Differential Geometry

James Emery

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1 Introduction

Differential Geometry is the study of curves and surfaces and their abstract generalization: the differential manifold. More generally it is the study of the calculus of curves and surfaces and involves definitions of curve tangents, normals, and curvature. In higher dimensions we study surfaces and manifolds and various kinds of curvature and so on. Some related subjects are Vector Analysis, Algebraic Geometry, Tensor Analysis, Differential Topology, Metrics and metric spaces, Riemanian Geometry, Differential Manifolds, and General Manifold Theory, and so on. Areas of physics that use differential geometry include General Relativity Theory, various subjects in Theoretical Physics, Mechanics, Particle theory, and String Theory.

2 Curvature and Elementary Differential Geometry

Curvature is the ratio of the change in turning to the distance traveled. Consider the circular path. As a point on the circle moves through an angle change $\Delta\theta$, it moves a distance $\Delta s = r\Delta\theta$. The ratio is a measure of the curvature

$$\frac{\Delta\theta}{\Delta s} = \frac{\Delta\theta}{r\Delta\theta} = \frac{1}{r}.$$

The angle change of the tangent $\Delta\phi$ is here equal to the angle change $\Delta\theta$, so we can use the tangent angle in our definition of the curvature. So suppose we are given a general curve in the plane

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}.$$

Suppose $dx/dt \neq 0$, then the angle of the tangent is

$$\phi = \tan^{-1}\left[\frac{dy/dt}{dx/dt}\right].$$

Let us write

$$\frac{dx}{dt} = \dot{x},$$
$$\frac{dy}{dt} = \dot{y},$$
$$\frac{d^2x}{dt^2} = \ddot{x},$$
$$\frac{d^2y}{dt^2} = \ddot{y}.$$

Then we have

$$\frac{d\phi}{dt} = \frac{1}{1+(\dot{y}/\dot{x})^2}\frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2} = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2+\dot{y}^2}.$$

We have

$$\frac{ds}{dt} = \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{\dot{x}^2+\dot{y}^2}$$

So the curvature κ is

$$\kappa = \frac{d\phi}{ds} = \frac{d\phi/dt}{ds/dt} = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2+\dot{y}^2}\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}.$$

$$= \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

We excluded the case where $dx/dt = 0$. However, we can just as well define the tangent angle as

$$\phi = \cot^{-1}\left[\frac{dx/dt}{dy/dt}\right].$$

If we carry out a similure derivation to the one above we shall find that we get the same formula for the curvature. Hence as long as at least one of dx/dt or dy/dt is not zero, the formula holds.

Suppose we have a function $y = f(x)$. This is a curve with $t = x$ as the parameter. Then $\dot{x} = 1$, $\ddot{x} = 0$, and the curvature formula reduces to

$$\kappa = \frac{d\phi}{ds} = \frac{\ddot{y}}{(1 + \dot{y}^2)^{3/2}},$$

where

$$\ddot{y} = \frac{d^2y}{dx^2},$$

and

$$\dot{y} = \frac{dy}{dx}.$$

Notice that in this formula the curvature can be either positive or negative. In the case of a 1-dimensional function the curvature determines if the function is concave up or down. In the more general definition in three space, the curvature is always positive. So the new curvature will be defined to be the absolute value of the 2d curvature defined above.

In three dimensional space there is no obvious tangent angle. So we must define curvature using another approch. Let

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

The velocity is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = dx/dt(t)\mathbf{i} + dy/dt(t)\mathbf{j} + dz/dt(t)\mathbf{k}.$$

The magnitude of the velocity is

$$v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \frac{ds}{dt}.$$

The unit tangent vector $\mathbf{T}$ is defined as

$$\mathbf{T} = \frac{\mathbf{v}}{v} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{d\mathbf{r}}{ds}.$$

$\mathbf{T}$ is a unit vector so

$$\mathbf{T} \cdot \mathbf{T} = 1.$$

We have

$$\frac{d(\mathbf{T} \cdot \mathbf{T})}{ds} = d\mathbf{T}/ds \cdot \mathbf{T} + \mathbf{T} \cdot d\mathbf{T}/ds = 0,$$

which implies that

$$2\mathbf{T} \cdot d\mathbf{T}/ds = 0.$$

So $\mathbf{T}$ and it derivative are orthogonal. Thus $d\mathbf{T}/ds$ is a vector normal to the curve. The unit normal vector $\mathbf{N}$ is defined as

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{\|d\mathbf{T}/ds\|}.$$

Thus

$$d\mathbf{T}/ds = \|d\mathbf{T}/ds\|\mathbf{N}.$$

We can define the curvature κ as

$$\kappa = \|d\mathbf{T}/ds\|,$$

because we can show that for a two dimensional curve this agrees with the two dimensional curvature.

Indeed, in two dimensions the unit tangent vector $\mathbf{T}$ can be written as

$$\mathbf{T} = \cos(\phi)\mathbf{i} + \sin(\phi)\mathbf{j},$$

where the tangent angle ϕ is a function of the arc length s. Then

$$d\mathbf{T}/ds = -\sin(\phi)(d\phi/ds)\mathbf{i} + \cos(\phi)(d\phi/ds)\mathbf{j}.$$

So

$$\|d\mathbf{T}/ds\| = |d\phi/ds|\sqrt{\sin^2(\phi) + \cos^2(\phi)} = |d\phi/ds| = \kappa.$$

Notice that here the curvature is always nonnegative. We can therefore write

$$d\mathbf{T}/ds = \kappa\mathbf{N}.$$

3 Frame Vectors

Let the curve $\mathbf{R}(s)$ in Euclidean 3-space be parameterized by arc length. Define the tangent vector by

$$\mathbf{T} = \frac{d\mathbf{R}}{ds},$$

and the curvature by

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|.$$

The normal vector is defined to be

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds},$$

and the binormal vector is defined by

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}.$$

The vectors $\mathbf{T}, \mathbf{N}$, and $\mathbf{B}$, are called the frame vectors.

4 The Serret-Frenet Formulas

There are a set of formulas relating the frame vectors $\mathbf{T}, \mathbf{N}$, and $\mathbf{B}$ and their derivatives.

The derivatives of the frame vectors with respect to arc length s are equal to linear combinations of the frame vectors themselves.

These are called the Serret-Frenet formulas. The Serret-Frenet formulas are derived from the facts that the frame vectors are mutually perpendicular, and that they have unit length. The dot product of any pair of frame vectors is zero. So the derivative of their dot product is also zero. Unit vectors are perpendicular to their derivatives, and $\mathbf{N}$ is a unit vector. So $d\mathbf{N}/ds$ is perpendicular to $\mathbf{N}$. Consequently $d\mathbf{N}/ds$ can be written as a linear combination of $\mathbf{T}$ and $\mathbf{B}$ only. Thus

$$\frac{d\mathbf{N}}{ds} = a_1\mathbf{T} + a_3\mathbf{B}.$$

Because $\mathbf{T}$ is perpendicular to $\mathbf{N}$,

$$a_1 = \frac{d\mathbf{N}}{ds} \cdot \mathbf{T} = -\mathbf{N} \cdot \frac{d\mathbf{T}}{ds}.$$

By definition the right hand expression is equal to $-\kappa$. So we conclude that a_1 is equal to the curvature κ.

Define the torsion τ to be a_3. Thus

$$\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}.$$

Let

$$\frac{d\mathbf{B}}{ds} = b_1\mathbf{T} + b_2\mathbf{N}.$$

$$b_1 = \frac{d\mathbf{B}}{ds} \cdot \mathbf{T} = -\mathbf{B} \cdot \frac{d\mathbf{T}}{ds} = -\mathbf{B} \cdot \mathbf{N} = 0.$$

Then

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\mathbf{B} \cdot \frac{d\mathbf{N}}{ds} = -\mathbf{B} \cdot (-\kappa\mathbf{T} + \tau\mathbf{B}) = -\tau.$$

Thus

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

Therefore we have the Serret-Frenet transformation,

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$

Notice that the matrix of the transformation is antisymmetric, and that the top right element of the matrix is zero. An antisymmetric matrix has a zero diagonal. The lower triangular part has negative elements. These facts might aid one in remembering the formula.

5 The Reflection of a Ray From a Surface Tangent Plane

Suppose the ray intersects the surface at a point $\mathbf{p}$. Let $\mathbf{n}$ be a unit normal to the tangent plane at point $\mathbf{p}$. Then $-\mathbf{n}$ is a second unit normal to the plane. It does not matter which of these normals we choose. Let the ray be represented by vector $\mathbf{a}$. The component of $\mathbf{a}$ in the direction of $\mathbf{n}$ is

$$\mathbf{b} = (\mathbf{n} \cdot \mathbf{a})\mathbf{n}.$$

Notice that if we use the other normal we get the same vector $\mathbf{b}$,

$$(-\mathbf{n} \cdot \mathbf{a})(-\mathbf{n}) = (\mathbf{n} \cdot \mathbf{a})\mathbf{n} = \mathbf{b}.$$

The vector $\mathbf{b}$ is actually the projection of vector $\mathbf{a}$ to the line orthogonal to the tangent plane.

We subtract this component from $\mathbf{a}$, getting

$$\mathbf{c} = \mathbf{a} - \mathbf{b},$$

which is the component of $\mathbf{a}$ orthogonal to $\mathbf{n}$. So

$$\mathbf{a} = \mathbf{b} + \mathbf{c}.$$

For the reflected ray, the component in the direction of $\mathbf{n}$ is the negative of the original component, but the orthogonal component is in the same direction as the original orthogonal component. Therefore the reflected ray is

$$\mathbf{d} = -\mathbf{b} + \mathbf{c} = \mathbf{c} - \mathbf{b}.$$

That is

$$\mathbf{d} = (\mathbf{a} - \mathbf{b}) - \mathbf{b} = \mathbf{a} - \mathbf{2b}.$$

As a check we see that

$$n \cdot d = n \cdot a - 2n \cdot b = -n \cdot a.$$

6 Curve Examples

6.1 The Circular Involute

The involute of a curve C is a curve generated by unwinding a string wrapped around C.

In the case of a circle of radius a centered at the origin, let t be the angle at the string tangent point on the circle, as the string is unwrapped to generate the involute. Then we find the parametric equation of the involute to be

$$x = a(\cos(t) + t\sin(t))$$

$$y = a(\sin(t) - t\cos(t)).$$

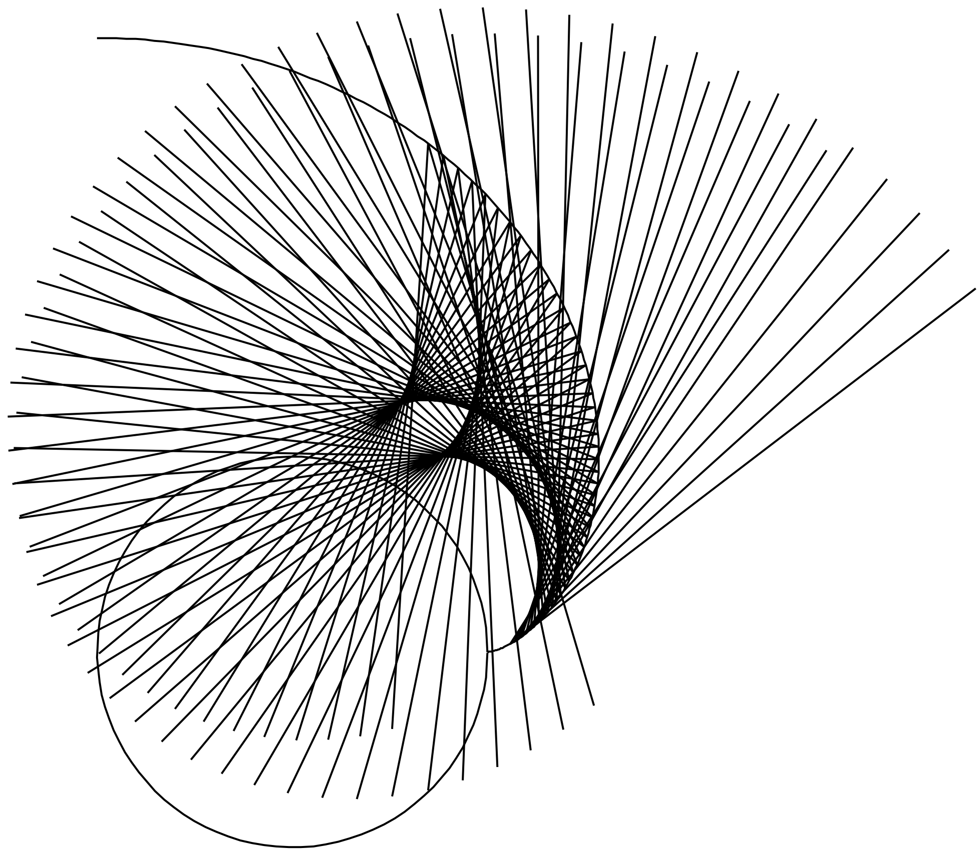

Figure 1: Two caustics (envelope curves of reflected rays) from the involute curve. The two curves are formed from two sets of parallel rays, the first making a zero angle with the horizontal, and the second making a twenty degree angle with the horizontal. The smaller caustic comes from the zero angle rays.

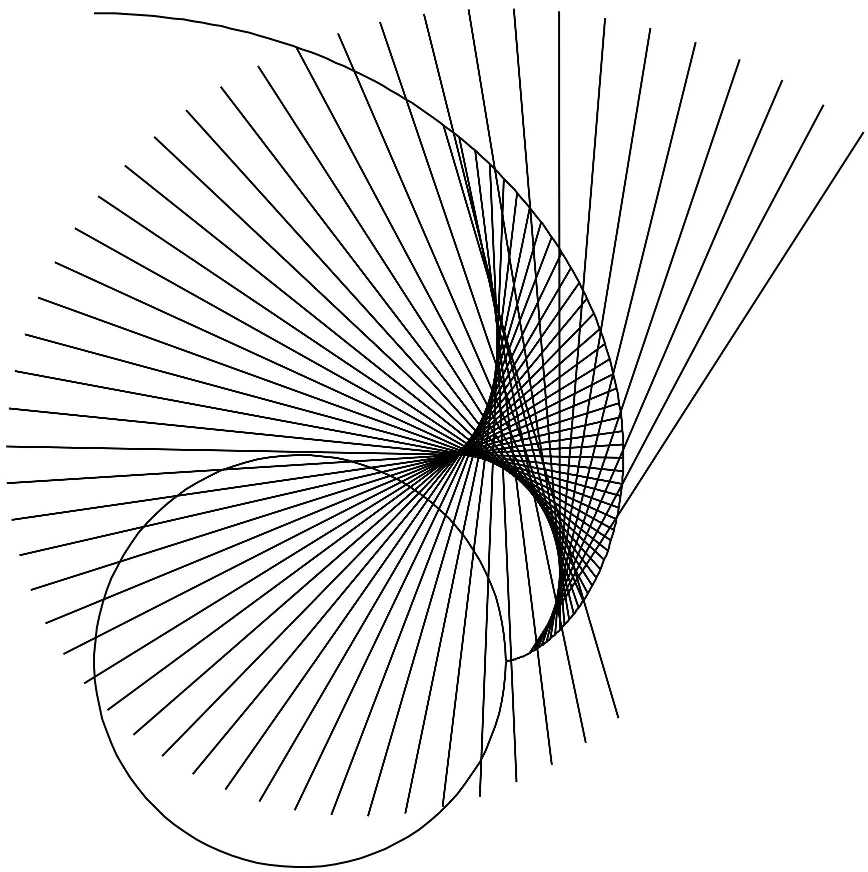

Figure 2: A caustic (envelope curve of reflected rays) from the involute curve. This curve is produced from reflections from a set of parallel rays making an angle of zero degrees with the horizontal axis. Notice that the cusp point occurs at a height equal to the height of the circle.

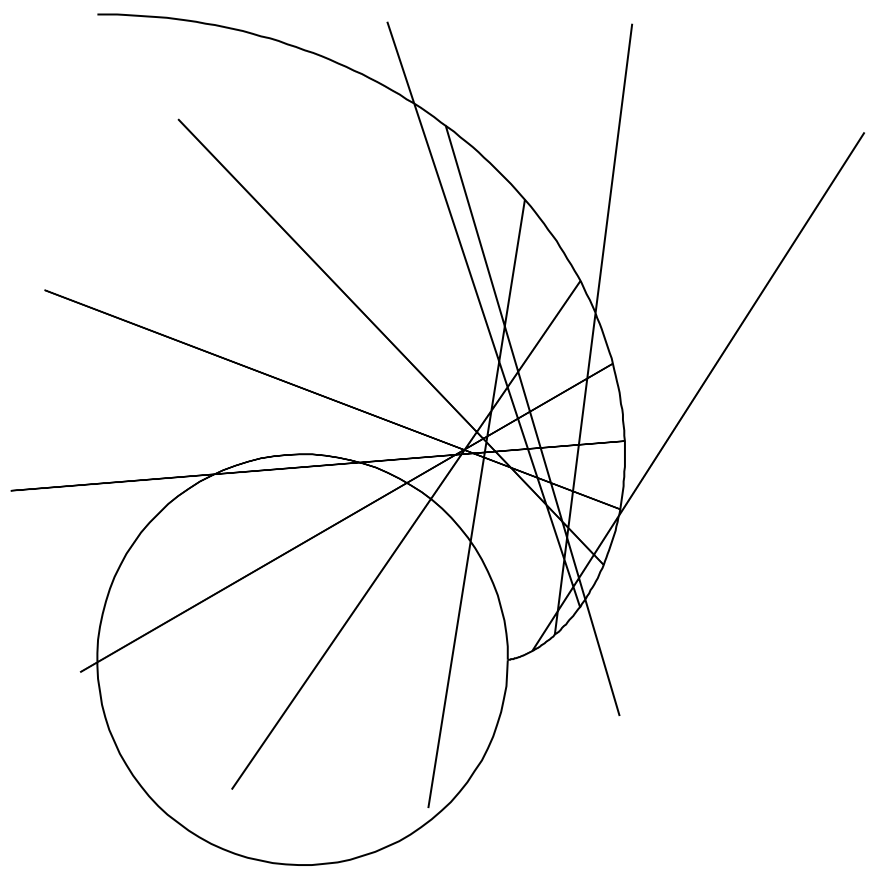

Figure 3: Reflections of parallel rays from the right at zero degrees from an involute curve.

The position vector of the curve is

$$\mathbf{R}(t) = x\mathbf{i} + y\mathbf{j}.$$

We have the derivatives

$$\dot{x} = \frac{dx}{dt} = r(-\sin(t) + \sin(t) + t\cos(t)) = rt\cos(t)$$

$$\dot{y} = \frac{dy}{dt} = r(\cos(t) - \cos(t) + t\sin(t)) = rt\sin(t)$$

$$\ddot{x} = \frac{d^2x}{dt^2} = r(\cos(t) - t\sin(t))$$

$$\ddot{y} = \frac{d^2y}{dt^2} = r(\sin(t) + t\cos(t)),$$

and the derivative of the arclength

$$\frac{ds}{dt} = \sqrt{\dot{x}^2 + \dot{y}^2} = rt$$

The tangent vector is

$$\mathbf{T} = \frac{d\mathbf{R}}{ds} = \frac{d\mathbf{R}}{dt}\frac{dt}{ds}$$
$$= \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$$

The normal vector is

$$\mathbf{N} = \kappa\frac{d\mathbf{T}}{ds} = \kappa\frac{d\mathbf{T}}{dt}\frac{dt}{ds}$$
$$= -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}.$$

```
c from involute.ftn
c+ function fx definition
      real*8 function  fx(t)
      implicit real*8(a-h,o-z)
       a=1.
       c=1.
       fx=a*(cos(t) + c*t*sin(t))
       return
      end

c+ function fy definition
      real*8 function  fy(t)
      implicit real*8(a-h,o-z)
       a=1.
       c=1.
       fy=a*(sin(t) - c*t*cos(t))
       return
      end
```

7 The Covariant Derivative in R^n

Given any curve α in R^n with $P = \alpha(0)$ and

$$\frac{d\alpha}{dt}(0) = X,$$

the covariant derivative of the vector field Y in the direction X at the point P is

$$\nabla_X Y(P) = \frac{d}{dt} Y(\alpha)(0).$$

The chain rule may be used to show that the definition does not depend on the curve α.

This definition may be applied to a two dimensional surface patch embedded in R^3. Let ϕ be a surface patch mapping an open set in R^2 to R^3. Suppose β is a curve in the domain of ϕ so that $\alpha(t) = \phi(\beta(t))$. Then

$$\begin{aligned} \nabla_X Y(P) = \frac{d}{dt} Y(\alpha) = \\ = \frac{dY(\phi(\beta(t)))}{dt} \\ \approx \frac{Y(\phi(\beta_+)) - Y(\phi(\beta_-))}{\triangle t}. \end{aligned}$$

This is a central difference approximation where

$$\beta_- = \beta(-\triangle t/2),$$

and

$$\beta_+ = \beta(\triangle t/2).$$

The tangent vector X is

$$\begin{aligned} X &= \frac{d\phi(\beta(t))}{dt} \\ &\approx \frac{\phi(\beta_+) - \phi(\beta_-)}{\triangle t}. \end{aligned}$$

Richardson extrapolation can be applied to these central difference approximations. An intrinsic covariant derivative may be defined on the patch, which is derived from the covariant derivative in R^3. This is accomplished by subtracting the normal component of the R^3 covariant derivative (see Hicks, p26).

8 The Weingarten Map, The Shape Operator

Let N be a unit normal vector field on the surface. The Weingarten map, sometimes called the shape operator, is the mapping from the tangent space at P to itself, defined by

$$W(X) = \nabla_X N(P),$$

where X is in the tangent space at P. The Weingarten map is linear and it is symmetric. To prove that it is symmetric, let $\phi(u, v)$ be a coordinate patch. Then ϕ_u and ϕ_v are linearly independent tangent vectors and so a basis for the tangent space. We have

$$\begin{aligned} W(\phi_u) \cdot \phi_v &= \nabla_{\phi_u} N(\phi) \cdot \phi_v \\ &= \frac{\partial N(\phi)}{\partial u} \cdot \phi_v \\ &= \frac{\partial N(\phi)}{\partial u} \cdot \frac{\partial \phi}{dv} \\ &= -N(\phi) \cdot (\frac{\partial}{\partial u} \frac{\partial \phi}{dv}) \\ &= -N(\phi) \cdot (\frac{\partial}{\partial v} \frac{\partial \phi}{du}) \\ &= \frac{\partial N(\phi)}{\partial v} \cdot \frac{\partial \phi}{du} \\ &= W(\phi_v) \cdot \phi_u. \end{aligned}$$

We have used the fact that N is perpendicular to the tangent space and hence to tangent vectors

$$\frac{\partial \phi}{\partial v},$$

and

$$\frac{\partial \phi}{\partial u}.$$

We will present a method for determining W numerically when surface patch values are available, but derivatives values are not. In addition we suppose that the surface normal is available (although it could be computed numerically from patch values).

Let β_1 be a curve in the uv coordinate space defined by

$$\beta_1(t) = (u_0, v_0) + tT_1,$$

where $T_1 = (1, 0)$ is a tangent vector in the uv coordinate space. Let $\alpha_1(t)$ be the curve $\phi(\beta_1(t))$. Let

$$X_1 = \frac{d\alpha_1(0)}{dt}.$$

Similarly define

$$X_2 = \frac{d\alpha_2(0)}{dt},$$

where the uv tangent vector is $T_2 = (0, 1)$. If Φ has nonzero Jacobian then X_1 and X_2 are independent vectors in the tangent space. X_1 and X_2 are not necessarily perpendicular to one another.

Let Z_1 and Z_2 be orthogonal vectors in the tangent space, which means that Z_1 and Z_2 are in R^3, and are perpendicular to N at P. Z_1 and Z_2 are a basis of the tangent space, so there exists b_{ij} so that

$$X_1 = b_{11}Z_1 + b_{12}Z_2$$

and

$$X_2 = b_{21}Z_1 + b_{22}Z_2.$$

We have

$$b_{ij} = \frac{X_i \cdot Z_j}{\|Z_i\|^2}.$$

Then using the linearity of W we get the vector equations

$$W(X_1) = b_{11}W(Z_1) + b_{12}W(Z_2)$$

and

$$W(X_2) = b_{21}W(Z_1) + b_{22}W(Z_2).$$

We may solve these equations for the x, y and z components of $W(Z_1)$ and $W(Z_2)$. Then let

$$W(Z_1) = c_{11}Z_1 + c_{12}Z_2$$

and

$$W(Z_2) = c_{21}Z_1 + c_{22}Z_2.$$

Z_1 and Z_2 are perpendicular, so we have

$$c_{ij} = \frac{Z_i \cdot W(Z_j)}{\|Z_i\|^2}.$$

The matrix representation of the Weingarten transformation in this basis is.

$$\begin{bmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{bmatrix}$$

9 Normal Curvature

Define the normal curvature of a surface in the direction of a curve α, at a point p, by

$$\kappa_n = -W(\alpha') \cdot \alpha',$$

where $a(0) = p$, and $\alpha' = \alpha'(0)$. This turns out to be the component of the curvature of the curve α in the direction of the surface normal, when α is parameterized by arc length. Let N be the surface normal restricted to α. Then N is a curve, and

$$W(\alpha') = \nabla_{\alpha'} N = \frac{dN(\alpha)}{ds} = N'.$$

We have

$$\alpha' \cdot N = 0.$$

So

$$\alpha'' \cdot N = -\alpha' \cdot N'.$$

But

$$\alpha'' = \frac{d\alpha'}{ds} = \frac{dt}{ds} = \kappa n.$$

So

$$\kappa_n = -N' \cdot \alpha' = \alpha'' \cdot N = \kappa n \cdot N = \kappa \cos(\psi),$$

where ψ is the angle between the surface normal N and the curve normal n. From the definition one sees that the normal curvature depends only upon the direction of the unit vector $t = \alpha'$. Hence notice that given two curves that pass through a given point p and have the same direction at that point have the same surface normal component of their curvature vector. Also if the curve lies in a plane which contains the surface normal, then the curve normal coincides with the surface normal so the normal curvature is the same as the curvature of the curve. If λ is an eigenvalue of W with eigenvector t then the normal curvature is

$$\kappa_n = -W(t) \cdot t = -\lambda t \cdot t = -\lambda.$$

The direction of t is called a principal direction. W is a symmetric transformation so the eigenvalues are real and the eigenvectors are orthogonal. The principal directions are well defined except at an umbilic point where the eigenvalues are equal and every vector is an eigenvector. The normal curvature is a quadratic form and the maximum and minimum values of the form occur at the eigenvalues of W. Thus the principal curvatures are the maximum and the minimum normal curvatures.

10 Bilinear Forms

The classical fundamental bilinear forms defined on pairs of tangent vectors include the first fundamental form

$$B_1(T_1, T_2) = T_1 \cdot T_2,$$

and the second fundamental form

$$B_2(T_1, T_2) = W(T_1) \cdot T_2.$$

The quadratic form derived from the first fundamental form is

$$Q_1(T) = T \cdot T.$$

This is the metric form.

In classical differential geometry a tangent vector is written as

$$T = ds = \sum_{i=1}^{n} \frac{\partial X}{\partial u_i} du_i.$$

Then the first fundamental quadratic form would be written as

$$Q_1(ds) = ds \cdot ds = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} du^i du^j.$$

g is called the metric tensor. Suppose $X(u,v)$ is a coordinate patch, then we have

$$ds^2 = Q_1(ds) = X_u \cdot X_u du^2 + 2X_u \cdot X_v dudv + X_v \cdot X_v dv^2.$$
$$= Edu^2 + 2Fdudv + Gdv^2.$$

As an example, consider a spherical patch of radius a,

$$X(u,v) = (a\cos(v)\sin(u), a\sin(v)\sin(u), a\cos(u)).$$

Then

$$X_u(u,v) = (a\cos(v)\cos(u), -a\sin(v)\cos(u), -a\sin(u)).$$

and

$$X_v(u,v) = (-a\sin(v)\sin(u), -a\cos(v)\sin(u), 0).$$

Thus

$$E = X_u \cdot X_u = a^2$$
$$F = X_u \cdot X_v = 0$$

and

$$G = X_v \cdot X_v = a^2\sin^2(u).$$

The second fundamental quadratic form is

$$Q_2(ds) \cdot ds = W(X_u) \cdot X_u du^2 + 2W(X_u) \cdot X_v dudv + W(X_v) \cdot X_v dv^2.$$
$$= edu^2 + 2fdudv + gdv^2.$$

Let N be a unit normal. Then

$$0 = X_u \cdot N = X_v \cdot N,$$

so

$$e = W(X_u) \cdot X_u = X_u \cdot N_u = -X_{uu} \cdot N,$$
$$f = W(X_v) \cdot X_u = X_u \cdot N_v = -X_{uv} \cdot N,$$

and

$$g = W(X_v) \cdot X_v = X_v \cdot N_v = -X_{vv} \cdot N.$$

The normal curvature in the direction of T is

$$\kappa_n = -W(T/\|T\|) \cdot T/\|T\| = -\frac{W(T) \cdot T}{T \cdot T} = -\frac{Q_2(T)}{Q_1(T)}.$$

Proposition. The Gaussian curvature is given by

$$K(X) = \frac{eg - f^2}{EG - F^2},$$

and the mean curvature by

$$H(X) = \frac{Ge - Eg - 2Ff}{2(EG - F^2)},$$

Proof. Let T_1 and T_2 be linearly independent tangent vectors. Let a be the coefficient matrix of W with respect to this basis. Write

$$W(T_i) = a_i^j T_j$$

Then

$$W(T_1) \times W(T_2) = \mathrm{Det}(a) T_1 \times T_2 = K T_1 \times T_2.$$

and

$$W(T_1) \times T_2 + T_1 \times W(T_2) = 2\mathrm{Trace}(a) T_1 \times T_2 = 2H T_1 \times T_2.$$

These equations can be solved for the Gaussian curvature K and the mean curvature H by dotting each equation by

$$T_1 \times T_2.$$

We have

$$(W_1 \times W_2) \cdot (T_1 \times T_2) = K(T_1 \times T_2) \cdot (T_1 \times T_2).$$

The left side is

$$(W_1 \times W_2) \cdot (T_1 \times T_2) = W_1 \cdot (W_2 \times (T_1 \times T_2))$$

$$= W_1 \cdot (T_1(W_2 \cdot T_2) - T_2(W_2 \cdot T_1))$$
$$= (W_1 \cdot T_1)(W_2 \cdot T_2) - (W_1 \cdot T_2)(W_2 \cdot T_1)$$
$$= (W_1 \cdot T_1)(W_2 \cdot T_2) - (W_1 \cdot T_2)^2.$$

We have used the $BAC - CAB$ rule. Similarly

$$(T_1 \times T_2) \cdot (T_1 \times T_2) = (T_1 \cdot T_1)(T_2 \cdot T_2) - (T_1 \cdot T_2)^2.$$

Thus

$$K = \frac{(W_1 \cdot T_1)(W_2 \cdot T_2) - (W_1 \cdot T_2)^2}{(T_1 \cdot T_1)(T_2 \cdot T_2) - (T_1 \cdot T_2)^2}$$

If

$$T_1 = X_u$$

and

$$T_2 = X_v,$$

then

$$K(X) = \frac{eg - f^2}{EG - F^2}.$$

The mean curvature equation is proved similarly.

11 Quadratic Approximation

Let p be a point on the surface. We may assume a coordinate system in which p is the origin and the normal is in the z direction. Thus we may write the surface as

$$z = f(x, y).$$

We expand f in a Taylor series. Keeping only second order terms we obtain a quadratic approximation to the surface. Now we could fit local data to a quadric of this form using least squares and then obtain approximations to the principal curvatures. However, the least squares quadratic goes to the quadratic approximation only as the data approaches p. In effect we are approximating second derivatives, while the previous technique requires only covariant derivatives of the first order.

12 The Tangent Vector as a Derivation

Let c be a curve with parameter t. A tangent to the curve is $v = dc/dt$. Let f be a function. We define

$$D_v(f) = \frac{df(c(t))}{dt} = \nabla f \cdot \frac{dc}{dt} = \nabla f \cdot v.$$

This defines a functional D_v mapping functions to real numbers. It depends only on the tangent vector v, not specifically on the curve c. It does not depend on the coordinate system because

$$\frac{df(c(t))}{dt}$$

is coordinate independent. From the theory of differential equations, given a tangent vector v at a point p, there exists a curve c so that

$$\frac{dc(0)}{dt} = v,$$

and $c(0) = p$. Thus for every tangent vector there is a functional D_v. The functional has the obvious properties

$$D_v(f+g) = D_v(f) + D_v(g)$$

and

$$D_v(ag) = aD_v(f),$$

where a is a scalar. D_v is a linear functional. We have

$$D_v(fg) = \frac{df(c)g(c)}{dt} = f(c)\frac{dg(c)}{dt} + \frac{df(c)}{dt}g(c) =$$

$$f(c)D_v(g) + D_v(f)g(c).$$

Functionals satisfying these properties are called derivations. Consider the example of the x coordinate curve in Cartesian three space. We have $c(t) = tu_x$ where u_x is the unit coordinate vector in the x direction. Then

$$v = \frac{dc}{dt} = u_x$$

and

$$D_v(f) = \nabla f \cdot u_x = \frac{\partial f}{\partial x}.$$

We have shown that

$$D_v = \frac{\partial}{\partial x}.$$

Similarly if ψ_i is a local surface coordinate, then

$$\frac{\partial}{\partial \psi_i},$$

is the derivation corresponding to its tangent vector. We have

$$\frac{\partial}{\partial \psi_j}(\psi_i) = \delta^i_j.$$

Hence

$$A = \{\frac{\partial}{\partial \psi_i} : 1 \leq i \leq n\}$$

is a linearly independent set of derivations. We shall show that tangent vectors and derivations can be identified. So A is a basis of the tangent space. Let v_1 and v_2 be tangent vectors ,then

$$D_{v_1+v_2} f = \nabla f \cdot (v_1 + v_2) =$$

$$\nabla f \cdot v_1 + \nabla f \cdot v_2 = D_{v_1} f + D_{v_2} f$$

If a is a scalar, then

$$D_{av_1} = a D_{v_1}.$$

Therefore the mapping

$$v \longmapsto D_v$$

is a homomorphism. Since the tangent space of an n-dimensional manifold is n-dimensional, and there are tangent vectors mapping to A, the mapping $v \longmapsto D_v$ is actually an isomorphism.

Now we no longer have to rely on a coordinate system in the embedding space, in which the surface lies, to define a tangent vector. Given any curve in the surface c, and a parameter point t_0, there is a derivation defined by

$$D_c f = \frac{df(c)}{dt}(t_0).$$

This is independent of the embedding. A local surface coordinate system can be used to find the components of D_c with respect to the local tangent basis.

The covariant derivative $\nabla_X Y$ can be given as the action of the derivation X on the vector field Y. Indeed let

$$X = x_1 \frac{\partial}{\partial u_1} + x_2 \frac{\partial}{\partial u_2} + x_3 \frac{\partial}{\partial u_3}$$

and

$$Y = y_1 \frac{\partial}{\partial u_1} + y_2 \frac{\partial}{\partial u_2} + y_3 \frac{\partial}{\partial u_3}$$

be tangent vectors in R_3 and let X act as a derivation. Then

$$\nabla_X Y = (X(y_1), X(y_2), X(y_3)).$$

To prove this let c be a curve so that $c' = X$. Then

$$\nabla_X Y = \frac{\partial Y(c(t))}{dt} =$$

$$(\frac{\partial y_1(c(t))}{dt}, \frac{\partial y_2(c(t))}{dt}, \frac{\partial y_3(c(t))}{dt}) = (X(y_1), X(y_2), X(y_3)).$$

Reference Abraham Goetz, "Introduction to Differential Geometry," Addison-Wesley 1970.

13 Envelope Surfaces

Given a one parameter family of surfaces

$$f(x, y, z, t) = 0,$$

the envelope surface is defined by eliminating t from the equations

$$f(x, y, z, t) = 0$$

and

$$\frac{\partial f(x, y, z, t)}{\partial t} = 0$$

(see Goetz p125). To establish this, let the envelope surface have parametric equation

$$r(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)).$$

Given the coordinates of a point $u_1 u_2$ of a point p on the envelope, there is an element of the family of surfaces that is tangent to the envelope at this point. Let the corresponding parameter of this tangent surface be $t(u_1, u_2)$. Now ∇f is normal to this tangent surface. Hence

$$\nabla f \cdot \frac{dr}{du_1} = 0.$$

But let

$$g(u_1, u_2) = f((x(u_1, u_2), y(u_1, u_2), z(u_1, u_2), t(u_1, u_2)).$$

Then $g = 0$, so

$$0 = \frac{dg}{du_1} = \nabla f \cdot \frac{dr}{du_1} + \frac{\partial f(x, y, z, t)}{\partial t} \frac{dt}{du_1}.$$

Thus

$$\frac{\partial f(x, y, z, t)}{\partial t} \frac{dt}{du_1} = 0.$$

Similarly

$$\frac{\partial f(x, y, z, t)}{\partial t} \frac{dt}{du_2} = 0.$$

We shall assume that

$$\frac{dt}{du_1}$$

and

$$\frac{dt}{du_2}$$

do not vanish simultaneously. Thus

$$\frac{\partial f(x, y, z, t)}{\partial t} = 0.$$

A canal surface is the envelope of a family of spheres. For example the torus is a canal surface, and Dupin cyclides are also canal surfaces. A spherical Milling cutter produces a canal surface, other cutter shapes produce other envelope surfaces. A rather general milling cutter may be considered the envelope of a family of flat disk like ellipsoids. Then a milled surface may be considered the envelope of a two parameter family of surfaces

$$f(x, y, x, s, t) = 0.$$

The envelope surface is found by eliminating t and s from

$$f(x, y, z, s, t) = 0, f_t(x, y, z, s, t) = 0, f_s(x, y, z, s, t) = 0.$$

See Goetz.

The milled internal screw thread is an envelope of surfaces generated by swept disks. It can also be considered a the envelope of a two parameter family of algebraic surfaces and is thus algebraic.

An envelope of a one-parameter set of planes is called a developable surface.

14 The Covariant Derivative On a Surface

Define the covariant derivative on the surface by

$$D_X Y = \nabla_X Y + < \nabla_X N, Y > N.$$

We will show that this is the projection of the R_3 covariant derivative to the tangent space. For if we take the dot product with the normal N, we get

$$D_X Y \cdot N =< \nabla_X Y, N > + < \nabla_X N, Y >= X < Y, N >= 0.$$

X, as a derivation, is simply a differential operator. Thus the expression on the right is a differential operator operating on an inner product, and Y is orthogonal to N, so it is zero. Then we have

$$< \nabla_X N, Y >= - < \nabla_X, N > .$$

Then we have

$$D_X Y = \nabla_X Y - < \nabla_X Y, N > N,$$

which shows that $D_X Y$ is the projection of $\nabla_X Y$ to the tangent space.

The covariant derivative has an explicit coordinate representation, given in terms of the soon to be introduced Christoffel symbols.

15 Christoffel Symbols

Let $\phi(u_1, u_2)$ be a surface patch (an inverse coordinate system). Write

$$\phi_{ij} = \frac{\partial^2 \phi}{\partial u_i \partial u_j}.$$

We may express these derivatives in the moving basis

$$\{\phi_1, \phi_2, N\}$$

We have

$$\phi_{ij} = \Gamma^k_{ij} \phi_k + \beta_{ij} N.$$

The Γ^k_{ij} are called the Christoffel symbols of the second kind. Taking inner products we find that the β_{ij} are the coefficients b_{ij} of the second fundamental form. Thus

$$\beta_{ij} =< \phi_{ij}, n >= b_{ij}, .$$

where

$$b_{11} = E, b_{12} = b_{21} = F, b_{22} = G.$$

Let us write the derivative of the normal as

$$n_i = -b_i^k \phi_k.$$

Then we get

$$< n_i, \phi_j >= -b_i^k g_{kj}.$$

By differentiating

$$< n, \phi_j >= 0,$$

we get

$$< n_i, \phi_j >= - < n, \phi_{ij} >= -b_{ij}$$

Thus

$$b_{ij} = b_i^k g_{kj}.$$

Then

$$b_{ij} g^{kj} = b_i^k g_{kj} g^{kj} = b_i^k \delta_k^l = b_i^l,$$

where

$$\{g^{ij}\} = \{g_{ij}\}^{-1}.$$

The Christoffel symbols are symmetric. In fact ϕ_{ij} and b_{ij} are symmetric, so

$$\Gamma_{ij}^k \phi_k = \phi_{ij} - b_{ij} n = \phi_{ji} - b_{ji} n = \Gamma_{ji}^k \phi_k.$$

Then

$$\Gamma_{ij}^k = \Gamma_{ji}^k,$$

because the ϕ_i are linearly independent.

The Christoffel symbols of the first kind are defined by

$$\Gamma_{ijl} =< \phi_{ij}, \phi_l >=< \Gamma_{ij}^k \phi_k + b_{ij} n, \phi_l >= \Gamma_{ij}^k < \phi_k, \phi_l >= \Gamma_{ij}^k g_{kl}.$$

They are symmetric in the first two indices because of the symmetry of ϕ_{ij}. It follows that

$$\Gamma_{ij}^k = \Gamma_{ilj} g^{lk}.$$

Theorem

$$\Gamma_{ijk} = \frac{1}{2}(\frac{\partial g_{jk}}{\partial u_i} + \frac{\partial g_{ki}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_k}),$$

and

$$\Gamma^k_{ij} = \frac{g^{kl}}{2}(\frac{\partial g_{jl}}{\partial u_i} + \frac{\partial g_{li}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_l}).$$

Proof. We have

$$\frac{\partial g_{ij}}{\partial u^k} = \frac{\partial < \phi_i, \phi_j >}{\partial u^k} =< \phi_{ik}, \phi_j > + < \phi_i, \phi_{jk} >=$$

$$\Gamma_{ikj} + \Gamma_{jki}.$$

The permutation

$$i \to k, j \to i, k \to j$$

gives

$$\frac{\partial g_{ki}}{\partial u^k} = \Gamma_{kji} + \Gamma_{ijk} = \Gamma_{jki} + \Gamma_{ijk},$$

and the permutation

$$i \to j, j \to k, k \to i$$

gives

$$\frac{\partial g_{jk}}{\partial u^i} = \Gamma_{jik} + \Gamma_{kij} = \Gamma_{ijk} + \Gamma_{ikj}.$$

Adding the last two equations and subtracting the first, we obtain the result which was to be proved.

16 The Intrinsic Covariant Derivative

Let $\phi(u_1, u_2)$ be a coordinate patch. Let X be the tangent to curve $c(t) = (u_1(t), u_2(t))$. Let

$$Y = y^1\phi_1 + y^2\phi_2 = y^i\phi_i$$

Then the covariant derivative is

$$\nabla_X Y = \frac{dY(c)}{dt} = \frac{dy^i}{dt}\phi_i + y^i\phi_{ij}\frac{du^j}{dt}$$

We may substitute the expressions involving the Christoffel symbols for the second derivatives, obtaining a linear combination of the basis vectors

$$\{\phi_1, \phi_2, N\}.$$

Taking the projection into the tangent space, (deleting terms involving N), we get an expression for the surface covariant derivative

$$D_X Y = \frac{dy^i}{dt}\phi_i + y_i \Gamma^k_{ij}\phi_k \frac{dy^j}{dt} =$$

$$\frac{dy^k}{dt}\phi_k + y^i \Gamma^k_{ij}\phi_k \frac{du^j}{dt} =$$

$$(\frac{dy^k}{dt} + y^i \Gamma^k_{ij}\frac{du^j}{dt})\phi_k$$

which involves the basis vectors

$$\{\phi_1, \phi_2\},$$

and the Christoffel symbols. The Christoffel symbols involve only the coefficients of the first fundamental form and their derivatives. So the surface covariant derivative is intrinsic, and depends only on the metric g_{ij}.

Let $X = \phi_m$ and $Y = \phi_n$. Then

$$\frac{dy^k}{dt} = 0$$

and

$$y^i = \delta^i_n,$$

and

$$\frac{du^j}{dt} = \delta^j_m.$$

Then

$$D_{\phi_m}\phi_n = \Gamma^k_{mn}\phi_k.$$

17 The Covariant Derivative in Riemmanian Geometry

In Hicks, the covariant derivative of vector field Y with respect to vector field X is written as

$$\bar{D}_X Y.$$

We shall continue to write it as

$$\nabla_X Y.$$

Above we introduced the derivation corresponding to vector v. A derivation D is a linear functional on a manifold that has the properties:

$$(1) D(f+g) = D(f) + D(g)$$

$$(2) D(af) = aD(f),$$

$$(3) D(fg) = f(c)D(g) + D(f)g(c).$$

The derivations play the roll of tangent vectors. The set of derivations constitute the tangent space. A derivation may also be defined as a parametric derivative along a curve.

In the Euclidean case suppose we have a vector

$$X = (a_1, a_2, ..., a_n)$$

Then a directional derivative in the direction of this vector is a derivation and the operation on a function f is given by

$$Xf = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial u_i}.$$

In this case it is easy to see that property (3) of the derivation follows from the rule for diferentiating the product of functions. Further in this Euclidean case one can show that every derivation is of this form. If Y is a vector field in this Euclidean space with ith coordinate y_i, then a covarient derivative is given by

$$\nabla_X Y = (Xy_1, ..., Xy_n).$$

Covariant derivatives have the following four characteristic properties:

$$(1) \nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$$

$$(2) \nabla_{X+W} Y = \nabla_X Y + \nabla_W Z$$

$$(3) \nabla_{f(p)X} Y = f(p) \nabla_X Y$$

$$(4) \nabla_X f(p) Y = (Xf) Y + f(p) \nabla_X Y$$

Property (4) follows from the property (3) of derivations.

Given a curve σ and a vector field Y defined along the curve, if T is the curve tangent and

$$\nabla_T Y = 0,$$

then the vector $Y(0)$ is said to be parallel translated along σ.

If

$$\nabla_T T = 0,$$

then the curve tangent is parallel translated along the curve σ and the curve is a geodesic, a shortest path connecting the starting point and the ending point.

Consider the case of translating a tangent vector around a path on a sphere, say the earth. Say one starts at the north pole and suppose we translate a vector along the zero degree meridian. Let the vector Y be perpendicular to the meridian as it is translated, that is it is always pointing east, always making the same angle with the meridian, i.e. the geodesic curve.

When we reach the equator the vector points east in the direction of the equator, which is another geodesic. We keep it in the direction of the equator until we get to say the 30thth degree meridian. The vector makes a 90 degree angle with the 30th degree meridian as we translate it back up to the north pole. When we reach the north pole we find that the translated vector is pointing in a different direction from the starting direction. So on the earth we have traveled along a spherical triagle path, always keeping our vector pointing in the same direction with respect to what we consider locally as straight lines. But when we return to our starting point we see that the vector has been turned. So this tells us that our surface is curved. Because on a flat surface this would not occur.

The geodesics are the paths of particles when they do not experience any force. Their motion is the analogue of classical uniform motion.

18 The Bracket

If X and Y are C^∞ vector fields, their bracket is a C^∞ vector field (Hicks P8) defined by

$$[X,Y]f = X(Yf) - Y(Xf).$$

The bracket has the following properties:

$$[X,Y] = -[Y,X].$$

$$[X, X] = 0.$$

$$[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y]$$

and

$$[fX, gY] = f(Xg)Y - g(Yf)X + fg[X, Y].$$

At a fixed point p on the manifold,

$$[X, Y]_p f$$

is a functional, mapping f to a real number. For a fixed p and a fixed f,

$$[X, Y]_p f$$

maps two vectors from the product tangent space to a real number. If it were multilinear with respect to X and Y, then it would be a second rank tensor. But property

$$[fX, gY] = f(Xg)Y - g(Yf)X + fg[X, Y],$$

shows that this is not the case.

The bracket satisfies the Jacoby identity

$$[X, [X, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.$$

The bracket comes from the Poisson bracket of mechanics, and was introduced by Sophus Lie in his work on Lie Groups (originally called continuous groups of transformations) and Lie Algebras in solving partial differential equations.

19 Differential Manifolds

A differential manifold is a topological space M and a set of charts, where each chart Φ is a one to one mapping from an open neighborhood of M to an open connected neighborhood in R^n. If Φ_1 and Φ_2 are overlapping charts, then $\Phi_1^{-1}\Phi_2$ is a C^∞ map. M is covered by the domains of the set of charts. The charts specify a set of local coordinates for the manifold. For example on the spherical earth, the mapping of a point on the earth to its latitude and longitude is a chart. It takes more than one chart to cover the earth. Let us take the example of a 2d unit sphere embedded in R^3. The use of spherical coordinates constitutes a chart. That is a point on the sphere is mapped to the ϕ, θ coordinates. The inverse of this chart is the mapping

$$x = \sin(\theta)\cos(\phi)$$

$$y = \sin(\theta)\sin(\phi)$$

$$z = \cos(\theta)$$

Tangent vectors in these coordinate directions are the functionals (derivations)

$$\frac{\partial}{\partial \phi}, \text{ and } \frac{\partial}{\partial \theta}.$$

20 Riemannian Geometry

A Riemannian Space is a differential manifold with a Riemannian metric. A Riemannian Metric is a function $< X, Y >$, where X and Y are elements of a tangent space. This is the distance between X and Y. In classical notation this is written as g(X,Y), and has tensor components g_{ij}. The Riemannian metric is real valued, bilinear, symmetric, and positive definite. The metric is a quadratic form and is represented by a symmetric matrix. Positive definite means that if X is not zero, then $g(X, X) > 0$. The eigenvalues of the matrix are all positive.

A Semi-Riemannian metric is real valued, bilinear, and symmetric. But is not necessarily positive definite, but must be nonsingular. The eigenvalues must be nonzero, but not necessarily positive. This is the case for the Minkowski metric of relativity theory. A positive definite matrix is clearly nonsinglular, so a Riemannian metric is a Semi-Riemannian metric. Clearly, there can be a nonsingular matrix that is not positive definite, namely a diagonal matrix. A Riemannian metric tensor is given classically as a symmetric quadratic form

$$ds^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}dx_i dx_j.$$

where ds is the infinitesimal distance.

21 The Unique Riemannian Connexion Defined by the Riemannian or semi-Riemannian Metric

The properties 1-6 of the standard connexion on R^n (Hicks pp19-20) are satisfied by this Riemannian connection properties 5 and 6 are additional properties not necessarily satisfied by a covariant derivative or connexion that is not Riemannian.

There exists a unique Riemannian connexion (pp 69-70) on a Riemannian or semi-Riemannian manifold. The Christoffel symbols are given in terms of the metric tensor. We have by definition

$$\nabla_{X_k}(X_j) = \sum_{i=1}^{n} \Gamma^i_{jk} X_i.$$

From the properties of the covariant derivative or connexion we have the Christoffel symbols in terms of the metric

$$\Gamma^k_{ij} = \frac{1}{2}\sum_r (g^{-1})_{kr}\left(\frac{\partial g_{rj}}{\partial x_i} + \frac{\partial g_{ri}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_r}\right).$$

The symbol g^{-1} is the inverse of the symmetric matrix of the metric quadratic form. This is where the nonsingularity of the semi-Riemannian metric is needed. This defines the connexion, because it has been defined for the basis vector fields X_i.

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