple proof of the Fundamental Theorem of Algebra using elementary facts from complex analysis. We give that proof in a later section. Elementary proofs that do not use complex analysis are long and involved.

24.4 Partial Fractions: Integrating Rational Functions

A rational function is of the form

$$r(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomials. If the degree of $p(x)$ is not less than the degree of $q(x)$ we may divide and get a polynomial plus a new rational function, where the degree of the numerator is less than the degree of the denominator. So we will consider only rational functions where the latter condition holds. If the denominator can be factored into first degree non-repeating factors then the rational function can be expanded in a partial fraction of the form

$$r(x) = \sum_{i=1}^{n} \frac{A_i}{x - x_i)},$$

where $\{x_1, x_2, ..., x_n\}$ are the roots of $q(x)$, and where the $a_1, A_2, ..., A_n$ are constants. Then the integral

$$\int r(x),$$

is equal to a sum of logarithms. If there are repeating roots, say $q(x$ has k roots x_1, then we must include terms of the form

$$\frac{A_1}{(x - x_1)}, \frac{A_2}{(x - x_1)^2}, ..., \frac{A_k}{(x - x_1)^k}.$$

If there are complex roots of $q(x)$ then they occur in pairs, and so we need fractions of the form

$$\frac{A_j + xB_j}{x^2 + b_j x + c_j}.$$

By completing the square and doing a substitution the denominator can be put in the form

$$u^2 + \alpha^2.$$

If we have repeated complex roots we must include powers in the denominator as for non-complex roots. Thus we get integrals of the form

$$\int \frac{A_j + uB_j}{(u^2 + \alpha^2)^m} du = \int \frac{A_j}{(u^2 + \alpha^2)^m} du + \int \frac{uB_j}{(u^2 + \alpha^2)^m} du.$$

The first can be evaluated using a recurance formula evaluating integrals of the form

$$\int \frac{1}{(u^2 + \alpha^2)^m} du.$$

The recurance formula comes from doing a substitution on the result of Example 5 in the section on substituion.

The recurance formula is

$$\int \frac{du}{(u^2 + \alpha^2)^m} = \frac{1}{2\alpha^2(m-1)} \frac{u}{(u^2 + \alpha^2)^{m-1}} + \frac{2m-3}{2\alpha^2(m-1)} \int \frac{du}{(u^2 + \alpha^2)^{m-1}}.$$

24.5 Rational Functions of Sines and Cosines

A rational function involving Sines and Cosines can be integrated by using the special substitution

$$z = \tan(x/2).$$

We find that

$$\cos(x) = \frac{1-z^2}{1+z^2}$$
$$\sin(x) = \frac{2z}{1+z^2}$$
$$dx = \frac{2dz}{1+z^2}.$$

So a rational function of $\sin(x)$ and $\cos(x)$ can be converted to a rational function in z. This can then be integrated by partial fractions.

24.6 Products of Sines and Cosines

The integral of the products of sines and cosines can be handled by using the identities

$$\sin(mx)\sin(nx) = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$
$$\sin(mx)\cos(nx) = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$$
$$\cos(mx)\cos(nx) = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

25 Hyperbolic Functions

Hyperbolic Functions

The hyperbolic sine is defined as

$$\sinh(z) = \frac{\exp(z) - \exp(-z)}{2}.$$

The hyperbolic cosine is

$$\cosh(z) = \frac{\exp(z) + \exp(-z)}{2}.$$

We have

$$\sinh^2(z) - \cosh^2(z) = 1.$$

The functions $\tanh(z), \coth(z), sech(z), csch(z)$ are defined in the obvious way.

By the usual Tayor series:

$$\sin(z) = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - ...$$

$$\cos(z) = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - ...$$

$$\sinh(z) = z + \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + ...$$

$$\cosh(z) = 1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + ...$$

$$\exp(z) = 1 + \frac{1}{1!}z^1 + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + ...$$

Then

$$\exp(iz) = 1 + i\frac{1}{1!}z - \frac{1}{2!}z^2 - i\frac{1}{3!}z^3 + ...$$
$$= \cos(z) + i\sin(z).$$

We also have

$$\sin(iz) = i\sinh(z)$$

$$\sinh(iz) = i\sin(z)$$

$$\cos(iz) = \cosh(z)$$

$$\cosh(iz) = \cos(z).$$

If $z = x + iy$ then

$$\sin(z) = \sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy)$$

$$= \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

$$\cos(z) = (\cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy)$$

$$= \cos(x)\cosh(y) - i\sin(x)\sin(y).$$

26 A Table of Elementary Derivatives

$f(x)$	Domain	Range	df/dx
$\sin(x)$	$(-\infty,\infty)$	$[-1,1]$	$\cos(x)$
$\cos(x)$	$(-\infty,\infty)$	$[-1,1]$	$-\sin(x)$
$\tan(x)$	x not $n\pi/2$	$(-\infty,\infty)$	$\sec^2(x)$
$\cot(x)$	x not $n\pi$	$(-\infty,\infty)$	$-\csc^2(x)$
$\sec(x)$	x not $n\pi/2$	$(-\infty,-1]\cup[1,\infty)$	$\sec(x)\tan(x)$
$\csc(x)$	x not $n\pi$	$(-\infty,-1]\cup[1,\infty)$	$-\csc(x)\cot(x)$
$\sin^{-1}(x)$	$[-1,1]$	$(-\pi/2,\pi/2)$	$1/\sqrt{1-x^2}$
$\cos^{-1}(x)$	$[-1,1]$	$(0,\pi)$	$-1/\sqrt{1-x^2}$
$\tan^{-1}(x)$	$(-\infty,\infty)$	$(-\pi/2,\pi/2)$	$1/(1+x^2)$
$\cot^{-1}(x)$	$(-\infty,\infty)$	$(0,\pi)$	$-1/(1+x^2)$
$\sec^{-1}(x)$	$(-\infty,-1]$	$(\pi/2,\pi)$	$1/(x\sqrt{x^2-1})$
$\sec^{-1}(x)$	$[1,\infty)$	$[0,\pi/2)$	$-1/(x\sqrt{x^2-1})$
$\csc^{-1}(x)$	$(-\infty,-1]$	$[-\pi/2,0)$	$-1/(x\sqrt{x^2-1})$
$\csc^{-1}(x)$	$[1,\infty)$	$(0,\pi/2]$	$1/(x\sqrt{x^2-1})$
$\ln(x)$	$(0,\infty)$	$(-\infty,\infty)$	$1/x$
$\log_a(x)=\log_a(e)\ln(x)$	$(0,\infty)$	$(-\infty,\infty)$	$\log_a(e)/x$
$\exp(x)$	$(-\infty,\infty)$	$(0,\infty)$	$\exp(x)$
$a^x=\exp(x\ln(a))$	$(-\infty,\infty)$	$(0,\infty)$	$a^x\ln(a)$
$sinh(x)=(e^x-e^{-x})/2$	$(-\infty,\infty)$	$(-\infty,\infty)$	$cosh(x)$
$cosh(x)=(e^x+e^{-x})/2$	$(-\infty,\infty)$	$[1,\infty)$	$sinh(x)$
$tanh(x)$	$(-\infty,\infty)$	$(-1,1)$	$sech^2(x)$
$coth(x)$	x not 0	$(-\infty,-1)\cup(1,\infty)$	$-csch^2(x)$
$sech(x)$	$(-\infty,\infty)$	$(0,1]$	$-sech(x)\tanh(x)$
$csch(x)$	x not 0	$(-\infty,0)\cup(0,\infty)$	$-csch(x)\coth(x)$
$sinh^{-1}(x)$	$(-\infty,\infty)$	$(-\infty,\infty)$	$1/\sqrt{x^2+1}$
$cosh^{-1}(x)$	$[1,\infty)$	$[0,\infty)$	$1/\sqrt{x^2-1}$
$cosh^{-1}(x)$	$[1,\infty)$	$(-\infty,0]$	$-1/\sqrt{x^2-1}$
$tanh^{-1}(x)$	$(-1,1)$	$(-\infty,\infty)$	$1/(1-x^2)$
$coth^{-1}(x)$	$(-\infty,-1)\cup(1,\infty)$	$(-\infty,0)\cup(0,\infty)$	$1/(1-x^2)$
$sech_1^{-1}(x)$	$(0,1]$	$(-\infty,0]$	$1/(x\sqrt{1-x^2})$
$sech_2^{-1}(x)$	$(0,1]$	$[0,\infty)$	$-1/(x\sqrt{1-x^2})$
$csch^{-1}(x)$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$	$-1/(\lvert x\rvert\sqrt{1+x^2})$

27 Inequalities

By the triangle inequality

$$|a+b| \leq |a| + |b|.$$

So if

$$a = b + c,$$

then

$$|a| \leq |b| + |c|.$$

And so

$$|b| \geq |a| - |c|.$$

We have

$$|a+b+c| \leq |a| + |b+c| \leq |a| + |b| + |c|$$

and so on. If

$$w = \sum_{k=1}^{n} z_k,$$

then

$$w - \sum_{k=2}^{n} z_k = z_1,$$

so

$$|w| \geq |z_1| - \sum_{k=2}^{n} |z_k|.$$

28 Convergence of Sequences

A sequence is a set indexed by the positive integers written

$$\{s_n\}_0^\infty.$$

The elements of the set might be numbers or functions. For example

$$s_n = 1/n.$$

As n goes to infinity, s_n goes to 0. A sequence of numbers $\{s_n\}_0^\infty$ converges to a if for each $\epsilon > 0$, there exists an integer N so that for all $n > N$,

$$|s_n - a| < \epsilon.$$

So the sequence $\{s_n\}_0^\infty$ given by

$$s_n = \frac{n + 3n^3}{3 + n^2 + 5n^3}$$

converges to 3/5, as can be seen by writing

$$s_n = \frac{2 + n + 3n^3}{3 + n^2 + 5n^3} = \frac{2/n^3 + 1/n^2 + 3}{3/n^3 + 1/n + 5}.$$

A sequence of functions $\{f_n\}_0^\infty$ defined on a domain D converges to a function f defined on D if every pointwise sequence $\{f_n(x)\}_0^\infty$ converges to $f(x)$. So for example if for $x \in (0, 1)$

$$f_n(x) = x^n,$$

then f_n converges to the zero function on $(0, 1)$.

A sequence with the property that for every $\epsilon > 0$ there exists an integer N so that

$$|s_n - s_m| < \epsilon$$

for all $n, m > N$ is called a Cauchy sequence.
proposition A convergent sequence is a Cauchy sequence.
Proof. So given an $\epsilon/2 > 0$ there exist an integer so that for all $n > N$ and $m > N$

$$|s_n - s_m| = |s_n - a - (s_m - a)| \leq |s_n - a| + |s_m - a| < \epsilon/2 + \epsilon/2 = \epsilon.$$

But a Cauchy sequence need not converge. A space in which every Cauchy sequence converges is called a complete space. There are Cauchy sequences of rational numbers that converge to $\sqrt{2}$, which is not a rational number. So the rational numbers are not a complete space. The real numbers are complete.

29 Infinite Series and Power Series

The nth partial sum of the infinite series

$$\sum_{k=0}^{\infty} a_k,$$

is

$$s_n = \sum_{k=0}^{n} a_k.$$

These partial sums form a sequence

$$\{s_n\}_0^\infty$$

The infinite series converges if the partial sums converge. If an infinite sequence converges then the sequence $\{|a_n|\}_0^\infty$ must converge to zero. This follows from the fact that the partial sums of the sequence, being a convergent sequence, are a Cauchy sequence. An alternating series (terms alternating in sign), where each term decreases in magnitude always converges.

A power series has the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

In general a power series has a radius of convergence R so that the series converges for all z such that $|z| < R$. Every power series converges for $z = 0$.

29.1 The Geometric Series

If

$$S = \sum_{k=0}^{n} x^k$$

then

$$Sx = S - 1 + x^{n+1}$$
$$S(1 - x) = 1 - x^{n+1}$$

or

$$S = \frac{1 - x^{n+1}}{x - 1}$$

If

$$0 < x < 1$$

then

$$\frac{x^{n+1}}{x - 1} \to 0$$

as $n \to \infty$. So

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}.$$

29.2 Comparison Test

If there exists an $0 < x < 1$ so that

$$0 < a_n < x^n,$$

then

$$\sum_{n=0}^{\infty} a_n$$

converges because the geometric series converges.

29.3 The Root Test

If

$$0 < a_n$$

and

$$\lim_{n\to\infty} (a_n)^{1/n} < 1,$$

then

$$\sum_{n=0}^{\infty} a_n$$

converges.

Proof. If the limit is less than 1, then there exists an $0 < x < 1$ and an integer N so that for all $n > N$

$$a_n^{1/n} < x$$

or

$$a_n < x^n.$$

and so the series converges by comparison with the geometric series.

Similarly if the limit is greater than 1, then the series diverges, because there exists an $x > 1$, and an N so that for all $n > N$

$$a_n > x^n,$$

and the geometric series diverges for $x > 1$.

29.4 The Ratio Test

If

$$0 < a_n$$

and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1,$$

then

$$\sum_{n=0}^{\infty} a_n$$

converges.

Proof. If the limit is less than 1, then there exists an $0 < x < 1$ and an integer N so that for all $n \geq N$

$$a_{n+1} < xa_n,$$

which implies that

$$a_n < x^{n-N}c,$$

for some constant c. So the series is eventually dominated by a convergent geometric series, and so must converge. And again if

$$\lim_{n\to\infty} \frac{a^{n+1}}{a^n} > 1,$$

then the series eventually dominates a divergent geometric series, so diverges.

30 Polar Coordinates

If $p = (x, y)$ is a point in the plane, let

$$\theta = \tan^{-1}(y/x),$$

and

$$r = \sqrt{x^2 + y^2}.$$

Then θ and r are called the polar coordinates of the point p. Given θ and r we have

$$x = r\cos(\theta),$$

and

$$y = r\sin(\theta).$$

A curve can be defined as a function $r(\theta)$. Example the spiral of Achimedies is defined as

$$r(\theta) = \alpha\theta.$$

31 Areas, Volumes, Moments of Inertia

Example Calculate the center of mass of a semicircular disk of unit thickness, unit density, and radius r. Let the semicircular disk lie above the x-axis. Moment about the x axis is

$$M_x = \int_A y dA = \int_0^r y 2x dy$$

$$= \int_0^r 2y\sqrt{r^2 - y^2} dy$$

$$= \frac{2r^3}{3}.$$

The y coordinate of the center of mass is

$$c_y = \frac{M_x}{(\pi/2)r^2} = \frac{4r}{3\pi}$$

32 The Elements of Complex Analysis

Let

$$w = f(z)$$

be a complex function of a complex variable $z = x + yi$. Write

$$w = u + vi.$$

The derivative of f at z_0 is defined as

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

If this limit is to exist, it must exist for z approaching z_0 from any direction. So it must exist in the special case where

$$z = x + y_0 i, z_0 = x_0 + y_o i.$$

Then since $w = u + vi$ and u and v are both functions of x and y, we see that

$$f'(z_0) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} i.$$

Similarly letting z approach z_0 in the y direction, we have

$$f'(z_0) = \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} i.$$

So a necessary condition for the existence of the derivative is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are called the Cauchy-Riemann equations. A function is called analytic or regular at a point z_0 if f has a derivative in an open neighborhood of z_0. We have shown that the Cauchy-Riemann equations are a necessary condition for f to be differentiable. This condition is also sufficient for a function to be analytic. That is if the four partial derivatives exist and are continuous in a neighborhood of z_0, and if the Cauchy-Riemann equations hold, then f is analytic at z_0.

An analytic function is given by a power series (Taylor Series) in a neighborhood of the point z_0

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

where the kth coefficient is defined by the kth derivative at z_0

$$a_k = \frac{f^{(k)}(z_0)}{k!}.$$

Cauchy's Theorem. The integral of an analytic function around a closed path is zero.

From this result we may deduce that the value of an analytic function at a point z_0 is given by **Cauchy's integral formula,**

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz,$$

where C is a closed path containing z_0 in its interior. Similarly all derivatives are defined by similar integrals.

To prove the integral formula we start with the result

$$2\pi i = \int_C \frac{1}{z - z_0} dz.$$

Then

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z_0)}{z - z_0} dz + \frac{1}{2\pi i} \int_C \frac{f(z) - f(z_0)}{z - z_0} dz.$$

The first integral is equal to

$$\frac{2\pi i f(z_0)}{2\pi i} = f(z_0).$$

According to Cauchy's theorem we may replace the path C by a circle K of arbitrarily small radius ρ. Then the second integral becomes

$$\frac{1}{2\pi i\rho} \int_K (f(z) - f(z_0)) dz.$$

Given an $\epsilon > 0$, we can find a radius ρ so that

$$|f(z) - f(z_0)| < \epsilon.$$

So the second interval is arbitrarily small and hence zero. This proves the integral formula.

We have an integral formula for the nth derivative

$$f^{(n)}(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

Cauchy's Inequality Given a powseries representation for an analytic function $f(z)$, the nth coefficient is given by

$$a_n = \frac{f^{(n)}(z_o)}{n!} = \frac{1}{n! 2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where C is a circle about Z_0 contained in the region of regularity of f. Let ρ be the radius of the circle, and let M be the maximum value of $|f(z)|$ on C. Then

$$|a_n| \leq \frac{M2\pi}{n!2\pi\rho^{n+1}} \leq \frac{M}{n!\rho^{n+1}}.$$

Theorem. A bounded entire function is a constant.
Proof. Let the entire function have a power series representation

$$f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k.$$

Suppose $f(z)$ is bounded by a number $M > 0$. Then using Cauchy's inequality

$$|a_k| \leq \frac{M}{k!\rho^{n+1}}.$$

But because f is an entire function, the radius ρ may be taken arbitrarily large, so the right side can be made arbitrarily small. Therefore a_k is zero for $k > 0$, so

$$f(z) = a_0,$$

a constant.

33 A Polynomial is Unbounded

Given a polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n,$$

we have

$$p(z) - (a_0 + a_1 z + a_2 z^2 + ... + a_{n-1} z^{n-1} = a_n z^n.$$

Thus

$$|p(z)| + |a_0| + |a_1||z| + ... + |a_{n-1}||z^{n-1}| \geq |a_n||z^n|.$$

Let $r = |z|$, then

$$|p(z)| \geq |a_n|r^n - (|a_0| + |a_1|r^n + ... + |a_{n-1}|r^{n-1}).$$

So

$$|p(z)| \geq r^n(|a_n| - (\frac{|a_0|}{r^n} + \frac{|a_1|}{r^{n-1}} + ... + \frac{|a_{n-1}|}{r})).$$

Clearly

$$\frac{|a_0|}{r^n} + \frac{|a_1|}{r^{n-1}} + ... + \frac{|a_{n-1}|}{r}$$

goes to zero as r goes to infinity. So there is some $R > 1$ so that if $r > R$ then

$$|a_n| - (\frac{|a_0|}{r^n} + \frac{|a_1|}{r^{n-1}} + ... + \frac{|a_{n-1}|}{r}) > |a_n|/2.$$

So if $r > R$, then

$$|p(z)| \geq r^n(|a_n|/2.$$

Then given an arbitrarily large M, an $r > R$ can be chosen so that

$$r^n|a_n|/2 > M.$$

Hence given any $M > 0$, there exists a circle with center at the origin with radius r so that for all z outside of this circle.

$$p(z) > M.$$

Theorem. Given a polynomial $p(z)$ and a number $M > 0$ there exists a circle about the origin so that $\forall z$ outside of this circle.

$$|p(z)| > M.$$

34 A Proof of the Fundamental Theorem of Algebra

A bounded entire function is a constant. Given a non-constant polynomial $p(z)$. Suppose $p(z)$ does not have a root. Then

$$\frac{1}{p(z)}$$

is an entire function. But because $p(z)$ is a polynomial, $1/|p(z|$ is say less than 1 for all points outside of some circle. That is it is bounded, and so a bounded entire function, and so a constant. This is a contradiction. Therefore $p(z)$ has a root.

35 Laurent Series

If f is analytic in $r_1 < |z - z_0| < r_2$ then

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z - z_0)^n,$$

where

$$A_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

36 The Residue Theorem

At an isolated singular point

$$A_{-1} = \frac{1}{2\pi i} \int_C f(z) dz.$$

So that

$$\int_C f(z) dz = 2\pi i A_{-1}$$

A_{-1} is called the residue of $f(z)$ at the isolated singular point z_0.

37 Calculating Residues

Let $f(z)$ have a simple pole of order n at z_0. Then

$$\phi(z) = (z - z_0)^n f(z),$$

is analytic in a neighborhood of z_0. Then the $n-1$ coefficient of the Taylor expansion of $\phi(z)$ is

$$A_{-1} = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1} \phi(z)}{dz^{n-1}}.$$

38 The Inversion of the Laplace Transform

We define the Fourier transform as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt.$$

The Fourier inversion theorem is

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega.$$

The double sided Laplace transform is

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt.$$

Let $s = \phi + i\omega$. Then $F(s)$ is the Fourier transform of $g_\phi(t) = f(t)e^{-\phi t}$, that is

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} f(t)e^{-\phi t}e^{-i\omega t}dt \\ &= \hat{g}_\phi(\omega). \end{aligned}$$

Formally applying the Fourier inversion theorem, we have

$$\begin{aligned} f(t)e^{-\phi t} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}_\phi(\omega)e^{i\omega t}d\omega. \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{i\omega t}d\omega. \end{aligned}$$

Then

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{\phi t}e^{i\omega t}d\omega. \\ &= \frac{1}{2\pi i}\int_{C_\phi} F(s)e^{st}ds, \end{aligned}$$

where C_ϕ is the Bromwich contour defined by

$$\{\phi + i\omega : -\infty < \omega < \infty\}.$$

Note that i appears in the expression $2\pi i$ because

$$ds = id\omega.$$

In general we will find that if we define a closed curve consisting of a finite line of length $2R$ on the bromwich contour, and a semicircle of radius R to the left, then as R goes to infinity, the integral over the semicircle goes to zero, so that the total integral over the curve is equal to the integral on the Bromwich line, which is thus equal to $2\pi i$ times the residues of $F(s)e^{st}$ in the left halfspace bounded by the contour. Our inversion expression is therefore

equal to the sum of the residues themselves. We get the single sided Laplace transform from the double when $f(t)$ is equal to zero for $t \leq 0$.
Example: Consider

$$F(s) = \frac{1}{s-1},$$

for $\Re(s) > 1$. The residue of $F(s)e^{st}$ is

$$\lim_{s \to 1} (s-1)F(s)e^{st} = e^t.$$

Therefore

$$f(t) = e^t.$$

Example: Consider

$$F(s) = \frac{1}{s^2+1} = \frac{1}{(s-i)(s+i)},$$

for $\Re(s) > 0$. The residues of $F(s)e^{st}$ are

$$\lim_{s \to i} (s-i)F(s)e^{st} = \frac{e^{it}}{2i},$$

and

$$\lim_{s \to -i} (s+i)F(s)e^{st} = \frac{e^{-it}}{-2i},$$

Therefore

$$f(t) = \frac{e^{it} - e^{-it}}{2i} = \sin(t).$$

39 Parametric Curves

A parametric curve is a vector function where each component is a function of some parameter t. For example

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

$$= a\cos(t)\mathbf{i} + b\sin(t)\mathbf{j} + t^2\mathbf{k},$$

is some kind of elliptical spiral.

40 Arc Length

Let a parametric curve be defined by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

The velocity is

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

The velocity is is tangent to the curve. Define the arclength as

$$s(t) = \int_0^t \|\mathbf{v}\| dt$$

$$= \int_0^t \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} dt.$$

If

$$x(t) = r\cos(t),$$
$$y(t) = r\sin(t).$$
$$z(t) = 0,$$

then

$$dx/dt = -r\sin(t),$$
$$dy/dt = r\cos(t).$$

So

$$s(t) = \int_0^t r dt = rt.$$

On the other hand if the curve is

$$x(t) = a\cos(t),$$
$$y(t) = b\sin(t).$$
$$z(t) = 0,$$

which is an ellipse, then

$$dx/dt = -a\sin(t),$$
$$dy/dt = b\cos(t).$$

So

$$s(t) = \int_0^t \sqrt{a^2\sin(t)^2 + a^2\cos(t)^2} dt.$$

This is not an elementary integral, but is a type of elliptic integral, whose values can be expressed as a standard elliptic integral, whose values in turn are computed numerically and tabulated in mathematical handbooks.

41 Curvature and Elementary Differential Geometry

Curvature is the ratio of the change in turning to the distance traveled. Consider the circular path. As a point on the circle moves through an angle change $\Delta\theta$, it moves a distance $\Delta s = r\Delta\theta$. The ratio is a measure of the curvature

$$\frac{\Delta\theta}{\Delta s} = \frac{\Delta\theta}{r\Delta\theta} = \frac{1}{r}.$$

The angle change of the tangent $\Delta\phi$ is here equal to the angle change $\Delta\theta$, so we can use the tangent angle in our definition of the curvature. So suppose we are given a general curve in the plane

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}.$$

Suppose $dx/dt \neq 0$, then the angle of the tangent is

$$\phi = \tan^{-1}\left[\frac{dy/dt}{dx/dt}\right].$$

Let us write

$$\begin{aligned} \frac{dx}{dt} &= \dot{x}, \\ \frac{dy}{dt} &= \dot{y}, \\ \frac{d^2x}{dt^2} &= \ddot{x}, \\ \frac{d^2y}{dt^2} &= \ddot{y}. \end{aligned}$$

Then we have

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{1}{1+(\dot{y}/\dot{x})^2}\frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2} \\ &= \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}. \end{aligned}$$

We have

$$\frac{ds}{dt} = \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{\dot{x}^2 + \dot{x}^2}$$

So the curvature κ is

$$\kappa = \frac{d\phi}{ds} = \frac{d\phi/dt}{ds/dt}$$

$$= \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2} \frac{1}{\sqrt{\dot{x}^2 + \dot{x}^2}}.$$

$$= \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

Now we excluded the case where $dx/dt = 0$. However we can just as well define the tangent angle as

$$\phi = \cot^{-1}\left[\frac{dx/dt}{dy/dt}\right].$$

If we carry out the derivation above we shall find that we get the same formula for the curvature. Hence as long as at least one of dx/dt or dy/dt is not zero, the formula holds.

Suppose we have a function $y = f(x)$. This is a curve with $t = x$ as the parameter. Then $\dot{x} = 1$, $\ddot{x} = 0$, and the curvature formula reduces to

$$\kappa = \frac{d\phi}{ds} = \frac{\ddot{y}}{(1 + \dot{y}^2)^{3/2}},$$

where

$$\ddot{y} = \frac{d^2y}{dx^2},$$

and

$$\dot{y} = \frac{dy}{dx}.$$

In three dimensional space there is no obvious tsngent angle. So we must define curvature using another approch. Let

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

The velocity is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = dx/dt(t)\mathbf{i} + dy/dt(t)\mathbf{j} + dz/dt(t)\mathbf{k}.$$

The magnitude of the velocity is

$$v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \frac{ds}{dt}.$$

The unit tangent vector $\mathbf{T}$ is defined as

$$\mathbf{T} = \frac{\mathbf{v}}{v} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{d\mathbf{r}}{ds}.$$

$\mathbf{T}$ is a unit vector so

$$\mathbf{T} \cdot \mathbf{T} = 1.$$

We have

$$\frac{d(\mathbf{T} \cdot \mathbf{T})}{ds} = d\mathbf{T}/ds \cdot \mathbf{T} + \mathbf{T} \cdot d\mathbf{T}/ds = 0,$$

which implies that

$$2\mathbf{T} \cdot d\mathbf{T}/ds = 0.$$

So $\mathbf{T}$ and it derivative are orthogonal. Thus $d\mathbf{T}/ds$ is a vector normal to the curve. The unit normal vector $\mathbf{N}$ is defined as

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{\|d\mathbf{T}/ds\|}.$$

Thus

$$d\mathbf{T}/ds = \|d\mathbf{T}/ds\|\mathbf{N}.$$

We can define the curvature κ as

$$\kappa = \|d\mathbf{T}/ds\|,$$

because we can show that for a two dimensional curve this agrees with the two dimensional curvature.

Indeed, in two dimension the unit tangent vector $\mathbf{T}$ can be written as

$$\mathbf{T} = \cos(\phi)\mathbf{i} + \sin(\phi)\mathbf{j},$$

where the tangent angle ϕ is a function of the arc length s. Then

$$d\mathbf{T}/ds = -\sin(\phi)(d\phi/ds)\mathbf{i} + \cos(\phi)(d\phi/ds)\mathbf{j}.$$

So

$$\|d\mathbf{T}/ds\| = |d\phi/ds|\sqrt{\sin^2(\phi) + \cos^2(\phi)} = |d\phi/ds| = \kappa.$$

Notice that here the curvature is always nonnegative. We can therefore write

$$d\mathbf{T}/ds = \kappa\mathbf{N}.$$

Let the curve $r(s)$ in Euclidean 3-space, be parameterized by arc length. Define the tangent vector

$$t = \frac{dr}{ds},$$

the curvature

$$\kappa = \left|\frac{dt}{ds}\right|,$$

the normal vector

$$n = \frac{1}{\kappa}\frac{dt}{ds},$$

and the binormal vector

$$b = t \times n.$$

The vectors t, n, and b, are called the frame vectors. The derivatives of these frame vectors with respect to arc length s are equal to linear combinations of the frame vectors themselves. These are called the Serret-Frenet formulas. The Serret-Frenet formulas are derived from the facts that the frame vectors are mutually perpendicular, and that they have unit length. The dot product of any pair of frame vectors is zero. So the derivative of their dot product is also zero. Unit vectors are perpendicular to their derivatives, and n is a unit vector. So dn/ds is perpendicular to n. Consequently dn/ds can be written as a linear combination of t and b only. Thus

$$\frac{dn}{ds} = a_1 t + a_3 b.$$

Because t is perpendicular to n,

$$a_1 = \frac{dn}{ds} \cdot t = -n \cdot \frac{dt}{ds}.$$

By definition the right hand expression is equal to $-\kappa$. So we conclude that a_1 is equal to the curvature κ.

Define the torsion τ to be a_3. Thus

$$\frac{dn}{ds} = -\kappa t + \tau b.$$

Let

$$\frac{db}{ds} = b_1 t + b_2 n.$$

$$b_1 = \frac{db}{ds} \cdot t = -b \cdot \frac{dt}{ds} = -b \cdot n = 0.$$

$$\frac{db}{ds} \cdot n = -b \cdot \frac{dn}{ds} = -b \cdot (-\kappa t + \tau b) = -\tau.$$

Thus

$$\frac{db}{ds} = -\tau n.$$

Therefore we have the Serret-Frenet transformation,

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

Notice that the matrix of the transformation is antisymmetric, and that the top right element of the matrix is zero. An antisymmetric matrix has a zero diagonal. The lower triangular part has negative elements. These facts might aid one in remembering the formula.

42 Curves and Surfaces

There are parametric curves and algebraic curves. The same for surfaces. A curves and surfacces may be defined piecewise.

43 Interpolation, Piecewise Curves, Splines

44 Vector Spaces

45 Linear Transformations and Matrices

See linearal.tex

46 Multivariable Calculus

47 Elementary Differential Equations

48 Elements of Vector Analysis

See vecana.tex Vector analysis is a classical subject dealing with those aspects of vectors which have application in applied mathematics and specifically to physics. Vector analysis deals largely with vector calculus. Linear Algebra also deals with vectors and vector spaces, but confines itself to algebra.

49 The Inner Product

We shall prove the law of cosines. Suppose we have three points

$$p_0 = (0,0), p_1 = (b,0), p_2 = (x,y) = (a\cos(\theta), a\sin(\theta)).$$

These points form a triangle with sides p_0p_2, p_0p_1, p_2p_1. These sides have lengths a, b, c. The angle between side p_0p_1 and side p_0p_2 is θ. We have

$$c^2 = (x-b)^2 + y^2$$

$$= (x-b)^2 + a^2 - x^2$$

$$= x^2 - 2xb + b^2 + a^2 - x^2$$

$$= a^2 + b^2 - 2xb = a^2 + b^2 - 2ab\cos(\theta).$$

Thus we have the law of cosines, namely the square of the side opposite an angle of a triangle, is equal to the sum of the squares of the adjacent sides, minus two times the product of the sides and the cosine of the angle. That is,

$$c^2 = a^2 + b^2 - 2ab\cos(\theta).$$

The inner product (dot product) of two vectors, A and B, is defined as

$$A \cdot B = a_1b_1 + a_2b_2 + a_3b_3.$$

Then the dot product of a vector with itself is the square of its length. That is,

$$A \cdot A = a_1a_1 + a_2a_2 + a_3a_3 = \|A\|^2.$$

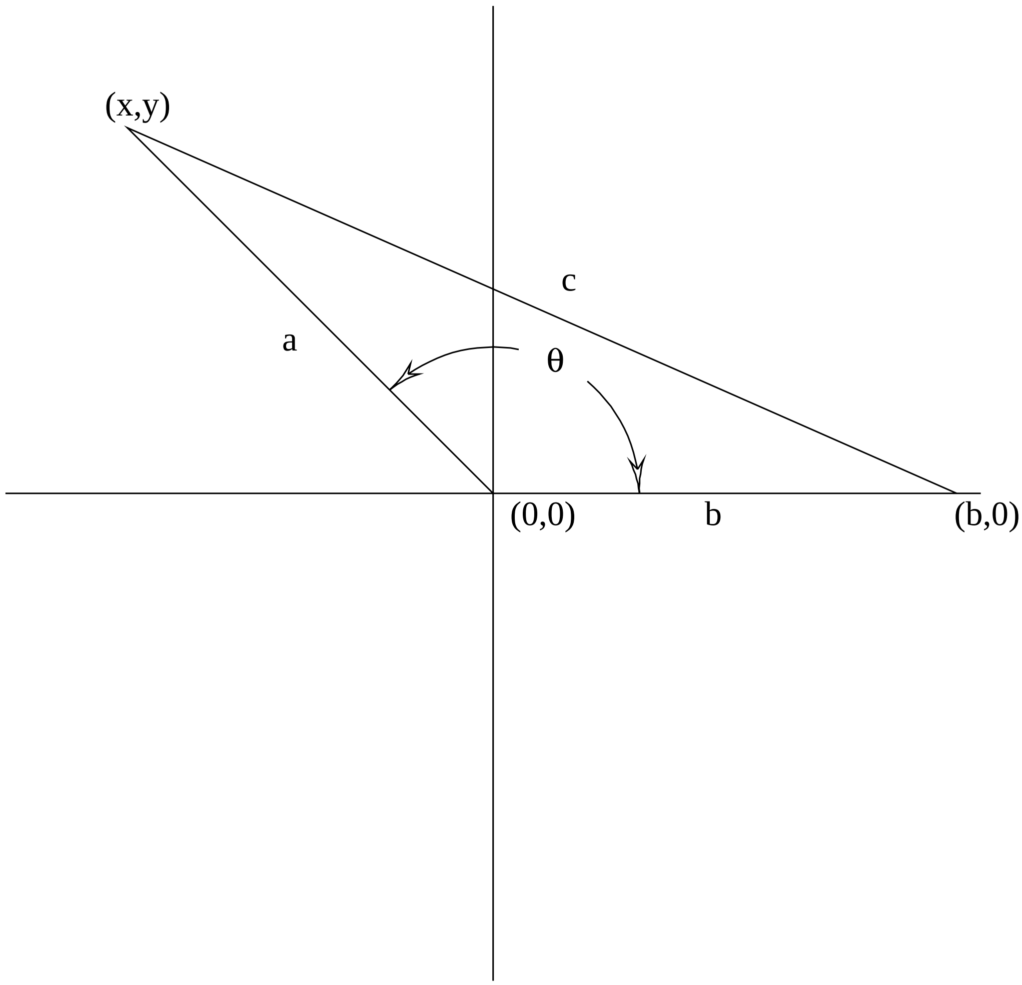

Figure 5: Derivation of the law of cosines. $x = \cos(\theta)$, $y = \sin(\theta)$. Computing c^2, we find that $c^2 = a^2 + b^2 - 2ab\cos(\theta)$.

Let

$$C = B - A.$$

Then

$$\begin{aligned} \|C\|^2 &= (B - A) \cdot (B - A) \\ &= B \cdot B - B \cdot A - A \cdot B + A \cdot A \\ &= \|B\|^2 - 2A \cdot B + \|A\|^2. \end{aligned}$$

From which it follows that

$$2A \cdot B = \|A\| + \|B\|^2 - \|C\|^2.$$

But the right hand side is, by the law of cosines,

$$2\|A\|\|B\| \cos(\theta),$$

where θ is the angle between vectors A and B. Hence

$$A \cdot B = \|A\|\|B\| \cos(\theta).$$

Thus if the the dot product is zero, then the cosine is zero, and so the angle between the vectors is plus or minus $\pi/2$, and the vectors are perpendicular.

50 The Vector Product

The vector product of two vectors A and B, (the cross product), is defined to be

$$A \times B = (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k,$$

where i, j, k are the unit coordinate vectors. This may be written as a determinant with i, j, k in the first row, the components of A in the second, and the components of B in the third row.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

When the rows of a determinant are interchanged, the sign of the determinant changes, hence

$$A \times B = -B \times A.$$

Then

$$A \times A = -A \times A.$$

But this can be true only if

$$A \times A = 0.$$

We have shown that the vector product of any two parallel vectors is zero.

Given three vectors A, B, C, we see that

$$A \cdot (B \times C),$$

is given as the determinant that has rows A, B, and C. By interchanging these rows twice, we see that

$$A \cdot (B \times C) = (A \times B) \cdot C.$$

That is, in the scalar triple product, the dot and the cross may be interchanged. Now using this result, we see that

$$(A \times B) \cdot B = A \cdot (B \times B) = A \cdot 0 = 0.$$

Then $A \times B$ is perpendicular to B. Similarly it is perpendicular to A. Therefore we have shown that the vector product of two vectors is perpendicular to each of them. This establishes the direction of the vector product, except possibly for sign. One may further establish the right hand rule. The direction of $A \times B$ is given by the right hand rule: Curl the fingers of your right hand from A to B, then $A \times B$ is in the direction of your thumb. One may verify directly that if V is a vector in the upper xy half plane that

$$i \times V$$

points in the positive z direction. This verifies the right hand rule in this case. One may also show the invariance of the cross product to a rigid motion, which establishes the right hand rule in general.

By direct computation one may verify that the vector triple product satisfies

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B).$$

This is the "Back Minus Cab Rule". We have established the direction of the cross product, now we shall find its magnitude. Let

$$C = A \times B.$$

Then

$$\begin{aligned}
\|C\|^2 &= C \cdot C \\
&= (A \times B) \cdot C \\
&= A \cdot (B \times C) \\
&= A \cdot (B \times (A \times B)) \\
&= A \cdot (A(B \cdot B) - B(B \cdot A) \\
&= (A \cdot A)(B \cdot B) - (A \cdot B)^2 \\
&= \|A\|^2 \|B\|^2 (1 - \cos^2(\theta)) = \|A\|^2 \|B\|^2 \sin^2(\theta).
\end{aligned}$$

The magnitude of the cross product is the product of the lengths of the vectors, times the sine of the angle between them,

$$\|A \times B\| = \|A\| \|B\| \sin(\theta).$$

Example The equation of a plane. Let the plane have a unit normal vector N. Let $P = (x, y, z)$ be a point on the plane. Let d be the distance from the origin to the plane. Then d is equal to the length of P times the cosine of the angle between P and the normal N. Hence

$$d = P \cdot N.$$

Therefore the equation of the plane is

$$P \cdot N - d = xn_1 + yn_2 + zn_3 - d = 0.$$

Suppose we are given three points P_1, P_2, P_3 and we wish to find the equation of the plane passing through these points. The normal to the plane is perpendicular to each of $P_2 - P_1$ and $P_3 - P_1$. Therefore

$$N = \frac{(P_2 - P_1) \times (P_3 - P_1)}{\|(P_2 - P_1) \times (P_3 - P_1)\|}$$

Also d is equal to the inner product of N with any one of the three points. For example

$$d = P_1 \cdot N.$$

Then the equation of the plane is

$$P \cdot N - P_1 \cdot N = xn_1 + yn_2 + zn_3 - d = 0.$$

51 Heron's Formula for the Area of a Triangle

Let T be the area of a triangle with sides given by vectors A, B, and C, and corresponding side lengths a, b and c. The area is one half of the magnitude of the cross product of the vectors A and B. That is,

$$2T = \|A \times B\|.$$

So

$$4T^2 = a^2b^2 \sin^2(\theta) = a^2b^2(1 - \cos^2(\theta)) = a^2b^2 - \|A \cdot B\|^2.$$

Also

$$c^2 = \|C\|^2 = \|A - B\|^2 = (A - B) \cdot (A - B) = a^2 - 2A \cdot B + b^2.$$

Then

$$\|A \cdot B\|^2 = \frac{(c^2 - (a^2 + b^2))^2}{4}.$$

Substituting this into the equation that we found above, namely

$$4T^2 = a^2b^2 - \|A \cdot B\|,$$

we get

$$\begin{aligned} 16T^2 &= 4a^2b^2 - (a^2 + b^2 - c^2)^2 \\ &= [2ab - (a^2 + b^2 - c^2)][2ab + (a^2 + b^2 - c^2)] \\ &= [c^2 - (a - b)^2][(a + b)^2 - c^2] \\ &= [c - (a - b)][c + (a + b)][a + b - c][a + b + c] \\ &= [c + b - a][c + a - b][a + b - c][a + b + c] \\ &= [a + b + c - 2a][a + b + c + -2b][a + b + c - 2c][a + b + c]. \end{aligned}$$

Dividing each product on the right by 2, we get

$$T^2 = (s - a)(s - b)(s - c)s,$$

where

$$s = \frac{a + b + c}{2},$$

is the half perimeter of the triangle. Taking the square root, we get Heron's formula,

$$T = \sqrt{(s - a)(s - b)(s - c)s}.$$

This derivation is suggested in a problem in Apostol's Calculus.

52 Line Integrals

A line integral is the integration of a vector function along a curve. So let C be a curve given by

$$\mathbf{r}(t) = x(r)\mathbf{i} + y(r)\mathbf{j} + z(r)\mathbf{k},$$

for $a \leq t \leq b$. Let $\mathbf{A}$ be a vector field. Then the line integral of $\mathbf{A}$ along curve C is

$$\int_C \mathbf{A} \cdot d\mathbf{r} = \int_a^b \mathbf{A}(\mathbf{r}) \cdot (d\mathbf{r}/dt)dt.$$

A vector field is a vector function defined in a region of space.

53 Curl, Divergence, Gradient

The curl of a vector field $\mathbf{A}$ in cartesian coordinates is

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{i}$$

$$- \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \mathbf{j}$$

$$\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{k}.$$

Divergence thereom. Stokes theorem. Directional derivative.
The divergence of a vector field $\mathbf{A}$ in cartesian coordinates is

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

If a surface S has bounding curve ∂S, Stokes theorem is

$$\int_S \nabla \times \mathbf{A} \cdot \mathbf{n} ds = \int_{\partial S} \mathbf{A} \cdot d\mathbf{r},$$

which allows a surface integral to be evaluated as a line integral around the boundary of the surface. The surface normal is $\mathbf{n}$.

The divergence theorem allows a volume integral to be evaluated as a surface integral. Let V be a volume and ∂V be it enclosing surface. Then

$$\int_V \nabla \cdot \mathbf{A} dv = \int_{\partial V} \mathbf{A} \cdot \mathbf{n} ds.$$

If S is an area in the x, y plane then a special case of Stokes Theorem gives

$$\int_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) ds$$

$$= \int_{\partial S} \mathbf{A} \cdot d\mathbf{r}$$

$$= \int_{\partial S} (A_x dx + A_y dy).$$

As an application of this formula we can find the area enclosed by a curve by evaluating a line integral around the curve. So let $A_x = -y/2$ and $A_y = x/2$, then

$$\int_S ds =$$

$$\int_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) ds$$

$$= \int_{\partial S} (A_x dx + A_y dy).$$

$$= (1/2) \int_{\partial S} (ydx - xdy).$$

54 Optimization: Lagrange Multipliers

See cnstrop.tex.

55 Numerical Analysis

See numanal.tex

56 Bibliography