Chaos

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1 Determinism

The problem of causality and determinism is an ancient topic in philosophy.

Laplace strongly believed in causal determinism. Here is a quote from his **Essay on Probabilities** of 1819:

"We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes. "

This intellect is often referred to as Laplace's demon. Note, however, that the description of the hypothetical intellect described above by Laplace as a demon does not come from Laplace, but from later biographers: Laplace saw himself as a scientist; and while hoping that humanity would progress to a better scientific understanding of the world, he recognized that, if and when such an understanding were eventually completed, a tremendous calculating power would still be needed to compute it all in a single instant. While Laplace saw foremost practical problems for mankind to reach this ultimate stage of knowledge and computation, later interpretations of quantum mechanics, which were adopted by philosophers defending the existence of free will, also leave the theoretical possibility of such an "intellect" contested.

2 Dynamical system

The Lorenz attractor is an example of a non-linear dynamical system. Studying this system helped give rise to Chaos theory.The dynamical system concept is a mathematical formalization for any fixed "rule" which describes the time dependence of a point's position in its ambient space. Examples include the mathematical models that describe the swinging of a clock pendulum, the flow of water in a pipe, and the number of fish each spring in a lake.

A dynamical system has a state determined by a collection of real numbers, or more generally by a set of points in an appropriate state space. Small changes in the state of the system correspond to small changes in the numbers. The numbers are also the coordinates of a geometrical spacea manifold. The evolution rule of the dynamical system is a fixed rule that describes what future states follow from the current state. The rule is deterministic: for a given time interval only one future state follows from the current state.

3 Differential Equations

A higher order differential equation may be put into the form of a first order system of differential equations

$$\frac{d\mathbf{x}}{dt} = f(t, \mathbf{x}),$$

where $\mathbf{x}$ is a vector. Given certain conditions on the function $f(t, \mathbf{x})$, there exists a unique local solution specified by an initial value of the vector x. Hence given an initial point the evolution of the solution over time is completely determined. However, an ever so slight change in the initial conditions may lead to large changes in the solution, which may appear to be chaotic and undergo strange oscillations.

4 Linear and Non-Linear Systems

An differential equation is called linear if x and its derivatives of all orders appear linearly in the equation. Thus

$$(t-2)^2 \frac{d^3x}{dt^3} + 5\sin(t)\frac{dx}{dt} = e^{-t^2}$$

is a linear differential equation, but

$$(t-2)^2\frac{d^3x}{dt^3} + 5\sin(t)(\frac{dx}{dt})^2 = e^{-t^2},$$

is not because the first derivative of x is squared. Differential equations that exhibit chaotic solutions are usually nonlinear.

5 Deterministic Chaos

Although chaotic systems appear to behave randomly, they are actually deterministic. So chaos theory is actually deterministic chaos.

6 The Lorenz Attractor

(Source Wikipedia)

Edward Norton Lorenz (May 23, 1917 - April 16, 2008) was an American mathematician and meteorologist, and a pioneer of chaos theory. He discovered the strange attractor notion and coined the term butterfly effect. Lorenz in around 1960 had a primitive vacuum tube computer in his office at MIT and created a simple model of atmospheric convective flow that he solved with this computer. He discovered this strange behavior of his iteration using this computer.

The attractor itself, and the equations from which it is derived, were introduced by Edward Lorenz in 1963, who derived it from the simplified equations of convection rolls arising in the equations of the atmosphere.

From a technical standpoint, the system is nonlinear, three-dimensional and deterministic. In 2001 it was proven by Warwick Tucker that for a certain set of parameters the system exhibits chaotic behavior and displays what is today called a strange attractor. The equations that govern the Lorenz attractor are:

$$\frac{dx}{dt} = \sigma(y-x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$

where σ is called the Prandtl number and ρ is called the Rayleigh number. All $\sigma, \rho, \beta >$, but usually $\sigma = 10$, $\beta = 8/3$ and ρ is varied.

The source code to simulate the Lorenz attractor in GNU Octave follows.

```
## Lorenz Attractor equations solved by ODE Solve
## x' = sigma*(y-x)
## y' = x*(rho - z) - y
## z' = x*y - beta*z
function dx = lorenzatt(X,T)
    rho = 28; sigma = 10; beta = 8/3;
    dx = zeros(3,1);
    dx(1) = sigma*(X(2) - X(1));
    dx(2) = X(1)*(rho - X(3)) - X(2);
    dx(3) = X(1)*X(2) - beta*X(3);
    return
end

## Using LSODE to solve the ODE system.
clear all
close all
lsode_options("absolute tolerance",1e-3)
lsode_options("relative tolerance",1e-4)
t = linspace(0,25,1e3); X0 = [0,1,1.05];
[X,T,MSG]=lsode(@lorenzatt,X0,t);
T
MSG
plot3(X(:,1),X(:,2),X(:,3))
view(45,45)
```

Lorenz attractor (Wikipedia).

7 The Dripping Faucet

8 The Weather

9 Stability

10 The Logistic Map

$$x_{n+1} = ax_n(1 - x_n).$$

See Holden Page 43.
Bifurcation and period doubling.

11 Limit Cycles

Many solutions to dynamical systems approach a closed cyclical curve.

12 Chaos in Chemical Reactions.

The Belousov-Zhabotinskii reaction in chemistry. Reference:
Ordinary Differential Equations, Oscillating Chemical Reactions, and Chaos , Niki Kittur, Modified by Roger Ahn.
emails: adkittur@princeton.edu and rogerahn@princeton.edu
http://www.cs.princeton.edu/courses/archive/fall00/cs323/assign/ass3/ass3.pdf
You Tube Oscillating Chemical Reaction (Belousov-Zhabotinsky)
You Tube Briggs-Rauscher Oscillating Reaction
You Tube Raction oscillante
Wikipedia http://en.wikipedia.org/wiki/Belousov-Zhabotinsky_reaction

13 Catastrophe theory

In mathematics, catastrophe theory is a branch of bifurcation theory in the study of dynamical systems; it is also a particular special case of more general singularity theory in geometry.

Bifurcation theory studies and classifies phenomena characterized by sudden shifts in behavior arising from small changes in circumstances, analysing how the qualitative nature of equation solutions depends on the parameters that appear in the equation. This may lead to sudden and dramatic changes, for example the unpredictable timing and magnitude of a landslide.

Catastrophe theory, which originated with the work of the French mathematician Ren Thom in the 1960s, and became very popular due to the efforts of Christopher Zeeman in the 1970s, considers the special case where the long-run stable equilibrium can be identified with the minimum of a smooth, well-defined potential function (Lyapunov function).

Small changes in certain parameters of a nonlinear system can cause equilibria to appear or disappear, or to change from attracting to repelling and vice versa, leading to large and sudden changes of the behavior of the system. However, examined in a larger parameter space, catastrophe theory reveals that such bifurcation points tend to occur as part of well-defined qualitative geometrical structures.

14 Complexity Theory

Complexity Theory is rather loosely defined. It is what they do at the Santa Fe Institute, it is related to chaos theory, evolutionary biology, complex behavior due to simple causes. See the pictures of the players in the Lewin book: Murry Gell-Mann, Cris Langton, Steven Jay Gould, Richard Dawkins, Daniel Dennett, and so on.

15 Fractals

The classic fractal, the Mandelbroot set. is generated by a quadratic iteration in complex space. So is a member of the class of Julia sets.

16 Sun Spots

17 The Reversal of the Magnetic Field of the Earth

There exist physical models that mimic this magnetic reversal in a chaotic manner.

18 Taylor-Couette Flow

The Taylor-Couette flow consists of a viscous fluid confined in the gap between two rotating cylinders. For low angular velocities, measured by the Reynolds number Re, the flow is steady and purely azimuthal. This laminar basic state is known as circular Couette flow, after Maurice Marie Alfred Couette who used this experimental device as a means to measure viscosity. Sir Geoffrey Ingram Taylor investigated the stability of the Couette flow in a ground-breaking paper which has been a cornerstone in the development of hydrodynamic instability theory [1]. He showed that when the angular velocity of the inner cylinder is increased above a certain threshold, Couette flow becomes unstable and a secondary steady state characterized by axisymmetric toroidal vortices, known as Taylor vortex flow, emerges. Subsequently increasing the angular speed of the cylinder the system undergoes a progression of instabilities which lead to states with greater spatio-temporal complexity,with the next state being called as Wavy Vortex Flow.If the two cylinders rotate in opposite sense then Spiral Vortex Flow arises.Beyond a certain Reynold's Number there is the onset of turbulence. Circular Couette Flow has wide applicatons ranging from desalination to Magneto-Hydrodynamics and also in viscosimetric analysis. Furthermore, when the liquid is allowed to flow in the annular space of two rotating cylinders along with the application of a pressure gradient then a flow called Taylor-Dean Flow arises.[

19 Bibliography

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[3]Gleick James, **Chaos: making a New Science**, Penguin, 1987.
[4]Holden Arun V. (Editor), **Chaos**, Princeton, 1986.
[5]Beltrami Edward, **Mathematics For Dynamic Modeling**, Academic press, 1987.
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[7]Lewin Roger, **Complexity**, Macmillan, 1992.
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[9]Strogatz Steven H., **Nonlinear Dynamics and Chaos, With applications to Physics, Biology, Chemistry, and Engineering**, Perseus Books, 1998.
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