What is Calculus?

James D Emery

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Contents

1 The Parts of Calculus

Traditionally Calculus was divided into two parts, Differential Calculus and Integral Calculus, and taught successively. Now this division is rather old fashioned and not made often. Rather the material for a course in Calculus is normally mixed and integrated, (no pun intended). Calculus used to be called "The Infinitesimal Calculus," which stressed its concern with the infinitely small.

Differential Calculus deals with the derivative, which is a limiting ratio of small changes. The most common example is instantaneous velocity, which is a ratio of a small distance change to a small time change.

Integral Calculus concerns itself with taking small bits and adding them together to get a whole. Thus for example, the determination of area or volume is, except for simple cases, a problem in Integral Calculus.

The word integral means "whole" or "oneness." Politicians often talk about "having integrity," which according to them means being truthful. The real meaning of integrity is a bit different. One has integrity, if he behave just "one" way to all people. He does not say one thing to one person while saying a different thing to a second person, behind the back. That is, one with integrity is "whole." In fact, one could have integrity by lying uniformly to everyone. One fact is indubitably clear, to have true integrity, one must do Calculus.

One good reason for mixing the two parts of calculus is that differentiation and integration are inverses of one another, one splits apart and analyzes, the other puts together and synthesizes. Calculus is also known as Analysis, especially in its advanced treatment and ramifications.

Often one might hear someone say, "What is Calculus used for?" as in the age old statement of students, "What do I have to learn this for? I will never use it! I am going to be a politician, a football player, an exotic dancer, or whatever." But calculus is about everything and used for everything. It is about how everything in the world works.

2 The Derivative

A derivative of a function $y = f(x)$ is the limit of ratio of a change in y, written Δy to a change in x written Δx as Δx goes to zero. Put another way, the derivative of f, at coordinate x, is the slope of the tangent line to the curve at the point (x, y). Refer to figure 1, where we see the slope of the chord joining the points (x, y) and (x_1, y_1), and we see that as $x_1 \to x$, this chord becomes the tangent line. The derivative is often given as

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

where $\lim_{h \to 0}$ means the limit as h goes to 0, the limiting value as h becomes very small. Here h plays the role of $\Delta x = x_1 - x$ and $f(x+h) - f(x)$ plays the role of Δy.

Let us attempt to calculate a derivative approximately with a computer program. We use the function $f(x) = \sin(x)$, and calculate the derivative at $x = 1$.

The Program

```
// derfun.c, compute the derivative of sin(x) at 1.
#include <stdio.h>
#include <math.h>
double f(double);
int main (){
 double x;
 double y;
 double h;
 double dydx,dydx2;
 int n;
 int i;
 n=45;
```

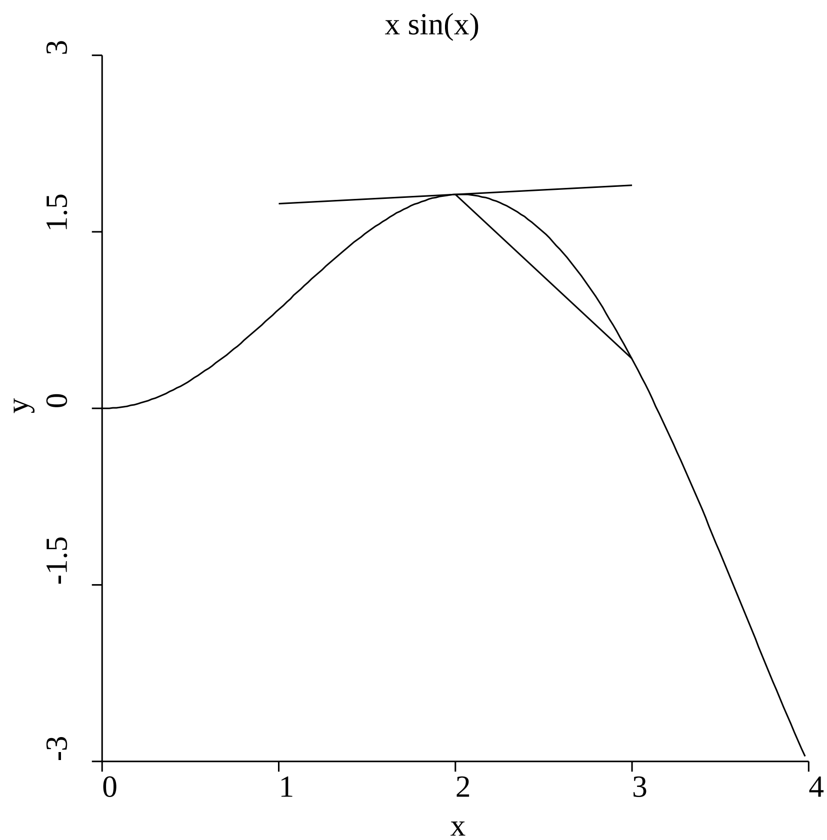

Figure 1: **The derivative is the slope of the tangent line**. This figure shows a plot of the function $y = f(x) = x \sin(x)$. The tangent line at the point on the curve where $x = 2$ is shown. A chord of the curve is a line joining two points of the curve, such as the line joining the points $(x, f(x))$ and $(x + h, f(x + h))$. Shown here is such a chord, where $x = 2$, and $h = 1$. The limit of the slope of this chord line as h goes to zero, is the derivative of f at x, which is the ratio of the change in y to the change in x, as h goes to zero.

```
 h=.5;
 x=1.;
  printf(" x = %21.15g \n",x);
  printf(" cos(x) = %21.15g \n",cos(x));
 for(i=0;i<n;i++){
  dydx=(f(x+h)-f(x))/h;
  dydx2=(f(x+h)-f(x-h))/(2.*h);
  printf(" h = %21.15g dydx = %21.15g dydx_2 = %21.15g\n",h,dydx,dydx2);
  h=h/2.;
 }

 return(1);
}
double f(double x){
 double y;
 y=sin(x);
 return (y);
}
```

The Output of the Program

```
x =                     1
 cos(x) =      0.54030230586814
 h =                   0.5 dydx =     0.312048003592316 dydx_2 =     0.518069447999851
 h =                  0.25 dydx =     0.430054538190759 dydx_2 =     0.534691718664504
 h =                 0.125 dydx =     0.486372874329589 dydx_2 =     0.538896367452272
 h =                0.0625 dydx =     0.513663205746793 dydx_2 =     0.539950615251025
 h =               0.03125 dydx =     0.527067456146781 dydx_2 =     0.540214370333548
 h =              0.015625 dydx =     0.533706462857715 dydx_2 =     0.540280321179402
 h =             0.0078125 dydx =     0.537009830329723 dydx_2 =     0.540296809645639
 h =            0.00390625 dydx =     0.538657435881987 dydx_2 =     0.540300931809369
 h =           0.001953125 dydx =     0.539480213605884 dydx_2 =     0.540301962353254
 h =          0.0009765625 dydx =     0.539891345517731 dydx_2 =     0.540302219989371
 h =         0.00048828125 dydx =      0.54009684715038 dydx_2 =     0.540302284398422
 h =        0.000244140625 dydx =     0.540199581875186 dydx_2 =     0.540302300500798
 h =       0.0001220703125 dydx =     0.540250945213302 dydx_2 =     0.540302304526449
 h =      6.103515625e-05 dydx =     0.540276625875777 dydx_2 =      0.54030230553235
 h =     3.0517578125e-05 dydx =     0.540289465956448 dydx_2 =      0.54030230578428
 h =    1.52587890625e-05 dydx =     0.540295885934029 dydx_2 =     0.540302305846126
 h =    7.62939453125e-06 dydx =     0.540299095911905 dydx_2 =     0.540302305867954
 h =   3.814697265625e-06 dydx =     0.540300700900843 dydx_2 =      0.54030230587523
 h =  1.9073486328125e-06 dydx =     0.540301503380761 dydx_2 =     0.540302305860678
 h =  9.5367431640625e-07 dydx =     0.540301904664375 dydx_2 =     0.540302305889782
 h = 4.76837158203125e-07 dydx =     0.540302105247974 dydx_2 =     0.540302305831574
 h = 2.38418579101562e-07 dydx =     0.540302205365151 dydx_2 =     0.540302305715159
 h = 1.19209289550781e-07 dydx =     0.540302256122231 dydx_2 =     0.540302305947989
 h = 5.96046447753906e-08 dydx =     0.540302280336618 dydx_2 =     0.540302305482328
 h = 2.98023223876953e-08 dydx =      0.54030229523778 dydx_2 =     0.540302306413651
 h = 1.49011611938477e-08 dydx =      0.54030229896307 dydx_2 =     0.540302306413651
 h = 7.45058059692383e-09 dydx =     0.540302306413651 dydx_2 =     0.540302306413651
 h = 3.72529029846191e-09 dydx =     0.540302306413651 dydx_2 =     0.540302306413651
 h = 1.86264514923096e-09 dydx =     0.540302276611328 dydx_2 =     0.540302276611328
 h = 9.31322574615479e-10 dydx =     0.540302276611328 dydx_2 =     0.540302276611328
 h = 4.65661287307739e-10 dydx =     0.540302276611328 dydx_2 =     0.540302276611328
```

```
h =   2.3283064365387e-10 dydx =     0.540302276611328 dydx_2 =     0.540302276611328
h =  1.16415321826935e-10 dydx =     0.540302276611328 dydx_2 =     0.540302276611328
h =  5.82076609134674e-11 dydx =     0.540302276611328 dydx_2 =     0.540302276611328
h =  2.91038304567337e-11 dydx =     0.540302276611328 dydx_2 =     0.540302276611328
h =  1.45519152283669e-11 dydx =     0.540306091308594 dydx_2 =     0.540302276611328
h =  7.27595761418343e-12 dydx =     0.540298461914062 dydx_2 =     0.540298461914062
h =  3.63797880709171e-12 dydx =     0.540313720703125 dydx_2 =     0.540313720703125
h =  1.81898940354586e-12 dydx =        0.540283203125 dydx_2 =        0.540283203125
h =  9.09494701772928e-13 dydx =        0.540283203125 dydx_2 =        0.540283203125
h =  4.54747350886464e-13 dydx =        0.540283203125 dydx_2 =        0.540283203125
h =  2.27373675443232e-13 dydx =         0.54052734375 dydx_2 =         0.54052734375
h =  1.13686837721616e-13 dydx =          0.5400390625 dydx_2 =          0.5400390625
h =   5.6843418860808e-14 dydx =           0.541015625 dydx_2 =           0.541015625
h =   2.8421709430404e-14 dydx =             0.5390625 dydx_2 =             0.5390625
```

The actual derivative of $\sin(x)$ is $\cos(x)$ and

$$\cos(1) = 0.54030230586814,$$

and our best approximation to this occurs where h is about 10^{-9}. As h decreases more the error actually increases. This is because of roundoff error, the fact that we are using numbers of only about 15 digits in the computer. Notice that the $dydx_2$ result is a bit more accurate. This is because we use a central difference approximation where the truncation error is of the order h^2, rather than of order h in the one sided difference.

3 The Derivative of $f(x) = x^n$

Consider the derivative of the function $f(x) = x$. From the definition

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = 1.$$

Consider the derivative of $f(x) = x^3$. We have

$$f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3.$$

So

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} =$$
$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$
$$= \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2.$$

As $h \to 0$, we are left with $3x^2$. In the same way we can compute the derivative of x^n using the binomial theorem

$$(x+h)^n = x^n + nx^{n-1}h + ... + h^n.$$

So to compute the derivative of x^n, we multiply by n and decrease the exponent by 1, getting nx^{n-1} . This also holds for $n < 0$.

We can use the derivative to find the maximum or minimum of a function. This is because at the minimum or maximum point of a function, it has a tangent line of zero slope. So we only to need find where the derivative of a function is zero.

Example. Suppose we want to build a cylindrical tank to hold a specified volume of liquid. The surface area is to be minimized. How should the radius r and the height h of the tank be chosen.

Let the constant volume be

$$V = \pi r^2 h$$

then

$$h = \frac{V}{\pi r^2}.$$

The area is

$$\begin{aligned} A &= 2\pi r^2 + 2r\pi h \\ &= 2\pi r^2 + 2r\pi \frac{V}{\pi r^2} \\ &= 2\pi r^2 + \frac{2V}{r} \end{aligned}$$

The derivative is

$$\frac{dA}{dr} = 4\pi r - \frac{2V}{r^2}$$

Setting the derivative to zero, we have

$$4\pi r^3 - 2V = 0.$$

So

$$r^3 = \frac{V}{2\pi}$$

and so

$$r = \left[\frac{V}{2\pi}\right]^{1/3}.$$

From above

$$h = \frac{V}{\pi r^2} = \frac{V}{\pi \left[\frac{V}{2\pi}\right]^{2/3}}$$

So

$$h/2 = \frac{\frac{V}{2\pi}}{\left[\frac{V}{2\pi}\right]^{2/3}} = \left[\frac{V}{2\pi}\right]^{1/3} = r.$$

The diameter of the tank is

$$d = 2r = h.$$

The tank of minimum surface area has a diameter equal to the height.

4 A Table of Elementary Derivatives

$f(x)$	Domain	Range	df/dx
$\sin(x)$	$(-\infty,\infty)$	$[-1,1]$	$\cos(x)$
$\cos(x)$	$(-\infty,\infty)$	$[-1,1]$	$-\sin(x)$
$\tan(x)$	x not $n\pi/2$	$(-\infty,\infty)$	$\sec^2(x)$
$\cot(x)$	x not $n\pi$	$(-\infty,\infty)$	$-\csc^2(x)$
$\sec(x)$	x not $n\pi/2$	$(-\infty,-1]\cup[1,\infty)$	$\sec(x)\tan(x)$
$\csc(x)$	x not $n\pi$	$(-\infty,-1]\cup[1,\infty)$	$-\csc(x)\cot(x)$
$\sin^{-1}(x)$	$[-1,1]$	$(-\pi/2,\pi/2)$	$1/\sqrt{1-x^2}$
$\cos^{-1}(x)$	$[-1,1]$	$(0,\pi)$	$-1/\sqrt{1-x^2}$
$\tan^{-1}(x)$	$(-\infty,\infty)$	$(-\pi/2,\pi/2)$	$1/(1+x^2)$
$\cot^{-1}(x)$	$(-\infty,\infty)$	$(0,\pi)$	$-1/(1+x^2)$
$\sec^{-1}(x)$	$(-\infty,-1]$	$(\pi/2,\pi)$	$1/(x\sqrt{x^2-1})$
$\sec^{-1}(x)$	$[1,\infty)$	$[0,\pi/2)$	$-1/(x\sqrt{x^2-1})$
$\csc^{-1}(x)$	$(-\infty,-1]$	$[-\pi/2,0)$	$-1/(x\sqrt{x^2-1})$
$\csc^{-1}(x)$	$[1,\infty)$	$(0,\pi/2]$	$1/(x\sqrt{x^2-1})$
$\ln(x)$	$(0,\infty)$	$(-\infty,\infty)$	$1/x$
$\log_a(x)=\log_a(e)\ln(x)$	$(0,\infty)$	$(-\infty,\infty)$	$\log_a(e)/x$
$\exp(x)$	$(-\infty,\infty)$	$(0,\infty)$	$\exp(x)$
$a^x=\exp(x\ln(a))$	$(-\infty,\infty)$	$(0,\infty)$	$a^x\ln(a)$
$sinh(x)=(e^x-e^{-x})/2$	$(-\infty,\infty)$	$(-\infty,\infty)$	$cosh(x)$
$cosh(x)=(e^x+e^{-x})/2$	$(-\infty,\infty)$	$[1,\infty)$	$sinh(x)$
$tanh(x)$	$(-\infty,\infty)$	$(-1,1)$	$sech^2(x)$
$coth(x)$	x not 0	$(-\infty,-1)\cup(1,\infty)$	$-csch^2(x)$
$sech(x)$	$(-\infty,\infty)$	$(0,1]$	$-sech(x)\tanh(x)$
$csch(x)$	x not 0	$(-\infty,0)\cup(0,\infty)$	$-csch(x)\coth(x)$
$sinh^{-1}(x)$	$(-\infty,\infty)$	$(-\infty,\infty)$	$1/\sqrt{x^2+1}$
$cosh^{-1}(x)$	$[1,\infty)$	$[0,\infty)$	$1/\sqrt{x^2-1}$
$cosh^{-1}(x)$	$[1,\infty)$	$(-\infty,0]$	$-1/\sqrt{x^2-1}$
$tanh^{-1}(x)$	$(-1,1)$	$(-\infty,\infty)$	$1/(1-x^2)$
$coth^{-1}(x)$	$(-\infty,-1)\cup(1,\infty)$	$(-\infty,0)\cup(0,\infty)$	$1/(1-x^2)$
$sech_1^{-1}(x)$	$(0,1]$	$(-\infty,0]$	$1/(x\sqrt{1-x^2})$
$sech_2^{-1}(x)$	$(0,1]$	$[0,\infty)$	$-1/(x\sqrt{1-x^2})$
$csch^{-1}(x)$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$	$-1/(\|x\|\sqrt{1+x^2})$

5 The Integral

The integral of a function

$$I = \int_a^b f(x)dx$$

is the area under the curve of the function $f(x)$ for x varying from a to b. Suppose we have a partition of the interval $[a, b]$

$$a = x_1 < \chi_1 < x_2 < \chi_2 < ... < \chi_{n-1} < x_n = b.$$

We define the definite integral to be

$$\int_a^b f(x)dx = \lim_{|x_{i+1}-x_i|\to 0} \sum_{i=1}^{n} f(\chi_i)(x_{i+1} - x_i).$$

The limit is taken as the distance between the mesh points goes to zero.

The First Fundamental Theorem of Integral Calculus. Define

$$G(x) = \int_A^x f(x)dx.$$

Then

$$\frac{dG(x)}{dx} = f(x).$$

Proof. We have, recalling that the integral is the area under the curve, that

$$G(x+h) - G(x) = \int_A^{x+h} f(x)dx - \int_A^x f(x)dx = \int_x^{x+h} f(x)dx = f(c)h,$$

for some c between x and $x + h$. Dividing by h and taking the limit we get the result. This says that $G(x)$ is an antiderivative of $f(x)$, which means that the derivative of $G(x)$ is $f(x)$.

The Second Fundamental Theorem of Integral Calculus. If $F(x)$ is any antiderivative of $f(x)$, then

$$\int_A^B f(x)dx = F(B) - F(A).$$

Proof. If $F(x)$ is any antiderivative of $f(x)$, and $G(x)$ is the antiderivative

$$G(x) = \int_A^x f(x)dx,$$

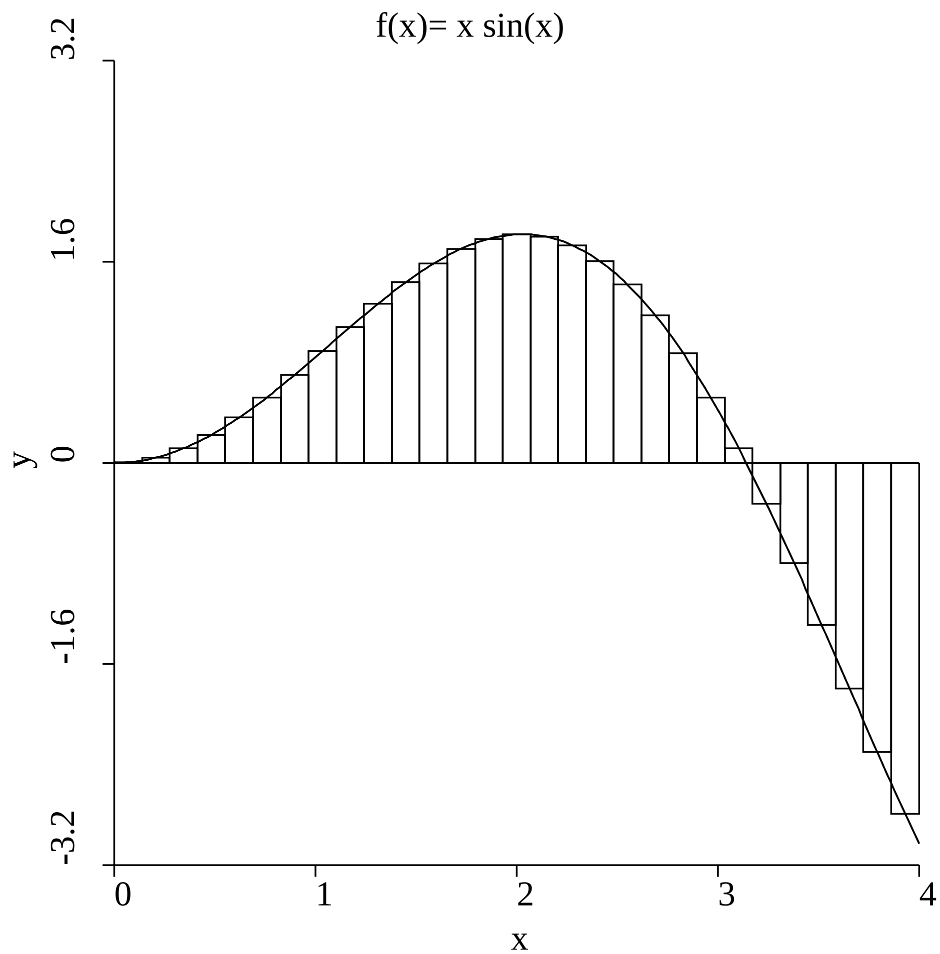

Figure 2: **The definite integral is the area under the curve.** . This figure shows a plot of the function $y = f(x) = x\sin(x)$. The integral of this function $\int_0^4 f(x)dx$ is approximated by the sum of the areas of the rectangles, where the areas of the rectangles under the $y = 0$ axis are negative. The integral is defined as the limit of such sums as the width of the rectangles goes to zero.

then $F(x)$ and $G(x)$ differ by some constant C. So

$$F(x) = G(x) + C.$$

$$G(B) = \int_A^B f(x)dx = F(B) - C.$$

$$F(A) = G(A) + C = \int_A^A f(x)dx + C = 0 + C = C.$$

So

$$\int_A^B f(x)dx = F(B) - F(A).$$

Example Let us calculate the volume of a sphere. A sphere has equation

$$x^2 + y^2 + z^2 = R^2,$$

where R is the radius of the sphere. Let us evaluate one half of the volume. We take a cylindrical slice of the sphere at some x value. The area of this slice is $\pi r^2(x)$, where $r(x)$ is the radius of the plane slice at x. If the thickness of the slice is dx, then the volume of a thin section is $\pi r^2(x)dx$. To compute the volume we add up an infinite number of infinitesimally small such sections. Then the volume of the half sphere is

$$\int_0^R \pi r^2(x)dx$$

We have

$$r^2(x) = R^2 - x^2.$$

So half the volume of the sphere is

$$V/2 = \int_0^R \pi(R^2 - x^2)dx = \int_0^R f(x)dx.$$

We need an antiderivative. Consider

$$F(x) = \pi(xR^2 - x^3/3)$$

Differentiating this we get

$$\pi(R^2 - \frac{3x^2}{3}) = \pi(R^2 - \frac{3x^2}{3}) = f(x)$$

So $F(x)$ is an antiderivative, so we have

$$V/2 = F(R) - F(0) = \pi((R^3 - R^3/3) - (0 - 0)) = \frac{2}{3}\pi R^3.$$

So the volume of the sphere is

$$V = \frac{4}{3}\pi R^3.$$

6 Snell's Law, the Early Bird Gets the Worm

Suppose that we can travel in one medium at velocity v_1 and in a second medium at a slower velocity v_2. Suppose we are to travel from point P in the first medium to point R in the second medium. What path results in the minimum travel time?

Solution

Referring to the **Snell's Law** figure, let v_1 be the velocity in the upper plane and v_2 the velocity in the lower plane, with $v_2 < v_1$. We are to find the position of the point $Q = (x, 0)$ to minimize the travel time from point P to point R. The length of the path in the upper plane is

$$\ell_1 = \sqrt{d^2 + x^2},$$

and the length of the path in the lower plane is

$$\ell_2 = \sqrt{e^2 + (c - x)^2}$$

The travel time as a function of x is

$$t = \frac{\ell_2}{v_1} + \frac{\ell_2}{v_2}.$$

The derivative of the time is

$$\begin{aligned} \frac{dt}{dx} &= \frac{(x/\sqrt{d^2 + x^2})}{v_1} - \frac{((c - x)/\sqrt{e^2 + (x - c)^2})}{v_2} \\ &= \frac{\sin(\theta_1)}{v_1} - \frac{\sin(\theta_2)}{v_2}. \end{aligned}$$

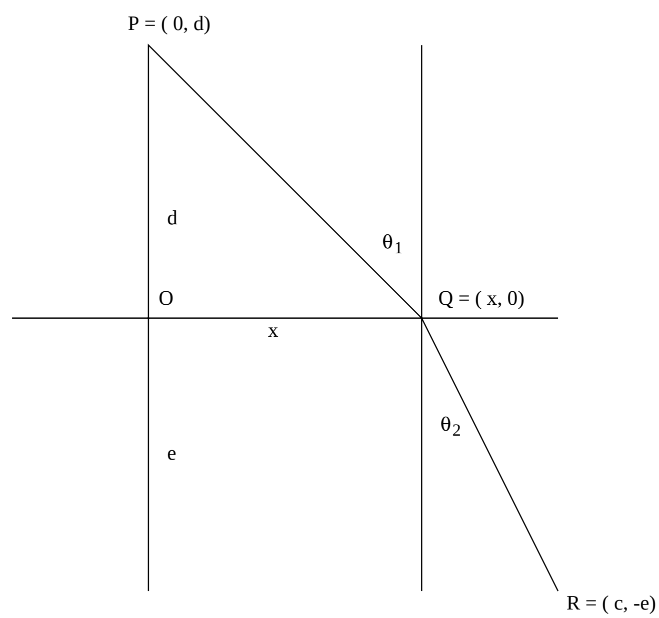

Figure 3: **Snell's Law.** Let two media be separated by the horizontal line. A particle in the upper media travels at velocity v_1, in the lower at velocity v_2, with $v_2 < v_1$. The travel time from P to R is minimized when x the coordinate of Q is selected to satisfy Snell's law.

Setting this to zero, we find that the condition for a minimum is

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2}.$$

In the case of optics we have the indices of refraction

$$n_1 = \frac{c}{v_1}, n_2 = \frac{c}{v_2},$$

where c is the velocity of light in a vacuum. So we obtain Snell's law of optical refraction.

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2).$$

7 Calculus On Vectors

Vectors in two or three space are objects that have x, y, and in the case of 3 dimensional vectors, z components. We write vectors in boldface. So suppose $\mathbf{R}$ is some vector

$$\mathbf{R} = (t^2 + 1, \sin(t), \ln(t))$$

Then the derivative of the vector is obtained by calculating the derivatives of the x, y, z components. So

$$\frac{d\mathbf{R}}{dt} = (2t, \cos(t), 1/t)$$

Vectors are commonly written using the unit coordinate basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

$$\mathbf{R} = (t^2 + 1, \sin(t), \ln(t)) = (t^2 + 1)\mathbf{i} + \sin(t)\mathbf{j} + \ln(t)\mathbf{k}$$

8 A Mass Point in Circular Motion

Let a mass point be specified in polar coordinates (θ, r). Let the point be constrained to lie on a circle of radius r. Let the polar coordinate unit vectors be

$$\mathbf{u}_r = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j},$$

$$\mathbf{u}_\theta = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}.$$

The first vector is perpendicular to the circle and the second is tangent to it. Let the position vector of the point be

$$\mathbf{p} = r\mathbf{u}_r.$$

The velocity is

$$\mathbf{v} = \frac{d\mathbf{p}}{dt} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\mathbf{u}_r}{dt} = r\frac{d\mathbf{u}_r}{dt},$$

because here r is constant. We have

$$\frac{d\mathbf{u}_r}{dt} = \frac{d\mathbf{u}_r}{d\theta}\frac{d\theta}{dt}$$

$$= \mathbf{u}_\theta \frac{d\theta}{dt}.$$

So

$$\mathbf{v} = r\frac{d\theta}{dt}\mathbf{u}_\theta = r\omega\mathbf{u}_\theta,$$

where ω is the angular velocity. The acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = r\frac{d\omega}{dt}\mathbf{u}_\theta + r\omega\frac{d\mathbf{u}_\theta}{dt}$$

$$= r\frac{d\omega}{dt}\mathbf{u}_\theta - r\omega^2\mathbf{u}_r$$

$$= r\frac{d\omega}{dt}\mathbf{u}_\theta - \frac{v^2}{r}\mathbf{u}_r,$$

where $d\omega/dt$ is the angular acceleration, and $v = r\omega$ is the tangential velocity. If the angular acceleration is zero then v^2/r is the magnitude of the centripetal acceleration directed toward the center of the circle.

9 The Period of an Earth Satellite

From the preceding section if a satellite is rotating around the earth, the magnitude of its centripetal acceleration is

$$r\omega^2,$$

where ω is the angular velocity in radians.

If its mass is m then the force keeping the mass in orbit is

$$F = r\omega^2 m.$$

This force is supplied by the earth's gravity and this force is given by

$$f = G\frac{mM}{r^2} = mG\frac{M}{r^2}$$

So the acceleration is

$$a = G\frac{M}{r^2}$$

Then

$$r\omega^2 = G\frac{M}{r^2},$$

or

$$\omega^2 = GM\frac{1}{r^3}.$$

The period of the orbit is the time it takes to rotate 2π radians

$$T = \frac{2\pi}{\omega} = \frac{2\pi r^{3/2}}{\sqrt{GM}}.$$

The period of the moon is about

$$T_m = (28)24) = 672$$

hours. The radius of the earth is about

$$r_e = 6371km$$

the distance to the moon is about

$$r_m = 384403km$$

So if a satellite could orbit right at the surface of the earth we would have

$$\frac{T_e}{T_m} = \left(\frac{r_e}{r_m}\right)^{3/2}.$$

So the satellite would orbit in about 1.43 hours. At a distance d from the earth the satellite period would be

$$T = T_m((r_e + d)/r_m)^{3/2}$$

A geosynchronous orbit where the period is about 24 hours is about $re + d =$ 42164 km.

10 Moment, Center of Gravity, Moment of Inertia

A moment is the twisting force due to a force acting on a lever arm. Given a plane area there is an area moment acting about the origin of the coordinate system. The x moment is given by an integral

$$M_x = \int x dm,$$

where dm is an element of mass. In the case of a plane object we have an area density σ, so that the element of mass $dm = \sigma dA$, where dA is an element of area.

Let us calculate the center of gravity of a triangle of base b and height a. We suppose the triangle base is on the x axis, with vertices $(0,0), (b,0)$ and (b,a). So let

$$y = (a/b)x$$

Consider the area under this curve on the interval $[0,b]$. We let the density $\sigma = 1$, so that $dm = dA$. We have $dA = ydx$, So the x moment is

$$M_x = \int_0^b xydx$$

$$= \int_0^b \frac{a}{b}x^2 dx$$

$$= \frac{a}{b}[x^3/3]_0^b$$

$$= ab^2/3.$$

The x coordinate of the center of gravity is obtained by dividing by the area $A = ab/2$ of the triangle,

$$\bar{x} = \frac{m_X}{A} = \frac{ab^2/3}{ab/2} = (2/3)b.$$

The y moment is

$$M_y = \int_0^a y(b-x)dy$$

$$= \int_0^a y(b - \frac{b}{a}y)dy$$

$$= \int_0^a (yb - \frac{b}{a}y^2)dy$$

$$= [by^2/2 - \frac{b}{a}y^3/3]_0^a$$

$$= ba^2/2 - ba^2/3$$

$$= ba^2(1/6)$$

So dividing by the area of the triangle $ab/2$, the y coordinate of the center of gravity is

$$\bar{y} = a/3.$$

So the center of gravity is at $(2/3b, 1/3a)$.

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be the vertices of any triangle. The midpoint of the side opposite A is

$$\frac{\mathbf{B} + \mathbf{C}}{2}$$

The line from $\mathbf{A}$ to the midpoint $\frac{\mathbf{B}+\mathbf{C}}{2}$ is called a median of the triangle. A point on the median 2/3 of the way from $\mathbf{A}$ to $\frac{\mathbf{B}+\mathbf{C}}{2}$ is

$$\mathbf{A} + \frac{2}{3}\left(\frac{\mathbf{B} + \mathbf{C}}{2} - \mathbf{A}\right)$$

$$= \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3}.$$

We get the same result starting from the other two vertices. Hence this point is the intersection of the three medians of a triangle. A calculation like the center of gravity argument above for the triangle shows that the intersection of the medians is the center of gravity of any triangle. So cut out a triangle draw the intersecting medians and try it.

The rotational moment of inertia of a body rotating about say the z axis is

$$I_z = \int (x^2 + y^2)dm,$$

where dm is an element of mass of the body. One can show that the rotational kinetic energy is

$$E_{rot} = \frac{1}{2}I_z\omega^2,$$

where ω is the angular velocity. Here ω plays the roll of velocity, and I_z plays the roll of the mass in the usual kinetic energy formula.

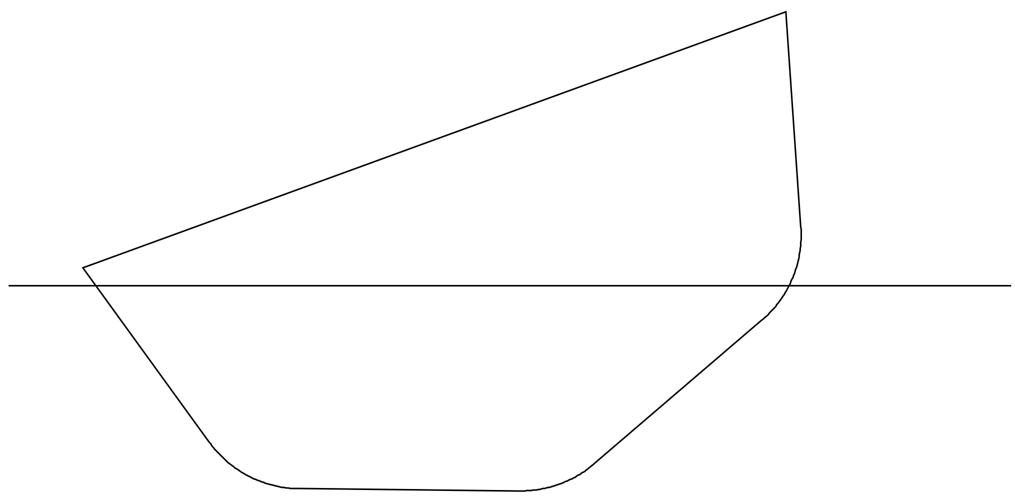

Figure 4: **Boat Stability**. As the boat is tipped to the left, the center of buoyancy lies to the left of the vertical center line of the boat. Assuming that the center of gravity lies on the center line, if the center of gravity is low enough, there will be a torque to level the boat. If the center of gravity is too high, it will lie to the left of the center of buoyancy in this tipped position, and the boat will capsize. This is because the force at the center of buoyancy acts upward and the force at the center of gravity acts downward.

11 Boat Stability

When a boat hull displaces water it produces antigravity. This antigravity has a center of antigravity called the center of buoyancy just like the mass and cargo of the boat has a center of mass. The antigravity force pushes upward, the gravity force pushes downward. If the boat is inclined at an angle, usually the antigravity force with the gravity force create a couple to right the boat. The relation of the center of gravity and the center of buoyancy determines how much side force, say on a sail can be tolerated without the boat capsizing.

12 Infinite Compound Interest

Let P be the principal, r the yearly interest, and A the current value. Then after one year, for simple interest

$$A = (1 + r)P$$

Suppose the interest is compounded twice a year. Then the half year interest rate is $r/2$, so at the end of the year we have

$$A = (1 + r/2)^2 P$$

Likewise if the amount is compounded n times per year we have

$$A = (1 + r/n)^n P$$

at the end of the year. What happens if we compound infinitely often? This is given by

$$A = \lim_{n\to\infty} (1 + r/n)^n P.$$

So let us compute

$$\lim_{n\to\infty} (1 + r/n)^n.$$

It is convenient to set $x = 1/n$ then the limit becomes

$$\lim_{x\to 0} (1 + rx)^{1/x}.$$

Let us consider the logarithm of our expression

$$f(x) = (1 + rx)^{1/x}$$

. Let
$$g(x) = \ln(f(x) = \ln((1+rx)^{1/x}) = \frac{\ln(1+rx)}{x}$$
As $x \to 0$ both numerator and denominator go to 0, and we get the indeterminant form 0/0. By L' Hospital's rule, in this case we may compute the limit as the limit of the derivative of the numerator divided by the derivative of the denominator. Thus

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{r}{1+rx} = r$$

So the limit of $g(x)$ is
$$\ln(f(x)) = r.$$
Thus the limit of $f(x)$ as $x \to 0$ is

$$e^r,$$

where $e = exp(1) = 2.718281828459045$. Most accounts are compounded daily, so $n = 365$. This is a large number, so the yearly amount is quite close to
$$A = e^r P.$$
The Annual Percentage Yield (APY) is then

$$e^r - 1$$

which is the effective yearly interest rate. So for example if the interest rate is $r = 5$ percent, then the APY is

$$e^{.05} - 1 = 0.05127109637602$$

or approximately 5.127 percent.

13 The Numerical Calculation of Integrals

One of the simplest ways of approximating an integral is to approximate the integral with many trapezoids. This is called the trapezoid method. It is not very accurate. But when combined with Richardson extrapolation, it becomes the Romberg method and is very accurate.
Trapezoid Method Program.

```
// trapezoid.c
#include <stdio.h>
#include <math.h>
double f(double);
double trapez(double (*fp)(double),double,double,int);
void main(){
 double a,b,v;
 int n,i;
 a=0.;
 b=4.;
 for(n=3;n < 1000000;n=2*n+1 ){
  v=trapez(f,a,b,n);
  printf(" v = %21.15g n= %d \n",v,n);
 }
 v= sin(b)-b*cos(b);
 printf(" sin(b)-b*cos(b) = %21.15g \n",v);
}
//c+ trapez trapezoid integration
double trapez(double (*fp)(double) ,double a,double b,int n){
//parameters
//    f-external function to be integrated
//  a,b-integration interval
//    n-interval divided into n-1 pieces
//    v-value returned for integral
 double x,y,v,h;
 int i;
 for(i=1,v=0.;i<=n;i++){
  x=(i-1)*(b-a)/(n-1)+a;
  y=(*fp)(x);
  if((i == 1)||(i == n)){
   y=y/2;
  }
  v=v+y;
 }
 h=(b-a)/(n-1);
 v*=h;
 return v;
 }
 //c+ f  function
 double f(double x){
  double v;
  v=x*sin(x);
  return v;
 }
```

Output of the Program.

```
 v =     0.609979726071014 n= 3
 v =      1.73154715222909 n= 7
 v =      1.83479212300388 n= 15
 v =      1.85277521009651 n= 31
 v =      1.85660247170371 n= 63
 v =      1.85748883881977 n= 127
 v =      1.85770231246565 n= 255
```

```
v =      1.85775470565523 n= 511
v =      1.85776768442241 n= 1023
v =      1.85777091431825 n= 2047
v =       1.8577717199517 n= 4095
v =      1.85777192113055 n= 8191
v =      1.85777197139663 n= 16383
v =      1.85777198395957 n= 32767
v =      1.85777198709985 n= 65535
v =      1.85777198788485 n= 131071
v =      1.85777198808109 n= 262143
v =      1.85777198813021 n= 524287
sin(b)-b*cos(b) =      1.85777198814652
```

Notice that to get good accuracy we had to take over 500000 steps with a very small step size. A very small step size often leads to roundoff error.

14 Explosions and Icebergs

There are frequent explosions where the air contains very small particles that can burn, such as at grain elevators. Why is that? When a body is heated or cooled, the amount of heat transferred is proportional to the surface area of the body. The temperature change of a body for a given supply of heat is inversely proportional to the mass of the body, that is, to the volume of the body. So the rate of temperature change of a heated or cooled body is proportional to the ratio of the surface area to the volume. This ratio for a spherical body is

$$\frac{4\pi r^2}{4\pi r^3/3}.$$

Therefore the heating rate is proportional to $1/r$. If a particle is very small, r is small and $1/r$ is large. So the body is very quickly heated to its combustion temperature. An explosion occurs. On the other hand, for a very large object, such as an iceberg, the heating rate is very slow. So an iceberg or a glacier can exist for a very long time.

15 The Rate of Chemical Reactions: We Are Chemical Machines.

Chemical reactions between two molecules occur when they collide and have sufficient energy to react. So in the simple case the reaction rate is proportional to the product of the concentrations of the two reactants multiplied by a rate constant. The rate constant is usually an exponential function of the temperature. Suppose a molecule with concentration A reacts with a molecule having concentration B to produce a new molecule with concentration C. Then the differential equation for the concentration of C might be

$$\frac{dC}{dt} = kAB,$$

where k is a rate constant, that typically depends on the temperature, faster reactions for higher temperature. At higher temperatures there are more molecules that have enough energy to react. So for a chemical system we will have a system of differential equations similar to this one above for each of the molecular concentrations.

A very complex such system is the metabolism in our bodies. Others concern the concentrations of enzymes and drugs in our blood system.

16 The Motion of an Electron in a Cathode Ray Tube

The Lorenz force on a moving charged particle such as an electron is a sum of an electric force and a magnetic force. The electric force is in the direction of the electric field. The magnetic force is perpendicular to the particle velocity and to the magnetic field.

The Lorentz force is the sum of the electric and magnetic forces

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

So to solve for the path of the electron we must solve a system of differential equations, which are obtained by setting the particle acceleration equal to the Lorenz force.

In a television CRT the electrons are accelerated toward the screen by the electric field $\mathbf{E}$ due to a very high voltage. The deflection of the electrons, vertical and horizontal, are caused by inductive windings on the neck of the CRT, creating a magnetic field $\mathbf{B}$. In the case of a uniform magnetic field, the electrons are forced to move in circles. So as the electron passes through the region of the coils they move in short arcs, thereby changing their direction.

17 Effective AC Voltage

In an alternating current circuit, the power given to a resistive load is given by i^2R where i is the instantaneous current

$$i = I\sin(\omega t).$$

The average power is

$$P_{ave} = \frac{1}{T} I^2 R \int_0^T \sin^2(\omega t)dt,$$

where the period is $T = \frac{2\pi}{\omega}$. Evaluating

$$\frac{1}{T}\int_0^T \sin^2(\omega t)dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin^2(u)du$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-\cos(2u)}{2}du$$

$$= \frac{1}{2},$$

after the change of variable $u = \omega t$. So the average power is

$$P_{ave} = \frac{I^2R}{2}$$

Therefore the effective constant current is

$$I_{eff} = \frac{I}{\sqrt{2}},$$

where I is the peak current. The effective constant voltage is similarly

$$V_{eff} = \frac{V}{\sqrt{2}}.$$

Thus the common AC line voltage of 110 volts corresponds to a peak voltage of

$$110\sqrt{2} = 155.56\text{v}olts.$$

18 Power Factor in Electric Power Transmission

The power taken by a load in an AC circuit is proportional to the average power taken over a cycle, which is the integral of the product of the voltage by the current divided by the period T. If a load is purely inductive, then the voltage and current are 90 degrees out of phase, so the integral will be zero. The amount of power taken by the load is proportional to $\cos(\phi)$, where ϕ is the phase angle difference between the voltage and current. This $\cos(\phi)$ proportionality is easily shown by looking at the average of the instantaneous power. This $\cos(\phi)$ is called the power factor. The instantaneous power is the product of the current and voltage. Large electric motors tend to have a large value of ϕ because the load is very inductive and are said to have a lagging power factor, because the current lags the voltage by ϕ. Capacitors can be added to the load to decrease ϕ, raise the power factor, and increase electrical efficiency. It does this by reducing the power line losses.

19 Spherical Mass

A spherical mass, for gravitational purposes can be treated as a point mass located at the center of the sphere. This can be established using a calculus argument. This is one of the calculations Newton had to do for his theory of gravity applied to the motion of the planets. Newton had to invent calculus in order to complete his theory.

20 Motion of Planets, Kepler's Laws

Kepler's three laws of planetary motion are:

1. **Planets move in elliptical orbits around the Sun with the Sun at a focus.**

2. **A line joining the Sun to the planet sweeps out equal areas in equal times.**

3. **The period of the orbit is proportional to the $3/2$ power of the major diameter of the ellipse.**

Kepler found these laws by observation. Newton proved the laws using calculus and his theory of gravitational force.

In the case of a body of mass m orbiting a body of mass M such as the sun, Newton's law is the following differential equation

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GmM\mathbf{r}}{r^3.}$$

Orbits are usually calculated by solving differential equations of this sort numerically with a computer.

21 The Deterministic Motion of All Particles and Things

All the particles of the universe are acted on by forces. These forces in turn are due to particles. Using these known forces there are differential equations for their motion obtained by setting the acceleration to the net force on each particle. Knowing at any one time the position and velocity of every particle in the universe, these differential equations determine all future motions and positions of all the particles in the universe. Thus the future of the universe, and of humans has been determined. Thus calculus determines everything. He who knows calculus is king.

According to Heisenberg's uncertainty principle, there is some uncertainty in measuring both position and velocity of a particle. So philosophers debate whether free will has been restored.