Differential Forms

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1 Introduction

Differential Forms and the Exterior Differential Calculus, as devised by Elie Cartan at the beginning of the 20th century, gives rigor and meaning to the manipulation of differentials. These differentials in the form of infinitesimals have been around since the beginning days of the invention of calculus. They are still widely used in applications of mathematics. Differential forms also generalize the techniques of vector analysis. Specifically, it generalizes those

Stokes-like theorems about line integrals, surface integrals, and volume integrals. For a taste of this generalization that is not too spicy, see the Oswald Wyler work in the bibliography. A simple introductory treatment of differential forms is in **Calculus on Manifolds** by Spivak. Walter Rudin also has an introduction to differential forms pp 208-226 in **Principles of Mathematical Analysis**. This might be the best place to get a view of what this subject is all about. He treats integration over chains, where the chains are based on the simplex. Spivak integrates over chains based on cubes or n-cells. Rudin does not mention the tensor character of differential forms, but introduces them as formal wedge products of differential one-forms, which are duals of tangent vectors. Basically one decomposes (triangulates) a manifold into a chain (a simplicial complex). A piece of surface (n-volume in general) of the manifold is an image of a standard simplex. The integral of this piece with respect to a differential form, which is of the same dimension as the simplex, is obtained by integrating the function times the product of the coordinate differentials in the coordinate domain, where one uses the Jacobian map for scaling. Alternately this may be regarded as a transformation of tensor components. The boundary operator ∂ maps the simplex to its "edge" simplices, which lowers the order of the simplex. On the other hand, the exterior derivative operator d maps the differential form to a higher order form. Thus we have Stokes theorem for integrals of differential forms over chains, i.e., simplicial complexes.

$$\int_{\partial c} \omega = \int_{c} d\omega.$$

This theory then is related to the homology theory of algebraic topology, and to the cohomology of differential forms (DeRahm Cohomology Groups). So this is rather deep in its generality, and allows integrating on higher dimensional abstract differential manifolds. The formalism can be relaxed to allow calculation techniques that are more easily remembered then are the manipulations of Vector Analysis. Or so it is claimed. I am a little skeptical about this. So for example consider the classical Stokes Theorem. Let the differential form for a line integral be

$$\omega = f_1 dx + f_2 dy + f_3 dz.$$

We use

$$df_1 = \frac{\partial f_1}{\partial x} dx + \frac{\partial f_1}{\partial y} dy + \frac{\partial f_1}{\partial z} dz$$

etc. And we use properties like, $dx \wedge dy = -dy \wedge dx$. Then we arrive at an expression for the exterior derivative

$$d\omega = (\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y})dx \wedge dy$$

$$+(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z})dx \wedge dz$$

$$+(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z})dy \wedge dz.$$

We can reorder this as

$$d\omega = (\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z})dy \wedge dz$$

$$-(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z})dz \wedge dx$$

$$+(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y})dx \wedge dy.$$

If we take the differentials dx, dy, dz to correspond to the coordinate vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and $dy \wedge dz$ to correspond to the cross product $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and so on, then this is the curl of f. So doing things like taking, $dx \wedge dy = dxdy$, and letting this be the normal element of surface $d\mathbf{S}$, and projecting to the xy plane, we get the classical Stokes theorem, which in vector analysis notation is

$$\int_C \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial C} \mathbf{F} \cdot d\mathbf{L},$$

where $d\mathbf{L}$ is the differential line element. Starting with the form

$$\omega = f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy,$$

we get the divergence theorem, which in vector analysis notation is

$$\int_C \nabla \cdot \mathbf{F} dV = \int_{\partial C} \mathbf{F} \cdot d\mathbf{S}.$$

For differential forms also see the books in the bibliography by Flanders, Singer and Thorpe, and Fleming.

2 Infinitesimals

Differentials were thought of originally as infinitesimals. These were supposed to be infinitely small quantities, whose ratios gave finite derivatives. The infinite sum of these infinetely small objects, appearing under an integral sign, magically produced a finite result. When the French analysts, Cauchy et al, introduced rigor into calculus, infinitesimals were more or less banned, but continued to be used in calculations, because they led to intuitive and correct solutions. There is a modern attempt to put infinitesimals back into the parlor. This is the work of Abraham Robinson called **Nonstandard Analysis**.

3 Tensor Products and Alternating Forms

An nth order tensor on a product space of n vector spaces is a multilinear functional on the space. This means that it is a linear function of each variable, whose value is a scalar, either a real or a complex number. If T is an m tensor on space U, and S is a n tensor on space V, and if u is in U and v is in V, then it is clear that

$$F(u, v) = T(u)S(v)$$

is a multilinear functional on space $U \times V$. So is a tensor of order $m + n$. A tensor of order one is just an element of the dual space of vector space V. A tensor for which

$$F(u_1, u_2, ..., u_i, ..., u_j...) = -F(u_1, u_2, ..., u_j, ..., u_i...),$$

that is for which interchanging variables changes the sign, is called alternating. The set of alternating tensors is written as $\Lambda^k(V)$. The determinant of order k of a vector space of dimension k operating on k column vectors is an alternating tensor. The product of two alternating tensors is not necessarily alternating. So there is introduced a product of two such vectors that is alternating. Given a tensor T , $\mathrm{Alt}(T)$ is defined like a determinant, being a sumation over the permutations of the arguments. It is defined as

$$\mathrm{Alt}(T)(v_1, v_2, ..., v_k) = \frac{1}{k!} \sum_{\sigma} \mathrm{sign}(\sigma) T(v_{\sigma(1)}, v_{\sigma(2)}, ..., v_{\sigma(k)}).$$

4 The Wedge Product

The wedge product is defined by applying the Alt function to a tensor product.

$$\omega \wedge \eta = \frac{(k+\ell)!}{k!\ell!} \mathrm{Alt}(\omega \otimes \eta).$$

5 Exterior Differential Calculus, Cartan

The exterior derivative of

$$\omega = \sum_{i_1 < ... < i_k} \omega_{i_1,...,i_k} dx_{i_1} \wedge ... \wedge dx_{i_k}$$

is

$$d\omega = \sum_{i_1 < ... < i_k} d\omega_{i_1,...,i_k} \wedge dx_{i_1} \wedge ... \wedge dx_{i_k}$$

6 Integration over Chains

A singular n-cube is a continuous mapping from a cube. A 1-cube is a mapping from an interval, thus is a curve. a 2-cube is a mapping from a square thus is a surface patch, et cetera. A chain is a sum of n-cubes with integer coefficients. Coefficients are usually 1 or -1. The boundary of an n-cube c written as ∂c is the set of (n-1)-cubes, consisting of the edges of the cube , the endpoints on the interval, or the faces of the 3-cube and so on. The integral of an n-form over an n-chain is the ordinary integral of the form

$$\int_c f(x_1, ..., x_k) dx_1 dx_2 ....dx_k$$

in the case where c is a real n-cube, that is, when the n-cube is the identity function. In general we use the coordinate system on the manifold to transform the integration region on the manifold back to real n-cubes in the euclidean space and so use the expression above for the integral.

7 Generalized Stokes Theorem

This theorem is

$$\int_c d\omega = \int_{\partial c} \omega,$$

where ω is a $(k-1)$ form, and c is a k chain.

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