# Complex Analysis

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## Contents

## 1 A Complex Function of a Complex Variable

We write a complex function as

$$f(z) = w,$$

where $z$ and $w$ are complex numbers

$$z = x + iy$$

$$w = u + iv.$$

$$f(x+yi) = u(x+iy) + v(x+iy)i = u(x,y) + iv(x,y)$$

where $u(x,y)$ and $v(x,y)$ can be considered real valued functions.

## 2 The Complex Derivative

The derivative is defined as

$$\frac{df}{dz} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Notice that for the derivative to exist the limits must be equal as $z$ approaches $z_0$ from all directions.

So if the derivative exists, then holding $y = y_0$, we have

$$\frac{df}{dz} = \lim_{x \to x_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{x \to x_0} \frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + i \lim_{x \to x_0} \frac{v(x,y_0) - v(x_0,y_0)}{x - x_0}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Similarly holding $x = x_0$, we have

$$\frac{df}{dz} = \lim_{y \to y_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{y \to y_0} \frac{u(x_0,y) - u(x_0,y_0)}{i(y - y_0)} + i \lim_{y \to y_0} \frac{v(x_0,y) - v(x_0,y_0)}{i(y - y_0)}$$

$$= \frac{1}{i}\left[\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}\right] = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Functions that have a derivative in an open region are called analytic. They then have derivatives of all orders, and thus have Taylor power series expansions.

## 3 The Cauchy-Riemann Equations

If the derivative of

$$f(x+yi) = u(x,y) + iv(x,y)$$

exists, by considering limits as $z = x + iy_0$ goes to $z_0$, and $z = x_0 + iy$ goes to $z_0$, as in the previous section, by equating the two expressions for the derivative, we find that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

## 4 The Real and Imaginary Parts of Complex Analytic Functions Solve Laplace's Equation in Two Dimensions

Differentiating the Cauchy-Riemann equations partially we find that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

## 5 The Elementary Functions

Define $\sin(z)$ , $\cos(z)$, $\sinh(z)$ , $\cosh(z)$ and $\exp(z)$ by the usual Tayor series:

$$\sin(z) = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - ...$$

$$\cos(z) = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - ...$$

$$\sinh(z) = z + \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + ...$$

$$\cosh(z) = 1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + ...$$

$$\exp(z) = 1 + \frac{1}{1!}z^1 + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + ...$$

Then

$$\exp(iz) = 1 + i\frac{1}{1!}z - \frac{1}{2!}z^2 - i\frac{1}{3!}z^3 + ...$$
$$= \cos(z) + i\sin(z).$$

We also have

$$\sin(iz) = i\sinh(z)$$
$$\sinh(iz) = i\sin(z)$$
$$\cos(iz) = \cosh(z)$$
$$\cosh(iz) = \cos(z).$$

If $z = x + iy$ then

$$\sin(z) = \sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy)$$
$$= \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

$$\cos(z) = (\cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy)$$
$$= \cos(x)\cosh(y) - i\sin(x)\sin(y).$$

# 6 Laurent Series

If $f$ is analytic in $r_1 < |z - z_0| < r_2$ then

$$f(z) = \sum_{n=-\infty}^{\infty} A_n(z - z_0)^n,$$

where

$$A_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}dz.$$

# 7 The Residue Theorem

At an isolated singular point

$$A_{-1} = \frac{1}{2\pi i} \int_C f(z) dz.$$

So that

$$\int_C f(z) dz = 2\pi i A_{-1}$$

$A_{-1}$ is called the residue of $f(z)$ at the isolated singular point $z_0$.

# 8 Calculating Residues

Let $f(z)$ have a simple pole of order $n$ at $z_0$. Then

$$\phi(z) = (z - z_0)^n f(z),$$

is analytic in a neighborhood of $z_0$. Then the $n - 1$ coefficient of the Taylor expansion of $\phi(z)$ is

$$A_{-1} = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}\phi(z)}{dz^{n-1}}.$$

# 9 The Inversion of the Laplace Transform

We define the Fourier transform as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt.$$

The Fourier inversion theorem is

$$f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega.$$

The double sided Laplace transform is

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt.$$

Let $s = \phi + i\omega$. Then $F(s)$ is the Fourier transform of $g_\phi(t) = f(t)e^{-\phi t}$, that is

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} f(t)e^{-\phi t}e^{-i\omega t}dt \\ &= \hat{g}_\phi(\omega). \end{aligned}$$

Formally applying the Fourier inversion theorem, we have

$$f(t)e^{-\phi t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}_\phi(\omega)e^{i\omega t}d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{i\omega t}d\omega.$$

Then

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{\phi t}e^{i\omega t}d\omega. \\ &= \frac{1}{2\pi i}\int_{C_\phi} F(s)e^{st}ds, \end{aligned}$$

where $C_\phi$ is the Bromwich contour defined by

$$\{\phi + i\omega : -\infty < \omega < \infty\}.$$

Note that $i$ appears in the expression $2\pi i$ because

$$ds = id\omega.$$

In general we will find that if we define a closed curve consisting of a finite line of length $2R$ on the bromwich contour, and a semicircle of radius $R$ to the left, then as $R$ goes to infinity, the integral over the semicircle goes to zero, so that the total integral over the curve is equal to the integral on the Bromwich line, which is thus equal to $2\pi i$ times the residues of $F(s)e^{st}$ in the left halfspace bounded by the contour. Our inversion expression is therefore equal to the sum of the residues themselves. We get the single sided Laplace transform from the double when $f(t)$ is equal to zero for $t \leq 0$.

**Example:** Consider

$$F(s) = \frac{1}{s-1},$$

for $\Re(s) > 1$. The residue of $F(s)e^{st}$ is

$$\lim_{s\to 1}(s-1)F(s)e^{st} = e^t.$$

Therefore

$$f(t) = e^t.$$

**Example:** Consider

$$F(s) = \frac{1}{s^2+1} = \frac{1}{(s-i)(s+i)},$$

for $\Re(s) > 0$. The residues of $F(s)e^{st}$ are

$$\lim_{s \to i}(s-i)F(s)e^{st} = \frac{e^{it}}{2i},$$

and

$$\lim_{s \to -i}(s+i)F(s)e^{st} = \frac{e^{-it}}{-2i},$$

Therefore

$$f(t) = \frac{e^{it} - e^{-it}}{2i} = \sin(t).$$