Continued Fractions

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11/6/2010

Contents

1 The Continued Fraction Representation of a Rational Number

A continued fraction is a fraction written in a form like

$$\frac{p}{q} = a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \frac{b_4}{a_4 + \frac{b_5}{a_5}}}},$$

where the a_i's and the b_i's are integers.

Any rational number $p/q > 1$ can be written in a form like

$$\frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{a_5}}}},$$

where the b's are all equal to 1. This is called a simple continued fraction. This is the same as

$$\frac{p}{q} = (a_1 + 1/(a_2 + 1/(a_3 + 1/(a_4 + 1/(a_5))))).$$

We write this in abreviated form as

$$\frac{p}{q} = a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\frac{1}{a_4+}\frac{1}{a_5},$$

and in an even more abreviated form as

$$[a_1, a_2, a_3, a_4, a_5].$$

For example consider finding the continued fraction representation of the fraction

$$\frac{22}{9}.$$

Dividing 22 by 9 we get 2 with a remainder of 4, so

$$\frac{22}{9} = 2 + \frac{4}{9} = 2 + \frac{1}{9/4}.$$

Dividing 9 by 4 we get

$$\frac{9}{4} = 2 + \frac{1}{4}.$$

So

$$\frac{22}{9} = 2 + \frac{4}{9} = 2 + \frac{1}{9/4} = 2 + \frac{1}{2 + \frac{1}{4}} = [2, 2, 4].$$

With the same repeated division we find for example that

$$\frac{75948}{24175} = [3, 7, 15, 1, 212, 1].$$

An infinite continued fraction goes on forever,

$$[a_1, a_2, a_3, a_4, a_5, ...].$$

In that case the sequence of continued fractions

$$\{[a_1], [a_1, a_2], [a_1, a_2, a_3], [a_1, a_2, a_3, a_4], ...\}$$

is called the sequence of partial quotients. The partial quotients are also called convergents. So the convergents are

$$[a_1] = a_1,$$

$$[a_1, a_2] = a_1 + \frac{1}{a_2},$$

$$[a_1, a_2, a_3] = a_1 + \frac{1}{a_2 + \frac{1}{a_3}},$$

and so on.

It turns out that such an infinite simple continued fraction converges to some real number, which is the value of the infinite continued fraction. This value is defined as the limit of the sequence of convergents.

In the case of expanding an irrational number into an infinite continued fraction

$$[a_1, a_2, a_3, a_4, a_5, ...],$$

the convergents can be used as rational approximations to the irrational number. For example π can be written as a partial fraction as

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, ...]$$

We will show that the convergents will converge to the value of the infinite continued fraction. Note that the second partial quotient for π is

$$[3, 7] = \frac{22}{7},$$

which is often used as an approximation to π.

We now show how to compute the partial quotients in an iterative way. For arbitrary integers p and q, let us write

$$p = a_1 q + r_1,$$

where $a_1 = [p/q]$ and r_1 is less than q. The brackets represent the greatest integer function. Then

$$\frac{p}{q} = a_1 + \frac{r_1}{q} = a_1 + \frac{1}{\frac{q}{r_1}},$$

where

$$\frac{q}{r_1} > 1.$$

This can be continued as follows

$$q = a_2 r_1 + r_2, \frac{q}{r_1} = a_2 + \frac{r_2}{r_1}, r_2 < q$$

$$r_1 = a_3 r_2 + r_3, \frac{r_1}{r_2} = a_2 + \frac{r_3}{r_2}, r_3 < r_2$$

$$\ldots$$

$$r_n = a_{n+2} r_{n+1} + r_{n+2}, \frac{r_n}{r_{n+1}} = a_{n+2} + \frac{r_{n+2}}{r_{n+1}}, r_{n+2} < r_{n+1},$$

until finally the remainder is zero. This algorithm for computing the partial quotients of a continued fraction is equivalent to the Euclidian algorithm for finding the GCD of integers p and q. The program **cf.java** computes the quotients of the continued expansion of a rational number.

```
//cf.java, continued fraction expansion of a rational number. 12/6/07
//
class cf {
  public static void main(String args[]) {
    int cnv[];
    int p;
    int q;
    int r;
    int a;
    int b;
    int c;
    int n;
    int gcd;
    int sign;
    cnv = new int[50];
    n=0;
    if(args.length <= 1){
     System.out.println(" Continued fraction expansion of  p/q " );
     System.out.println(" Usage: java cv p q " );
     return;
    }
    if(args.length > 1){
     //System.out.println(" The first argument is " + args[0]);
     p=Integer.parseInt(args[0]);
     //System.out.println(" The second argument is " + args[1]);
     q=Integer.parseInt(args[1]);
     sign=1;
     if(q < 0){
      sign=sign*(-1);
```

```
 q=-q;
}
if(p < 0){
 sign=sign*(-1);
 p=-p;
}
System.out.println(" The partial quotients of " + sign*p +"/" + q + " are:");
//p = 101;
//q = 59;
a=p;
b=q;
gcd=q;
// p/q = a + r/q
for(int i=1;i<50;i++){
 c = a/b;
 r= a-c*b;
 // a/b = c + r/b
 //System.out.println(" c = " + c + " r = " + r);
 a=b;
 b=r;
 cnv[n]=c;
 n++;
 if(r == 0)break;
 gcd=r;
}
System.out.print(" [");
for(int i=0;i<n;i++){
 if(i != (n-1)){
 System.out.print(sign*cnv[i] + " ");
 }
 else{
  System.out.println(sign*cnv[i] + "]");
 }
}
if(cnv[n-1] > 1 || n > 1){
 System.out.println(" or");
 System.out.print(" [");
 if(cnv[n-1] > 1){
  cnv[n-1]=cnv[n-1]-1;
  cnv[n]=1;
  n++;
 }
 else{
  n--;
  cnv[n-1]=cnv[n-1]+1;
 }
 for(int i=0;i<n;i++){
  if(i != (n-1)){
   System.out.print(sign*cnv[i] + " ");
  }
  else{
   System.out.println(sign*cnv[i] + "]");
  }
 }
}
```

```
      System.out.println(" gcd=" + gcd);
    }
  }
}
```

2 The Continued Fraction Representation of a Real Number

Suppose x is a real positive number. We shall compute the continued fraction for x. We shall later show that the the convergents of the continued fraction converge to x,

$$x = [a_1, a_2, a_3, ...]$$

Let us write

$$x_0 = x$$

$$x_1 = 1/(x_0 - a_1),$$

where

$$a_1 = [x_0],$$

is the greatest integer in x_0. So that

$$x_0 = a_1 + 1/x_1.$$

Next we expand x_1 in the same way, and continuing in this way, we find

$$x_n = 1/(x_{n-1} - a_n)$$

So that

$$x_{n-1} = a_n + 1/x_n.$$

If we ignore

$$1/x_n,$$

then we get the nth convergent of the continued fraction representation of x,

$$[a_1, a_2, a_3, ..., a_n],$$

which in general is an approximation to x. If x is irrational, the continued fraction will be infinite, for otherwise x would be rational.

Examples.

$$\pi = 3.14159265358979323846264338327 95...$$

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, ...]$$

$$e = 2.71828182845904523536028747135 27...$$

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, ...]$$

The program **cfxr.java** computes the quotients of the continued fraction expansion of a real number.

```
//cfxr.java, continued fraction expansion of a real number. 12/9/07
//
class cfxr {
  public static void main(String args[]) {
    int cnv[];

    int a1;
    int n;
    double x,x0,x1;
    cnv = new int[50];
    n=0;
    if(args.length <= 1){
     System.out.println(" Continued fraction expansion of a real number  x " );
     System.out.println(" x is the real number > 0, n is the number of desired quotients. " );
     System.out.println(" Usage: java cv x n " );
     return;
    }
    if(args.length > 1){
     //System.out.println(" The first argument is " + args[0]);
     x=Double.parseDouble(args[0]);
     //System.out.println(" The second argument is " + args[1]);
     n=Integer.parseInt(args[1]);


     System.out.println(" The partial quotients of " + x + " are:");
//   algorithm:
//   a_{i+1} = [x_i], i=0,1,2,3,4,..., where [x_i] is the greatest integer in x_i
//   x0=x
//   x_1 = 1/(x_0-a_1)
//   So x_0 = a_1 + 1/x_1
//   ........
//   x_{n+1} = 1/(x_n - a_{n+1})
//   So x_n = a_{n+1} + 1/x_{n+1}
//
//   pi=3.14159265358979323846264338327 95
//   quotients of pi= 3 7 15 1 292 1 1 1 2 1 3 1 14 2 1 1 2 2 2 2 1 84 2
//   e=2.71828182845904523536028747135 27
//   quotients of e= 2 1 2 1 1 4 1 1 6 1 1 8 1 1 10 1 1 12 .....
     x0=x;
```

```
      for(int i=0;i<n;i++){
       a1=(int)x0;
       x1=1/(x0-a1);
       cnv[i]=a1;
       x0=x1;
      }
      System.out.print(" [");
      for(int i=0;i<n;i++){
       if(i != (n-1)){
        System.out.print(cnv[i] + " ");
       }
       else{
        System.out.println(cnv[i] + "]");
       }
      }

    }
  }
}
```

3 Computing the Convergents

Define

$$p_0 = 1, q_0 = 0, p_1 = a_1, q_1 = 1.$$

The first convergent is

$$c_1 = a_1 = \frac{p_1}{q_1}.$$

The second convergent is

$$c_2 = a_1 + \frac{1}{a_2} = \frac{a_2 a_1 + 1}{a_2} = \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0} = \frac{p_2}{q_2},$$

where

$$p_2 = a_2 a_1 + 1, q_2 = a_2.$$

The third convergent is

$$\frac{p_3}{q_3} = c_3 = a_1 + \frac{1}{a_2 + \frac{1}{a_3}}$$

$$= a_1 + \frac{a_3}{a_2 a_3 + 1}$$

$$= \frac{a_1(a_2 a_3 + 1) + a_3)}{a_2 a_3 + 1}$$

$$= \frac{a_3(a_2a_1+1)+a_1)}{a_2a_3+1}$$
$$= \frac{a_3p_2+p_1}{a_3q_2+q_1}.$$

Theorem. The nth convergent c_n of a simple continued fraction is given by

$$c_n = \frac{p_n}{q_n},$$

where p_n and q_n are defined iteratively by

$$p_n = a_n p_{n-1} + p_{n-2},$$

and

$$q_n = a_n q_{n-1} + q_{n-2},$$

with initial values

$$p_0 = 1, q_0 = 0, p_1 = a_1, q_1 = 1.$$

Proof.

Assume that for all $k < n$ the kth convergent is

$$c_k = \frac{p_k}{q_k},$$

where

$$p_k = a_k p_{k-1} + p_{k-2},$$

and

$$q_k = a_k q_{k-1} + q_{k-2}.$$

Then write

$$c_n = [a_1, a_2, ..., a_n] = [a_1, a_2, ..., a_{n-1} + (1/a_n)] = c'_{n-1},$$

which is the $n-1$ convergent of the partial fraction with only the $n-1$ quotient changed to

$$a'_{n-1} = a_{n-1} + (1/a_n).$$

So we can write

$$\frac{p'_{n-1}}{q'_{n-1}} = \frac{a'_{n-1}p_{n-2} + p_{n-3}}{a'_{n-1}q_{n-2} + q_{n-3}}$$

$$= \frac{(a_{n-1} + 1/a_n)p_{n-2} + p_{n-3}}{(a_{n-1} + 1/a_n)q_{n-2} + q_{n-3}}$$

$$= \frac{(a_n a_{n-1} + 1)p_{n-2} + a_n p_{n-3}}{(a_n a_{n-1} + 1)q_{n-2} + a_n q_{n-3}}.$$

The numerator is

$$(a_n a_{n-1} + 1)p_{n-2} + a_n p_{n-3}$$

$$= a_n(a_{n-1}p_{n-2} + p_{n-3}) + p_{n-3}$$

$$= a_n p_{n-1} + p_{n-2}.$$

Similarly, the denominator is

$$a_n q_{n-1} + q_{n-2}.$$

Hence

$$p_n = a_n p_{n-1} + p_{n-2},$$

and

$$q_n = a_n q_{n-1} + q_{n-2}.$$

The program **convergents.java** compute the convergents of a continued fraction.

```
//convergents.java, convergents of a continued fraction. 12/7/07
class convergents {
 public static void main(String args[]) {
  int cnv[];
  int a;
  int a1=1;
  int a2=1;
  int p1=1;
  int p2=1;
  int p3=1;
  int q1=1;
  int q2=1;
  int q3=1;
  int n;
  int k;
  double v;
  cnv = new int[50];
  n=args.length;
  if(args.length < 1){
   System.out.println(" Convergents of a continued fraction " );
   System.out.println(" Usage: java convergents a1 a2 a3 a4 .... " );
   return;
  }
  for(int i=0;i<n;i++){
   cnv[i]=Integer.parseInt(args[i]);
```

```
}
for(int i=0;i<n;i++){
 a=cnv[i];
 k=i+1;
 if(i == 0){
  a1=a;
  p1=a1;
  q1=1;
  v=a1;
  System.out.print(" Convergent " + k);
  System.out.print(" [");
  for(int j=0;j<k;j++){
   if(j != (k-1)){
    System.out.print(cnv[j] + " ");
   }
   else{
    System.out.print(cnv[j] + "] ");
   }
  }
  System.out.println("= " + p1 + "/" + q1+ " = " + v);
 }
 if(i == 1){
  a2=a;
  p2=a2*a1+1;
  q2=a2;
  v=p2;
  v=v/q2;
  System.out.print(" Convergent " + k);
  System.out.print(" [");
  for(int j=0;j<k;j++){
   if(j != (k-1)){
    System.out.print(cnv[j] + " ");
   }
   else{
    System.out.print(cnv[j] + "] ");
   }
  }
  System.out.println("= " + p2 + "/" + q2 + " = " + v);
 }
 if(i > 1){
  p3=a*p2+p1;
  q3=a*q2+q1;
  v=p3;
  v=v/q3;
  System.out.print(" Convergent " + k);
  System.out.print(" [");
  for(int j=0;j<k;j++){
   if(j != (k-1)){
    System.out.print(cnv[j] + " ");
   }
   else{
    System.out.print(cnv[j] + "] ");
   }
  }
  System.out.println("= " + p3 + "/" + q3 + " = " + v);
```

```
      p1=p2;
      p2=p3;
      q1=q2;
      q2=q3;
    }
  }
 }
}
```

4 Properties of the Convergents

Theorem. For all n

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^n.$$

Proof. We shall prove this by induction. So first we have

$$p_1 = a_1, q_1 = 1,$$

$$p_2 = a_2 a_1 + 1, q_2 = a_2.$$

Thus

$$p_2 q_1 - q_2 p_1 = a_2 a_1 + 1 - a_2 a_1 = 1 = (-1)^2.$$

Now

$$p_n q_{n-1} - q_n p_{n-1}$$
$$= (a_n p_{n-1} + p_{n-2}) q_{n-1} - (a_n q_{n-1} + q_{n-2}) p_{n-1}$$
$$p_{n-2} q_{n-1} - q_{n-2} p_{n-1} = -(-1)^{n-1} = (-1)^n.$$

Theorem. Converents are in lowest terms.
Proof. We have

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^n.$$

So if there is an integer $k > 1$ that divides both p_n and q_n, then it must divide $(-1)^n$, which it can't do.
Theorem. The convergents c_n and c_{n-1} satisfy

$$c_n - c_{n-1} = \frac{(-1)^n}{q_n q_{n-1}}.$$

Proof.

$$c_n - c_{n-1} = \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}$$

$$= \frac{p_n q_{n-1} - q_n p_{n-1}}{q_n q_{n-1}}$$
$$= \frac{(-1)^n}{q_n q_{n-1}}.$$

Theorem. The convergents c_n and c_{n-2} satisfy

$$c_n - c_{n-2} = \frac{a_n(-1)^{n-1}}{q_n q_{n-2}}.$$

Proof.

$$c_n - c_{n-2} = \frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}}$$

$$= \frac{p_n q_{n-2} - q_n p_{n-2}}{q_n q_{n-2}}$$
$$= \frac{(a_n p_{n-1} - p_{n-2})q_{n-2} - (a_n q_{n-1} - q_{n-2})q_{n-2}}{q_n q_{n-2}}$$
$$= \frac{a_n(p_{n-1}q_{n-2} - q_{n-1}p_{n-2})}{q_n q_{n-2}}$$
$$= \frac{a_n(-1)^{n-1}}{q_n q_{n-2}}.$$

Theorem. Every even convergent is greater than its preceeding convergent.

$$c_{2n} > c_{2n-1}.$$

Proof. We have

$$c_{2n} - c_{2n-1} = \frac{(-1)^{2n}}{q_{2n} q_{2n-1}} > 0.$$

Theorem. The even convergents form a decreasing sequence.

$$c_{2n} < c_{2n-2}.$$

Proof. We have

$$c_{2n} - c_{2n-2} = \frac{a_n(-1)^{2n-1}}{q_{2n} q_{2n-2}} = \frac{-1}{q_{2n} q_{2n-2}} < 0.$$

Theorem. The odd convergents form an increasing sequence.

$$c_{2n+1} > c_{2n-1}.$$

Proof. We have

$$c_{2n+1} - c_{2n-1} = \frac{a_n(-1)^{2n}}{q_{2n+1}q_{2n-1}} = \frac{1}{q_{2n+1}q_{2n-1}} > 0.$$

Theorem. Each odd convergent is less than each even convergent.
Proof. Let c_{2k-1} be an odd convergent and c_{2j} be an even convergent. Let n be greater than both k and j. We have

$$c_{2n-1} < c_{2n}.$$

Because the odd converbents are increasing

$$c_{2k-1} < c_{2n-1}.$$

Because the even convergents are decreasing

$$c_{2n} < c_{2j-1}.$$

So

$$c_{2k-1} < c_{2n-1} < c_{2n} < c_{2j}.$$

The odd convergents are an increasing sequence bounded above, so they converge to some number ℓ_o. Likewise the even convergents are a decrasing sequence bounded below so converge to a number ℓ_e.
Theorem. The sequence of convergents of an infinite simple continued fraction converges.
Proof. We have

$$c_{2n} - c_{2n-1} = \frac{(-1)^2}{q_{2n}q_{2n-1}} = \frac{1}{q_{2n}q_{2n-1}}.$$

The iterative formula for the q_k show that they increase without limit. Therefore

$$\lim_{n\to\infty}(c_{2n} - c_{2n-1}) = 0.$$

So

$$\ell_o = \ell_e.$$

Theorem. If $[a_1, a_2, ...]$ is the continued fraction expansion for x, then the convergents of $[a_1, a_2, ...]$ converge to x.
Proof. See Olds.
Theorem. A continued fraction has repeating quotients iff it is a quadratic surd (a root of a quadratic equation with integer cefficients).
Proof. See Olds.

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