Electromagnetic Theory

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Contents

0.1 Maxwell's Equations

The Maxwell Equations in MKS form are

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
$$\nabla \cdot \mathbf{D} = \rho,$$
$$\nabla \cdot \mathbf{B} = 0.$$

0.2 Coulomb's Law

Let n charges q_i be placed at positions $\mathbf{r'}_i$. Let $\mathbf{a} = \mathbf{r} - \mathbf{r'}.$, Then

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i \mathbf{a}_i}{a_i^3}.$$

0.3 Potential

Because

$$\nabla \times \mathbf{E} = 0,$$

a line integral of $\mathbf{E}$ is independent of the path and there exists a potential ϕ so that

$$\mathbf{E} = -\nabla\phi.$$

We have

$$\phi = \int \mathbf{E} \cdot d\mathbf{l}.$$

For a point charge

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{a}.$$

0.4 Gauss's Law

Let S be a sphere. Let q be a point charge at the center of S. Then

$$\int_S \mathbf{E} \cdot \mathbf{n} ds = q/\epsilon_0$$

Let S be surrounded by an arbitrary surface G. Integrating the volume bounded by S and G we deduce that that the integral over G equals the integral over S. The integral of $\mathbf{E}$ over the surface of a volume not containing sources is zero. This follows because in such a volume

$$\nabla \cdot \mathbf{E} = 0$$

We conclude that the integral of a field $\mathbf{E}$ over a surface G, which is due to point charges, is equal to the sum of the point charges contained within the surface, divided by the permitivity of free space.

0.5 Charge Distribution

Divide a bounded space into small volumes ΔV_j. Let ρ_j be the charge per volume. Let ρ_d be a linear combination of products of characteristic functions of ΔV_j's and ρ_j's. Let the charge distribution (charge density) ρ be a continuous function that approximates this step function. If ρ is continuous, from the divergence theorem and Gauss's law we deduce that

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Now let ρ be a generalized function (i.e. a distribution). Then we define E to be a solution to this differential equation in the distributional sense.

0.6 Electric Polarization

Two charges of magnitude q and differing signs, which are separated by a vector r, create an electric field. This field depends only on the product of the charge and the separation vector, and the field point location. We let $p = q\mathbf{r}$. p is called the electric dipole moment. We can think of a point dipole moment, where as **r** shrinks, the charge q increases proportionately. We may thus consider a vector field P, which is a continuous distribution of point dipoles. P then is a dipole moment density. It is the dipole moment per unit volume. The continuous vector field P is called the polarization. The dipole moment of a volume Δv is

$$P\Delta v.$$

Given a polarized region, we may integrate with respect to the volume to get the electric field at a point due to the polarized region. As in the case of a charge distribution, the electric field will be defined both inside and outside of the polarized region.

The potential due to any localized charge distribution in a volume ΔV may be written as an infinite sum of multipole sources. We retain only monopole and dipole terms. The monopole moment is

$$q = \int_{\Delta V} \rho dv,$$

and the dipole moment is

$$\mathbf{p} = \int_{\Delta V} \mathbf{r} \rho dv.$$

Let $\mathbf{P}$ be the dipole moment per unit volume, which in general is a distribution. The dipole potential due to volume element dv is

$$d\phi = \frac{1}{4\pi\epsilon_0} P \cdot \frac{\mathbf{a}}{a^3} dv =$$

$$\frac{1}{4\pi\epsilon_0} P \cdot (-\nabla f) dv =$$

$$\frac{1}{4\pi\epsilon_0} (f\nabla \cdot P - \nabla \cdot (fP)) dv$$

Integrating over a charged isolated volume V, we get

$$\phi = \int_{\partial V} \frac{\sigma_p}{a} ds + \int_V \frac{\rho_p}{a} dv,$$

where

$$\sigma_p = \mathbf{P} \cdot \mathbf{n}$$

and

$$\rho_p = -\nabla \cdot \mathbf{P}.$$

If we integrate over all space and assume P is zero at infinity, we have

$$\phi = \int_V \frac{\rho_p}{a} dv.$$

Since $\mathbf{P}$ may not be differentiable in the classical sense we take $\mathbf{P}$ to be distribution. We may approximate $\mathbf{P}$ with a smooth function. When we have a finite volume V, we may replace part of the volume integration by surface integration on the boundary of V. This can be done by integrating over a thin shell A that contains the boundary of V. For the thin shell we have

$$\phi_A = \int_A \frac{-\nabla \cdot \mathbf{P}}{a} dv =$$

$$\int_A (-\nabla \cdot (\mathbf{P}/a) + P \cdot \nabla f) dv =$$

$$\int_{\partial A} \frac{-\mathbf{P} \cdot \mathbf{n}}{a} ds + \int_A \mathbf{P} \cdot \nabla f dv$$

If $\mathbf{P}$ is bounded, then the second integral goes to zero as the volume of the thin shell goes to zero. The first integral is over two parallel surfaces, one of which is outside of V, assuming P is zero outside of V, we get back our surface polarization charge density integral.

$$\phi_A = \int_{\partial A} \frac{-\mathbf{P} \cdot \mathbf{n}}{a} ds =$$

$$\int_{\partial V} \frac{\sigma_p}{a} ds$$

0.7 Electric Displacement

From Gauss's law integrating over a surface S we have

$$\int_S \mathbf{E} \cdot \mathbf{n} ds = (q + q_p)/\epsilon_0$$

$$\int_V \nabla \cdot \mathbf{E} dv = (q + q_p)/\epsilon_0$$

$$q = \int_V \frac{\rho}{a} dv,$$

$$q_p = \int_V \frac{\rho_p}{a} dv,$$

$$= \int_V \frac{-\nabla \cdot \mathbf{P}}{a} dv,$$

It follows that if **D** is defined by

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

then

$$\nabla \cdot \mathbf{D} = \rho.$$

0.8 Electric Susceptibility, Permitivity, and Dielectric Tensors

The polarization can usually be taken to be a linear function of the average applied field. We write the components of the polarization as

$$P_i = \epsilon_0 \chi_{ij} E_j.$$

The tensor χ depends on the material. For an isotropic material it becomes just a constant. The number ϵ_0 is called the permitivity of free space. In terms of the displacement D we have

$$D = \epsilon_0 E + P = \epsilon_0 (I + \chi) E = \epsilon_0 K E$$

where

$$K_{ij} = I + \chi_{ij}.$$

The tensor K is called the dielectric constant, I is the identity matrix. The tensor

$$\epsilon_{ij} = \epsilon_0 K_{ij},$$

is called the permitivity.

0.9 Energy of a Charge Distribution

Assembling point charges from infinity we find

$$U = \frac{1}{2}\sum_{i=1}^{n} q_i\phi_i.$$

Raising a charge density linearly from zero to full value, we find

$$U = \frac{1}{2}\int \rho(\mathbf{r})\phi(\mathbf{r})dv.$$

Using $\nabla \cdot \mathbf{D} = \rho$ and the divergence theorem we find that the energy density is

$$u = \frac{\mathbf{D}\cdot\mathbf{E}}{2}$$

0.10 Coefficients of Potential

Given n conductors we define p_{ij} to be the potential of conductor i when there is unit charge on conductor j and the other conductors are uncharged.
Proposition. If a potential is multiplied by a constant c, then the charges are multiplied by c.
Proof. Use $E_n = \sigma/\epsilon_0$ and $\nabla\phi = -\mathbf{E}$.

$Q_j p_{ij}$ is the potential on conductor i, when Q_j is the charge on conductor j and the other charges are zero. In the general case, by linear superposition, the potential on conductor i is

$$\phi_i = \sum_{i=1}^{n} p_{ij}Q_j$$

when the charges are Q_j, j=1...n .

The energy of the conductors is

$$U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}Q_iQ_j.$$

The coefficients of potential are symmetric,

$$p_{ij} = p_{ji}$$

This may be shown by using the expression for the energy of the conductors and taking the differential of the energy. Suppose only the charge Q_1 is nonzero. We get

$$dU = \frac{1}{2}\sum_{j=1}^{n}(p_{1j} + p_{j1})Q_j$$

This is also equal to

$$\phi_1 dQ_1 = \sum_{j=1}^{n} p_{1j}Q_j dQ_1.$$

Equating these two expressions, we find that

$$p_{1j} = p_{j1},$$

and so in general

$$p_{ij} = p_{ji}.$$

The coefficients of potential are positive (Reitz and Milford, 3rd ed., p 121).

Suppose there are only two conductors in a capacitor. The charges are equal, thus

$$C = \frac{1}{p_{11} + p_{22} - 2p_{12}}.$$

When

$$\phi_i = \sum_{i=1}^{n} p_{ij}Q_j$$

is inverted, we get

$$Q_i = \sum_{i=1}^{n} c_{ij}\phi_j.$$

The c_{ij} are called the coefficients of capacitance, and are elements of a symmetric matrix, being the inverse of a symmetric matrix. This follows by the finite spectral theorem. A symmetric matrix can be diagonalized by an orthogonal transformation, that is the eigenvectors of a symmetric matrix are orthogonal.

0.11 Properties of Harmonic Functions

(1) A maxima or minima must occur on a boundary. (2) The average over a spherical surface equals the value at the center.

0.12 Shielded Conductors

Let uncharged conductors be located inside another conductor. The charge densities of the inside conductors and on the inside surface of the bounding conductor are zero everywhere. Otherwise we may trace a line of flux from a positive charge on a conductor back to a negative charge on the same conductor, possibly traveling through other conductors. The potential drops in some portion of the path and never increases anywhere. This is a contradiction, because we return to the same conductor. Solving the Neumann problem we find the potential constant. It follows that if i and j are two of these conductors and k is an outside conductor, then for shielded conductors

$$p_{ik} = p_{jk}.$$

Proposition. $p_{ij} = p_{ji}$.
Proposition. $p_{ij} > 0$ and $p_{ii} \geq p_{ij}$.

0.13 Capacitors

Consider two conductors, one shielded by another. Number them 1 and 2. Using Gauss's law, the charges on the matching surfaces are equal. Call this charge Q. If i is not equal to 1 or 2, then $p_{1i} = p_{2i}$. So

$$\Delta\phi = (p_{11} + p_{22} - 2P_{12})Q = \frac{Q}{C}.$$

C is called the capacitance. The energy of a capacitor is

$$U = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} p_{ij}Q_iQ_j$$

$$= \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C\Delta\Phi^2.$$

0.14 Forces

Suppose the orientations and positions of a set of conductors depends on a set of m generalized coordinates, $u_1, ..., u_m$. Let charge be held fixed. And let the system do work dW. Using the first law of thermodynamics we find

$$dW = -dU$$

Let the ith generalized forces be F_i. We have

$$\sum_{i=1}^{m} F_i dU_i = dW = -\sum_{i=1}^{m} \frac{\partial U}{\partial u_i} dU_i.$$

Thus

$$F_i = -\frac{\partial U}{\partial u_i}.$$

If the potential is held fixed by a battery, then the battery does work $2dU$, and thus

$$F_i = \frac{\partial U}{\partial u_i}.$$

0.15 Current Density

Let N be the number of charged particles per unit volume. Let each particle have charge q and velocity $\mathbf{v}$. Define the current density

$$\mathbf{J} = Nq\mathbf{v}.$$

Let a surface have normal $\mathbf{n}$. Suppose $\mathbf{J}$ makes an angle θ with $\mathbf{n}$. Consider a tube of flowing charge of cross sectional area da'. In time dt, charge

$$dq = Nqdv = Nqvdtda' = Nqvdtcos(\theta)da$$

crosses the surface. da is the surface area through which the charge flows. Thus

$$\frac{dq}{dt} = \mathbf{J} \cdot \mathbf{n} da.$$

0.16 The Equation of Continuity

Consider a closed surface bounding a volume where the charge density is ρ. Using the fact that charge is conserved and the divergence theorem, and assuming continuity, we obtain the equation of continuity

$$\nabla \cdot \mathbf{J} + \frac{d\rho}{dt} = 0.$$

0.17 Ohm's Law

$$\mathbf{J} = g\mathbf{E},$$

where g is the conductivity. The resistence of a wire of cross section A and length L, assuming uniform current flow, is

$$R = \frac{V}{I} = \frac{EL}{JA} = \frac{L}{gA}.$$

0.18 Steady Currents

When time derivatives are zero, the equation of continuity becomes

$$\nabla \cdot \mathbf{J} = 0.$$

Using $\mathbf{E} = -\nabla\phi$ we obtain Laplaces's equation

$$\nabla^2\phi = 0.$$

Conservation of charge gives the boundary condition between two media

$$g_1\frac{\partial\phi}{\partial n} = g_1\frac{\partial\phi}{\partial n}.$$

0.19 Magnetic Induction

The magnetic force on a charge q_1 due to a charge q_2 is

$$\mathbf{F}_1 = q_1\mathbf{v}_1 \times (\frac{\mu_0}{4\pi}q_2\mathbf{v}_2 \times \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3})$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the respective velocities and $\mathbf{r}_1$ and $\mathbf{r}_2$ the positions of the charges. We define the magnetic induction field $\mathbf{B}$ by

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

where $\mathbf{B}$ is due to moving charges as in the first equation. The Lorentz force is the sum of the electric and magnetic forces

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

By definition

$$\mu_0 = 4\pi 10^{-7}.$$

We have

$$\mu_0 \epsilon_0 == \frac{1}{c^2}$$

Note: ϵ_0 is approximately

$$\frac{1}{36\pi} 10^{-9}.$$

0.20 Biot-Savart Law

$$\mathbf{B}(\mathbf{r}_1) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J} \times (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} dx_2 dy_2 dz_2.$$

Taking the divergence we find

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int_V ((\nabla \times \mathbf{J}) \cdot \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} - \mathbf{J} \cdot (\nabla \times \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3})) dv_2.$$

The first term is zero because $\mathbf{J}$ is not a function of $\mathbf{r}_1$. The second term is zero because the curl of a gradient is zero. Thus for current sources,

$$\nabla \cdot \mathbf{B} = 0.$$

If monopoles do not exist, this is a general result.

0.21 Torque on a Circuit

$$d\tau = \mathbf{r} \times d\mathbf{F} = \mathbf{r} \times (I d\mathbf{l} \times \mathbf{B})$$

and

$$\tau = I \oint \mathbf{r} \times (d\mathbf{l} \times \mathbf{B}).$$

Define the magnetic moment of the circuit as

$$\mathbf{m} = \frac{I}{2} \oint \mathbf{r} \times d\mathbf{l}.$$

We will prove:
Proposition.

$$\tau = \mathbf{m} \times \mathbf{B}.$$

Lemma 1.

$$\oint \mathbf{r} \cdot d\mathbf{r} = 0$$

and

$$\oint xdx = \oint ydy = \oint ydy = 0.$$

Proof.

$$\oint \mathbf{r} \cdot d\mathbf{r} = \int \nabla \times \mathbf{r} da = 0.$$

Lemma 2.

$$\oint (xdy + ydx) = \oint (xdz + zdx) = \oint (zdy + ydz) = 0.$$

Proof. For example, let

$$\mathbf{U} = y\mathbf{i} + x\mathbf{j}.$$

Then

$$\nabla \times \mathbf{U} = 0,$$

and the result follows from Stokes's Theorem.
Proof of the proposition.

$$\tau = I \oint \mathbf{r} \times (d\mathbf{l} \times \mathbf{B}).$$

$$= I(\oint(\mathbf{r}\cdot\mathbf{B}d\mathbf{r}) - \oint \mathbf{B}(\mathbf{r}\cdot d\mathbf{r}))$$

$$= I\oint d\mathbf{r}(\mathbf{r}\cdot\mathbf{B}).$$

The second integral vanishes by the Lemma. On the other hand

$$\mathbf{m}\times\mathbf{B} = \frac{I}{2}\oint \mathbf{r}\times d\mathbf{l}\times\mathbf{B}$$

$$= \frac{I}{2}(\oint d\mathbf{r}(\mathbf{r}\cdot\mathbf{B}) - (\oint \mathbf{r}(\mathbf{B}\cdot d\mathbf{r}).$$

When one expands these two terms, they are seen to be equal. For example

$$\oint d\mathbf{r}(\mathbf{r}\cdot\mathbf{B}) = \oint(B_y y dx + B_z z dx)\mathbf{i} + \oint(B_x x dy + B_z z dy)\mathbf{j} + \oint(B_x x dz + B_y y dz\mathbf{k}).$$

The equality is seen by using the lemmas. We get

$$\mathbf{m}\times\mathbf{B} = I(\oint d\mathbf{r}(\mathbf{r}\cdot\mathbf{B}) = \tau.$$

0.22 Amperes' Law

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}.$$

Proof. We take the Curl of the Biot-Savart Law

$$\mathbf{B}(\mathbf{r}_1) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}\times(\mathbf{r}_1-\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|^3}dx_2dy_2dz_2.$$

Let

$$\mathbf{G} = \frac{(\mathbf{r}_1-\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|^3}.$$

Then

$$\nabla\times\mathbf{B}(\mathbf{r}_1) = \frac{\mu_0}{4\pi}\int_V \nabla_1\times(\mathbf{J}\times\mathbf{G})dv_2.$$

We use the identity

$$\nabla_1\times(\mathbf{J}\times\mathbf{G}) = (\nabla_1\cdot\mathbf{G})\mathbf{J} - (\nabla_1\cdot\mathbf{J})\mathbf{G} + (\mathbf{G}\cdot\nabla_1)\mathbf{J} - (\mathbf{J}\cdot\nabla_1)\mathbf{G}$$

$$= (\nabla_1 \cdot \mathbf{G})\mathbf{J} - (\mathbf{J} \cdot \nabla_1)\mathbf{G}.$$

Terms 2 and 3 are zero because $\mathbf{J}$ is a function of $\mathbf{r}_2$. We have

$$(\mathbf{J} \cdot \nabla_1)\mathbf{G} = -(\mathbf{J} \cdot \nabla_2)\mathbf{G}.$$

So

$$\nabla \times \mathbf{B}(\mathbf{r}_1) = \frac{\mu_0}{4\pi} \int_V ((\nabla_1 \cdot \mathbf{G})\mathbf{J} + (\mathbf{J} \cdot \nabla_2)\mathbf{G}) dv_2.$$

The first term is

$$= \frac{\mu_0}{4\pi} 4\pi \int_V \delta(\mathbf{r}_2 - \mathbf{r}_1)\mathbf{J} dv_2 = \mu_0 \mathbf{J}.$$

The second term is zero. Consider for example the x component. We have

$$\int_v (\mathbf{J} \cdot \nabla_2) G_x dv = \int_v \nabla_2 G_x \cdot \mathbf{J} dv$$

$$= \int_v \nabla_2 \cdot (G_x \mathbf{J}) dv - \int_v G_x \nabla_2 \cdot \mathbf{J} dv$$

$$= \int_{\partial v} G_x \mathbf{J} \cdot \mathbf{n} da = 0.$$

We have assumed that there are no point current sources, i.e. $\nabla \cdot \mathbf{J} = 0$. The last integral is zero because all currents are zero outside of a bounded region contained in V.

0.23 The Vector Potential

If there are no magnetic monopoles, then

$$\nabla \cdot \mathbf{B} = 0.$$

Then $\mathbf{B}$ is given by the curl of a vector field A

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Then

$$\mu_0 \mathbf{J} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\nabla^2 \mathbf{A},$$

provided we select a guage so that $\nabla \cdot \mathbf{A} = 0$. The fundamental solution of Poission's equation gives

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}}{|\mathbf{r} - \mathbf{r}'|} dv'.$$

The vector potential for a distant circuit is obtained by using $id\mathbf{r} = \mathbf{J}dv$ and by expanding

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

using the binomial theorem. Keeping linear terms we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{m}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}.$$

This is the potential of a magnetic dipole.

0.24 Magnetization

The magnetization vector $\mathbf{M}$ is defined to be the magnetic dipole moment per unit volume. We have

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \mathbf{M} \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}dv'$$

$$= \frac{\mu_0}{4\pi}\int \mathbf{M} \times \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|}dv'$$

$$= -\frac{\mu_0}{4\pi}\int \nabla' \times (\mathbf{M}\frac{1}{|\mathbf{r} - \mathbf{r}'|})dv'$$

$$+\frac{\mu_0}{4\pi}\int \frac{\nabla' \times \mathbf{M}}{|\mathbf{r} - \mathbf{r}'|}dv'$$

The first integral can be transformed to a surface integral using the divergence theorem. In general, if $\mathbf{F}$ is an arbitrary constant vector, then

$$\mathbf{F} \cdot \int \nabla \times \mathbf{G}dv$$

$$= -\int \nabla \cdot (\mathbf{F} \times \mathbf{G})dv$$

$$= -\int (\mathbf{F} \times \mathbf{G}) \cdot \mathbf{n}da$$

$$= -\mathbf{F} \cdot \int \mathbf{G} \times \mathbf{n}da.$$

$\mathbf{F}$ is an arbitrary vector, so we have the general identity

$$\int \nabla \times \mathbf{G} dv = -\int \mathbf{G} \times \mathbf{n} da.$$

So

$$-\frac{\mu_0}{4\pi}\int \nabla' \times (\mathbf{M}\frac{1}{|\mathbf{r}-\mathbf{r}'|})dv'$$

$$= \frac{\mu_0}{4\pi}\int \frac{\mathbf{M}\times\mathbf{n}}{|\mathbf{r}-\mathbf{r}'|}da'.$$

As the bounding surface goes to infinity, where all current sources are zero, the integral goes to zero. Then

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\nabla' \times \mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}dv'$$

It follows in general that

$$\nabla \times \mathbf{M} = \mathbf{J}_m.$$

$\mathbf{J}_m$ is the current density of the magnetic material. If currents are not zero on the surface of a volume, then we have

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}_m}{|\mathbf{r}-\mathbf{r}'|}dv' + \frac{\mu_0}{4\pi}\int \frac{\mathbf{j}_m}{|\mathbf{r}-\mathbf{r}'|}da',$$

where

$$\mathbf{j}_m = \mathbf{M}\times\mathbf{n}.$$

0.25 Magnetic Intensity

Ampere's law gives

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J}_m + \mathbf{J}) = \mu_0(\nabla \times \mathbf{M} + \mathbf{J}).$$

$\mathbf{J}$ is the free current density. We define a magnetic intensity vector

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}.$$

Then

$$\nabla \times \mathbf{H} = \mathbf{J}.$$

0.26 Magnetostatics

Suppose the free current density is zero. Then

$$\nabla \times \mathbf{H} = 0$$

so $\mathbf{H}$ is the gradient of a magnetic scalar potential ϕ_m.

$$\mathbf{H} = -\nabla \phi_m.$$

Because

$$\nabla \cdot \mathbf{B} = 0$$

we have

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$$

Thus the magnetic scalar potential ϕ_m satisfies the poisson equation

$$\nabla^2 \phi_m = \rho_m$$

where

$$\rho_m = -\nabla \cdot \mathbf{M}.$$

0.27 Sources of H

When there are magnetic materials and free currents we have

$$\mathbf{H}(\mathbf{r}_1) = \frac{1}{4\pi} \int_V \frac{\mathbf{J} \times (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} dx_2 dy_2 dz_2 \ - \nabla \phi_m.$$

0.28 Boundary Conditions

The divergence of $\mathbf{B}$ is zero so the normal component of $\mathbf{B}$ is continuous across a surface separating two media. The boundary conditions on $\mathbf{H}$ are more complex. Let a surface separate material 1 from material 2. Suppose in general there is a surface current. Let $\mathbf{j}$ be the surface current density. This is the current per unit length on the surface. Let $\mathbf{n}_i$ be the surface normal that points into material i. Let C be a rectangular path with a short side δ that tends to zero and a long side h. One long side is in material 1 and the other in material 2. The plane containing C is perpendicular to the original separating surface. Let $\mathbf{t}$ be a unit tangent to c. Let $\mathbf{n}$ be the normal to the plane containing C. Then applying the right hand rule

$$\mathbf{n}_i \times \mathbf{t}_i = \mathbf{n}$$

Neglecting the contribution of the short sides to the line integral we have

$$\oint_c \mathbf{H} \cdot d\mathbf{r} = (\mathbf{H}_1 \cdot \mathbf{t}_1 + \mathbf{H}_2 \cdot \mathbf{t}_2)h = h\delta \mathbf{J} \cdot \mathbf{n}$$

$$= h\delta \mathbf{J}_{||} \cdot \mathbf{n}$$

$$= h\mathbf{j} \cdot \mathbf{n}.$$

$\mathbf{J}_{||}$ is the component of $\mathbf{J}$ parallel to the separating surface. Hence

$$\mathbf{H}_1 \cdot \mathbf{t}_1 + \mathbf{H}_2 \cdot \mathbf{t}_2 = \mathbf{j} \cdot (\mathbf{n}_i \times \mathbf{t}_i) = (\mathbf{j} \times \mathbf{n}_i) \cdot \mathbf{t}_i.$$

We have

$$\mathbf{t}_1 = -\mathbf{t}_2,$$

so when the surface current is zero, the tangential component of $\mathbf{H}$ is continuous across the surface.

0.29 Magnetic Susceptibility and Permeability

For isotropic and linear materials

$$\mathbf{M} = \chi_m \mathbf{H}.$$

Then

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H} = \mu \mathbf{H}.$$

χ_m is the susceptibility and μ the permiability. Ferromagnetic materials can have a permanent magnetization and the relation between $\mathbf{B}$ and $\mathbf{H}$ depends upon the magnetization history.

0.30 Magnetic Circuits

A continuous tube of flux Φ forms a magnetic circuit. Let the circuit pass through a coil containing N turns and current i. For a path around the circuit

$$Ni = \oint \mathbf{H} \cdot d\mathbf{r} = \sum \; H_i L_i = \sum \frac{L_i \phi}{\mu_i A_i} = \Phi \sum \Re_i.$$

This equation is an approximation. $\mathbf{H}_i$ is an assumed constant value of $\mathbf{H}$ in the ith piece of the circuit. The reluctance of the ith piece is $\Re_i$. L is the length of the piece and A is the cross sectional area. The magnetomotive force mmf is Ni. We have

$$mmf = \Phi \Re.$$