# Cave Model

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# 1 Model

The model consists of a chain of hexahedrons (polyhedrons with six faces). This chain contains basic hexahedrons defined by cave measurements, and transition hexahedrons joining pairs of basic hexahedrons.

# 2 Basic Hexahedron

A basic hexahedron is defined by a center point $C$, an axis unit vector $u_a$, a vertical unit vector $u_v$ (which is usually vertical, but could be off vertical), an up distance to the ceiling $d_u$, a down distance to the floor $d_d$, a forward distance $d_f$, a backward distance $d_b$, a right distance $d_r$ ( distance to the right wall), and a left distance $d_l$. A right unit vector is defined by

$$u_r = u_a \times u_v.$$

The eight points of the hexahedron are defined as

$$p_1 = c - d_b u_a - d_d u_v - d_l u_r,$$

$$p_2 = p_1 + (d_l + d_r)u_r,$$
$$p_3 = p_2 + (d_u + d_d)u_v,$$
$$p_4 = p_1 + (d_u + d_d)u_v,$$
$$p_5 = p_1 + (d_f + d_b)u_a,$$
$$p_6 = p_2 + (d_f + d_b)u_a,$$
$$p_7 = p_3 + (d_f + d_b)u_a,$$
$$p_8 = p_4 + (d_f + d_b)u_a.$$

So the back face has vertices

$$p_1, p_2, p_3, p_4,$$

the front face

$$p_5, p_6.p_7, p_8,$$

the bottom face

$$p_1, p_2.p_6, p_5,$$

the top face

$$p_4, p_3 . p_7, p_8,$$

the right face

$$p_2, p_3 . p_7, p_6,$$

and the the left face is

$$p_1, p_4 . p_8, p_5.$$

We triangulate the hexahedron as follows with 12 oriented triangles so that outward normals are given by the right hand rule. We have for the two triangles of the back face

$$p_1, p_3, p_4$$

and

$$p_2, p_3 p_1.$$

For the top

$$p_3, p_7, p_4,$$

and

$$p_4, p_7, p_8.$$

For the left face

$$p_1, p_8, p_5,$$

and

$$p_4, p_8, p_1.$$

For the right face

$$p_2, p_6, p_7,$$

and

$$p_3, p_2, p_7.$$

For the bottom face

$$p_1, p_5, p_6,$$

and

$$p_2, p_1, p_6.$$

For the front face

$$p_5, p_8, p_7,$$

and

$$p_6, p_5, p_7.$$

The axial direction of the tunnel is specified by the unit vector $u_a$, which is defined by succesive centers of basic hexahedrons. That is if $C_i$ is the center of the $i$th basic hexahedron and $c_{i+1}$ the center of the next basic hexahedron, then the axial unit vector for the $i$th hexahedron is

$$a_u = \frac{C_{i+1} - C_i}{\|C_{i+1} - C_i\|}$$

Letting

$$d = \|C_{i+1} - C_i\|$$

we might use as the forward distance $d_f$ of the $i$th hexahedron some fraction of $d$, say $d_f = \alpha d$, where $\alpha = 1/4$. These shortened basic hexahedrons are then joined by transition hexahedrons.

# 3 Transition Hexahedron

A transiton hexahedron joining the $i$th hexahedron to the $i+1$ hexahedron is constructed by using the four vertices of the front face of hexahedron $i$ as the back face of the transition hexahedron, and the four vertices of the back face of hexahedron $i+1$ as the front face of the transition hexahedron. Then the transition hexahedron is constructed from these eight vertices in the same manner of the basic hexahedron construction.

# 4 A Complete Polyhedron Model of the Cave

We can use the chain of hexahedrons to produce a polyhedral model of the entire cave by removing the triangles of the mating front and back faces of the hexahedrons in the chain. However, for the purposes of water level computation int will probably be easier to retain the chain of hexahedrons.

## 5 Clipping the Model With A Plane to Model Water Level

To clip a hexahedron with a plane we intersect all edges with the plane getting new vertices of a clipped face. We also retain the vertices below the plane. We triangulate the new clipped face and get a polyhedral model of the volume containing water. This will require a bit of work. Triangulating the clipped face will be made easier by the fact that the hexahedrons are convex and hence the clipped face will be convex.

## 6 Curl, Divergence, Gradient

The curl of a vector field $\mathbf{A}$ in cartesian coordinates is

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{i}$$

$$- \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \mathbf{j}$$

$$\left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{k}.$$

The divergence of a vector field $\mathbf{A}$ in cartesian coordinates is

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

## 7 Stokes's Theorem, The Divergence Theorem, Green's Theorem

If a surface $S$ has bounding curve $\partial S$, Stokes theorem is

$$\int_S \nabla \times \mathbf{A} \cdot \mathbf{n} ds = \int_{\partial S} \mathbf{A} \cdot d\mathbf{r},$$

which allows a surface integral to be evaluated as a line integral around the boundary of the surface. The surface normal is $\mathbf{n}$.

The divergence theorem allows a volume integral to be evaluated as a surface integral. Let $V$ be a volume and $\partial V$ be it enclosing surface. Then

$$\int_V \nabla \cdot \mathbf{A} dv = \int_{\partial V} \mathbf{A} \cdot \mathbf{n} ds.$$

**Green's Theorem in the Plane.**

If $S$ is an area in the $x, y$ plane then a special case of Stokes Theorem gives

$$\int_S \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) ds$$

$$= \int_{\partial S} \mathbf{A} \cdot d\mathbf{r}$$

$$= \int_{\partial S} (A_x dx + A_y dy)$$

$$= \int \mathbf{r} \cdot d\mathbf{r},$$

where

$$\mathbf{r}(t) = A_x(t)\mathbf{i} + A_y(t)\mathbf{j}$$

is the boundary curve bounding this area $A$. This is called Green's Theorem in the Plane.

**The Area**. As an application of Green's Theorem we can find the area enclosed by a curve by evaluating a line integral around the curve. So let $A_x = -y/2$ and $A_y = x/2$, then

$$\int_S ds =$$

$$\int_S \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) ds$$

$$= \int_{\partial S} (A_x dx + A_y dy).$$

$$= (1/2) \int_{\partial S} (-y dx + x dy).$$

**Area Example**. Let the region $S$ be bounded by the curve

$$\mathbf{r} = r\cos(t)\mathbf{i} + r\sin(t)\mathbf{j},$$

for

$$0 \le t \le 2\pi.$$

Then

$$dx = -r\sin(t)dt$$

$$dy = r\cos(t)dt.$$

Then the area $\alpha$ is

$$\alpha = (1/2)\int_{\partial S}(-ydx + xdy).$$

$$= \frac{r^2}{2}\int_0^{2\pi}(\sin^2(t) + \cos^2(t))dt = \frac{r^2}{2}2\pi = r^2\pi.$$

**The Center of Mass** . Now let us find the $x$ coordinate of the center of mass of a region $S$ bounded by the curve $\mathbf{r}(t)$. Let

$$A_x = 0,$$

and

$$A_y = \frac{x^2}{2}.$$

Then

$$\int_S xds$$

$$= \int_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) ds$$

$$= \int_{\partial S}(A_x dx + A_y dy)$$

$$= \int_{\partial S}\frac{x^2}{2}dy.$$

So the center of mass $x$ coordinate is

$$x_{cm} = \frac{1}{\alpha}\int_{\partial S}\frac{x^2}{2}dy,$$

where $\alpha$ is the area of region $S$. Similarly using

$$A_x = -\frac{y^2}{2},$$

and

$$A_y = 0,$$

we find

$$y_{cm} = -\frac{1}{\alpha} \int_{\partial S} \frac{y^2}{2} dx.$$

**A Center of Mass Example** . Let the area be the right half circle of radius $r$. Let the area be bounded by the curve

$$\mathbf{r} = r\cos(t)\mathbf{i} + r\sin(t)\mathbf{j},$$

for

$$-\pi/2 \leq t \leq 2\pi,$$

and by the straight line from $(0, r)$ to $(0, -r)$.

Now

$$dy = r\cos(t)dt,$$

So

$$\int_{\partial S} \frac{x^2}{2} dy$$

$$= \int_{-\pi/2}^{\pi/2} \frac{r^3}{2} \cos^3(t)dt + \int_{-r}^{r} 0dy = \frac{2}{3} r^3.$$

Hence the $x$ coordinate of the center of mass is

$$x_{cg} = \frac{(2r^3/3)}{\pi r^2/2} = \frac{4r}{3\pi}$$

**The Area Moment of Inertia** . Letting

$$A_x = 0$$

and

$$A_y = x^3/3$$

We get for the moment of inertia about the $y$ axis

$$I_y = \int_S x^2 ds$$

$$= \int_S \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} ds$$

$$= \int_{\partial S} A_x dx + A_y dy$$

$$= \int_{\partial S} (x^3/3)dy$$

**Moment of Inertia Example** . Let the area be a circle of radius $r$. Let the area be bounded by the curve

$$\mathbf{r} = r\cos(t)\mathbf{i} + r\sin(t)\mathbf{j},$$

for

$$0 \le t \le 2\pi.$$

Then

$$x = (r\cos(t))^3$$

and

$$dy = rcos(t)dt$$

So

$$I_y = r^4 \int_0^{2\pi} \cos^4(t)/3dt$$

$$= \frac{\pi r^4}{4}.$$

By the parallel axis theorem, the moment of inertia about an axis through the center of mass may be obtained from the moment of inertia about a parallel axis at a distance $d$ from the center of mass axis.

These formulas allow us to compute areas, centers of mass, and moments of inertia for areas bounded by piecewise defined curves.

# 8 Volume as a Surface Integral

A volume integral can be changed to a surface integral by using the divergence theorem. Define a vector field

$$H = \frac{r}{3} = \frac{1}{3}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Let $B$ be a set and $\partial B$ its bounding surface. Then the volume of $B$ is

$$\int_B dV = \int_B \nabla \cdot H dV$$

$$= \int_{\partial B} H \cdot n dS$$

$$= \frac{1}{3} \int_{\partial B} (x n_x + y n_y + z n_z) dS$$

Note that the integrand is the distance from the origin to the plane containing the area element $dS$. In particular for any plane area, say $A$, the volume of the pyramid, with vertex at the origin and base $A$, is

$$V = \frac{hA}{3},$$

where $h = n \cdot R$, is the height of the pyramid. The volume of a polyhedron consisting of $k$ polygons, is therefore

$$V = \sum_{i=1}^{k} A_i \frac{P_i \cdot n_i}{3},$$

where $A_i$ is the area of the $i$th polygon, $P_i$ is some point on the polygon, and $n_i$ is the outward normal to the polygon. The divergence theorem can also be used to compute moments. For example, applying the divergence theorem to the vector field

$$G = \frac{1}{2} \begin{bmatrix} x^2 \\ 1 \\ 1 \end{bmatrix},$$

gives a surface calculation of the $x$ moment.

$$m_x = \frac{1}{2} \int_{\partial B} x^2 n_x dS.$$

We shall calculate polygon areas. Let a closed curve $\gamma$ have position vector $\mathbf{r}(t), 0 \leq t \leq 1$. The area enclosed by the curve is

$$\mathbf{A} = \frac{1}{2} \oint_{\gamma} \mathbf{r} \times d\mathbf{r}.$$

This of course is a vector. It is the magnetic moment of a circuit with unit current. When the curve lies in a plane, the magnitude of this integral is equal to the area enclosed by the curve. This is obvious from the definition of the cross product when the plane passes through the origin. When the plane does not pass through the origin, the vector $\mathbf{r}$ can be written as a sum of a constant vector normal to the plane, and a vector that lies in the plane. The constant vector is everywhere normal to the line element, and so does not make a contribution to the integral. This case reduces to the former case.

As an aside, this calculation can be used to find the inner region enclosed by a plane curve. In the case that the curve lies in the $xy$ plane, the vector integral will be in the $z$ direction. If the component is positive then the inner region is to the left. This is obvious if the curve is the unit circle with the counterclockwise orientation. For a general proof, note that a curve which bounds an inner region can be continuously deformed to a unit circle in such a way that the area never vanishes. Thus the sign of the area must be maintained.

If the sign is negative then the inner region lies to the right. The curve direction is defined by the increasing parameter value.

Returning to the polygon area problem, the integral over a line segment is

$$\mathbf{A} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{2},$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the starting and ending points. This is true because the line element has the same direction as both $\mathbf{r}_2 - \mathbf{r}_1$ and $\mathbf{r} - \mathbf{r}_1$. The cross product of parallel vectors is zero, so the integral is

$$\frac{1}{2}\mathbf{r_1} \times \int d\mathbf{r}$$

$$= \frac{1}{2}\mathbf{r_1} \times (\mathbf{r_2} - \mathbf{r_1})$$

$$= \frac{1}{2}\mathbf{r_1} \times \mathbf{r_2}.$$

Thus the area of a polygon $A_i$ with $m$ sides equals the magnitude of

$$\frac{1}{2}\sum_{j=1}^{m} \mathbf{r}_j \times \mathbf{r}_{j+1}.$$

This sum is clearly normal to the polygon, since each term is. A unit normal is obtained by dividing by the area $A_i$. The number of cross products that need to be calculated can be reduced. For example for a triangle with $m = 3$ we have

$$(\mathbf{r}_2 - \mathbf{r}_3) \times (\mathbf{r}_3 - \mathbf{r}_2)$$

$$= \mathbf{r}_2 \times \mathbf{r}_3 - \mathbf{r}_2 \times \mathbf{r}_2 - \mathbf{r}_1 \times \mathbf{r}_3 + \mathbf{r}_1 \times \mathbf{r}_2$$

$$= \mathbf{r}_1 \times \mathbf{r}_2 - \mathbf{r}_2 \times \mathbf{r}_3 + \mathbf{r}_3 \times \mathbf{r}_1.$$

As another example we may decompose a hexagon into 4 triangles and write the area as

$$\frac{1}{2}[\sum_{i=1}^{3}(\mathbf{r}_{i+1} - \mathbf{r}_i) \times (\mathbf{r}_{i+2} - \mathbf{r}_{i+1}) + (\mathbf{r}_3 - \mathbf{r}_1) \times (\mathbf{r}_5 - \mathbf{r}_3)].$$

# 9 Bifurcation

It is probably not worth the effort to attempt to make a single model including bifurcations. So when we come to a fork in the road, the model adds a new branch model.

# 10 Flow of Water

# 11 The Program Compass

The compass program allows one to create cave models, do calculations, and generate graphic views of the cave.

# 12 Carroll Cave

# 13 Bibliography


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