Electrical Engineering and Electrical Circuits

James Emery

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Contents

1 Introduction

The physics underlying the behavior of electricity and magnetism is summarized in Maxwell's Equations. To fully understand these equations one must seriously study Electromagnetic Theory, that is one must either take a course in Electromagnetic Theory, or obtain this knowledge through self-study, given the proper sophistication in physics and mathematics, and having the study discipline to work many problems. However, Maxwell's equations can be related to the laws of electricity, which are due to physicists like Faraday, Ampere, and Gauss. These are discussed in a general physics course. We shall summarize this relationship and these equations in the next section.

2 Maxwell's Equations

A vector field is a function defined on a domain of points in space that assigns a vector $\mathbf{V}$ to each point $p = (x, y, z)$ of the domain.

$$\mathbf{V} = f(p) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit coordinate vectors in the x, y, z directions respectively. For example, if every point in a medium has a velocity, the set of velocity vectors is a vector field. Given a vector field

$$\mathbf{C} = C_x\mathbf{i} + C_y\mathbf{j} + C_z\mathbf{k},$$

the curl of $\mathbf{C}$ is defined to be

$$\nabla \times \mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ C_x & C_y & C_z \end{vmatrix}$$

$$= (\frac{\partial C_z}{\partial y} - \frac{\partial C_y}{\partial z})\mathbf{i} + (\frac{\partial C_x}{\partial z} - \frac{\partial C_z}{\partial x})\mathbf{j} + (\frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y})\mathbf{k}$$

The divergence is defined by

$$\nabla \cdot \mathbf{C} = \frac{\partial C_x}{\partial x} + \frac{\partial C_y}{\partial y} + \frac{\partial C_z}{\partial z}.$$

The gradient of a function f is

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

The Maxwell Equations in MKS form are

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{D} = \rho,$$

$$\nabla \cdot \mathbf{B} = 0.$$

$\mathbf{H}$ and $\mathbf{B}$ are magnetic fields, $\mathbf{E}$ and $\mathbf{D}$ are electric fields, $\mathbf{J}$ is the current density, and ρ is the charge density.

The fields $\mathbf{E}$ and $\mathbf{B}$ may be defined by the forces they exert on a charged particle of charge q. The Lorentz force on a charge q is the sum of the electric and magnetic forces

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

When the curl of a vector field is zero,

$$\nabla \times \mathbf{C} = 0,$$

the field is called irrotational. In that case the line integral of the field from a point A to B is independent of the path and there exists a potential function ϕ. So for example in electrostatics where there are no magnetic fields,

$$\nabla \times \mathbf{E} = 0.$$

Then $\mathbf{E}$ is equal to the negative gradient of a potential function ϕ, called the electrical potential.

$$\mathbf{E} = -\nabla\phi.$$

Ampere's Law is given by part of the First Maxwell Equation

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Ampere's Law says that each infinitesimal line element of current flow produces a magnetic field. So for example the current in a loop produces a magnetic field through the loop. The first Maxwell equation is Ampere's Law plus the addition of a displacement current term

$$\frac{\partial \mathbf{D}}{\partial t}.$$

Maxwell showed that this additional term is necessary. So Ampere's Law says that each portion of current flow produces a magnetic field, but more is required. When there is no changing field $\mathbf{D}$, which is a modification of the $\mathbf{E}$ field caused by the presence of electrically polarized materials, then the first Maxwell equation becomes

$$\nabla \times \mathbf{H} = \mathbf{J}.$$

The vector field $\mathbf{D}$ is defined by

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P},$$

where $\mathbf{P}$ is the electric dipole moment per unit volume in a dielectric material. The number ϵ_0 is called the permittivity of free space. The magnetic field vectors $\mathbf{B}$ and $\mathbf{H}$ are related by

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M},$$

where $\mathbf{M}$ is the magnetic dipole moment per unit volume. Circulating currents inside a material give rise to magnetic dipoles, just as separated charges in a material give rise to electric dipoles. For many materials there are linear relationships

$$\mathbf{D} = \epsilon \mathbf{E}$$

and

$$\mathbf{B} = \mu \mathbf{H}.$$

We return to showing how Ampere's law is related to the first Maxwell equation. Applying Stoke's Theorem we have

$$\int_C \mathbf{H} \cdot d\mathbf{R} = \int_S \nabla \times \mathbf{H} \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} = i.$$

That is, the line integral of the magnetic intensity $\mathbf{H}$ around a path C equals the amount of current i flowing through the surface S that is bounded by C. This is Ampere's law. Let us remark about the displacement term. Suppose there were no displacement current term. Then if there were a capacitor placed in our wire, there would be current flowing through the wire, but no actual charge flowing between the capacitor plates. Hence, if we let our surface S pass between the capacitor plates then there would be a zero J, and thus a zero current i flowing through the surface. And so our line integral of $\mathbf{H}$ around the magnetic circuit would be zero. So depending on where we place our surface we get zero or not zero for the line integral. This is why the displacement term

$$\frac{\partial \mathbf{D}}{\partial t}$$

must be added to the first Maxwell equation. The displacement current term is nonzero between the capacitor plates.

Faraday's Law of Induction, the second Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

The second Maxwell equation is Faraday's law of induction. Using Stoke's theorem we have

$$\int_C \mathbf{E} \cdot d\mathbf{R} = \int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\frac{\partial \Phi}{\partial t}.$$

That is, the electric potential (MMF) around a circuit C is equal to the rate of magnetic flux change through the circuit.

If the material is soft iron and essentially linear with little hysteresis we may write

$$\mathbf{B} = \mu \mathbf{H},$$

where μ is a constant called the permeability. Such material forms a linear magnetic circuit.

Coulomb's's Law, the third Maxwell equation

$$\nabla \cdot \mathbf{D} = \rho.$$

The third Maxwell equation arises from Coulomb's law, which gives the forces between charges. So suppose we have a small volume of charge located at the origin and a larger spherical volume V surrounding it of radius r. Also assume that we are in free space so that

$$\mathbf{D} = \epsilon_0 \mathbf{E}.$$

We have

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}.$$

Integrating this over the volume V we find

$$\begin{aligned} \frac{q}{\epsilon_0} &= \int_V \nabla \cdot \mathbf{E} dV \\ &= \int_S \mathbf{E} \cdot d\mathbf{S} \\ &= E 4\pi r^2, \end{aligned}$$

where we have used the divergence theorem to convert from a volume integral to a surface integral, q is the charge in the small volume, and E is the magnitude of the radial electric field on the spherical surface. So we have the electric field at a distance r^2 from a charge q is given by

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}.$$

This is a form of Coulombs law. The force on a charge q by an electric field E is by definition Eq. Thus given two charges q_1 and q_2, we obtain Coulombs law for the force between two charges

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}.$$

The Absence of Magnetic Monopoles, the fourth Maxwell equation

$$\nabla \cdot \mathbf{B} = 0.$$

The fourth Maxwell equation is has some similarity with the third law provided there is no magnetic charge density, no isolated magnetic charges. So the divergence of the $\mathbf{B}$ field is zero. Using the divergence theorem to convert a volume integral to a an integral on the bounding surface, we have

$$0 = \int_V \nabla \cdot \mathbf{B} dV = \int_S \mathbf{B} \cdot d\mathbf{S},$$

which means that every flux line entering a volume, leaves the volume. Thus there are no sources of magnetic flux lines, no isolated magnetic poles, and so flux lines form continuous loops.

3 Coulomb's Law

Let n charges q_i be placed at positions $\mathbf{r'}_i$. Let $\mathbf{a} = \mathbf{r} - \mathbf{r}'$., Then

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i \mathbf{a}_i}{a_i^3}.$$

We have

$$\frac{1}{4\pi\epsilon_0} = 8.987551787388872 \times 10^{-9},$$

which is approximately 9×10^{-9}.

4 Potential

Because

$$\nabla \times \mathbf{E} = 0,$$

a line integral of $\mathbf{E}$ is independent of the path. So there exists a potential ϕ so that

$$\mathbf{E} = -\nabla\phi.$$

We have

$$\phi = \int \mathbf{E} \cdot d\mathbf{l}.$$

For a point charge q_1 located at the origin

$$\phi(r) = -\frac{1}{4\pi\epsilon_0}\frac{q_1}{r}.$$

So the potential energy to transport a charge q_2 from a point on a circle of radius r_A to a point on a circle of radius r_B is

$$T = q_2\Delta\phi,$$

that is the difference in potential is the potential energy to transport a unit charge between the two potential positions.

5 Ohm's Law and Resistance

Ohm's law says that the current density $\mathbf{J}$, the current flow per unit area, is proportional to the electric field. So

$$\mathbf{J} = g\mathbf{E},$$

where g is the conductivity.

For uniform flow in a conductor of cross section A and length L, this becomes

$$\frac{I}{A} = J = gE = g\frac{V}{L},$$

or

$$R = \frac{V}{I} = \frac{L}{gA}.$$

R is called the resistance.

So the resistance of a wire of cross section A and length L is

$$R = \frac{V}{I} = \frac{EL}{JA} = \frac{L}{gA}.$$

So the more well known special form of Ohm's law is

$$V = RI.$$

The conductivity is sometimes written using the letter σ, and the reciprocal of the conductivity, called the resistivity, written as ρ. Then the resistance is given by

$$R = \frac{V}{I} = \frac{EL}{JA} = \frac{L}{\sigma A} = \frac{\rho L}{A}.$$

The unit of resistivity is the ohm meter.

6 Gauss's Law

Let S be a sphere. Let q be a point charge at the center of S. Then

$$\int_S \mathbf{E} \cdot \mathbf{n} ds = q/\epsilon_0$$

Let S be surrounded by an arbitrary surface G. Integrating the volume bounded by S and G we deduce the integral over G equals the integral over S. The integral of $\mathbf{E}$ over the surface of a volume not containing sources is zero. This follows because in such a volume

$$\nabla \cdot \mathbf{E} = 0$$

We conclude that the integral of a field $\mathbf{E}$ over a surface G, which is due to point charges, is equal to the sum of the point charges contained within the surface, divided by the permittivity of free space.

7 Constants

The unit of permeability μ is the Henry per meter or $kg \cdot m \cdot s^{-2} \cdot A^{-2}$, where A stands for Ampere. The permeability of free space is $\mu_0 = 4\pi \times 10^{-7}$. Relative permeability is

$$\mu_r = \frac{\mu}{\mu_0}$$

A few values of permeability are: for soft Iron $\mu_r = 64$, for ferrite $\mu_r = 16$ to $\mu_r = 640$, for Permalloy $\mu_r = 8000$, for steel $\mu_r = 100$, and for copper $\mu_r = 0.999994$.

8 An Illustration of Maxwell's Equations: The AC Clamp Meter

The clamp meter for measuring alternating current in a wire is an annular ring of soft iron that closes around the wire. The clamp meter measures the alternating current in the wire by sensing the magnetic field around the wire. The line integral of the H field along a path located in the soft iron is equal to the amount of current that passes through the ring of soft iron. This defines a B field along each path, and hence the total flux in the magnetic circuit. This magnetic flux is proportional to the current i. A coil is wound around the magnetic path and the induced voltage is given by Faraday's law as the negative derivative of the flux with respect to time, times the number of turns in the coil. Thus the induced voltage in the coil is proportional to the current in the wire. Alternating current is sinusoidal, as is the derivative of the flux induced by the current. So the induced voltage in the sensing coil is also sinusoidal.

9 Inductance

Suppose we have n circuits linked by magnetic flux. Then the voltage induced in circuit i according to Faraday's law is given by

$$\xi_i = \sum_{j=1}^{n} \frac{d\Phi_{ij}}{dt},$$

where Φ_{ij} is the flux linking circuit i due to current in circuit j. We can write

$$\frac{d\Phi_{ij}}{dt} = \frac{d\Phi_{ij}}{dI_j}\frac{dI_j}{dt} = M_{ij}\frac{dI_j}{dt},$$

where I_j is the current in circuit j, and

$$M_{ij} = \frac{d\Phi_{ij}}{dI},$$

is called the mutual inductance.

Example. Suppose we have a toroid and two windings around the toroid. Then according to ampere's law, the mean value of the magnitude of the magnetic induction $B = \mu H$, is

$$B = \frac{\mu N_1 I}{\ell},$$

where N_1 is the number of turns in winding 1, and ℓ is the mean length of the path around the inside of the toroid. If we assume that B is approximately constant throughout the cross section of the toroid, then the flux in the toroid caused by current I_1 is

$$\Phi_1 = \frac{\mu N_1 I A}{\ell},$$

where A is the cross sectional area of the toroid. In such a toroid made of soft iron all of the flux is confined to the interior of the toroid, and the voltage induced in circuit 2 is

$$\xi_2 = \frac{d\Phi_1}{dt} = N_2\frac{d}{dI_1}\left(\frac{\mu N_1 I A}{\ell}\right)\frac{dI_1}{dt}$$

$$= N_2\frac{\mu N_1 A}{\ell}\frac{dI_1}{dt}.$$

$$= M_{21}\frac{dI_1}{dt}.$$

So the mutual inductance is

$$M_{21} = N_2 N_1\frac{\mu A}{\ell}$$

A single winding induces a voltage in itself with mutual inductance

$$M_{11} = N_1 N_1 \frac{\mu A}{\ell}.$$

Such an inductance is called the self-inductance, which is written as

$$L_1 = M_{11}.$$

For an inductor L the voltage induced across the inductor is given as

$$\xi = L \frac{dI}{dt}.$$

This is the voltage drop across an inductor in a circuit where a current I flows. In the case of the toroid with a linear soft iron core, we see that the inductance is essentially constant and independent of the current I, and proportional to the square of the number turns N^2. This sort of relationship holds for most practical inductors.

For a toroid we have approximately

$$L = \mu_0 n^2 A/\ell.$$

10 Flux Linkages and the Calculation of Inductance

The flux lines linking a winding are the lines or tubes of flux that intersect the surface that has the winding curve as its boundary. Now a winding could be some kind of complex knot, which is not obviously the boundary of any surface. So in complicated cases there is some difficult mathematics here. However in simpler cases to calculate the flux linking a winding, perhaps for the purpose of calculating inductance, one could decompose the total flux into a large set of tubes of flux of small diameter. We are helped here by the absence of magnetic monopoles, so that all magnetic flux lines make continuous loops and don't originate at sources. So suppose we have found a bounding surface for the winding and decompose it into small elements of area, considered as cross sections of flux tubes. We may compute the magnetic flux density from the Biot-Savart law in each elemental area, then sum up to get the total flux.

Consider the very simple case of a winding in the form of a circular helix. Assuming the simple case where every flux line passes through each circular turn, and where we assume that the flux density is essentially the same throughout the length of the helix, we could compute only the flux in one circular turn, and multiply by the number of turns. In general this problem could be very complicated.

11 Magnetic Circuits

A continuous tube of flux Φ forms a magnetic circuit. Let the circuit pass through a coil containing N turns and current i. For a path around the circuit

$$Ni = \oint \mathbf{H} \cdot d\mathbf{r} = \sum H_i L_i = \sum \frac{L_i \phi}{\mu_i A_i} = \Phi \sum \Re_i.$$

This equation is an approximation. $\mathbf{H}_i$ is an assumed constant value of $\mathbf{H}$ in the ith piece of the circuit. The reluctance of the ith piece is $\Re_i$. L is the length of the piece and A is the cross sectional area. The magnetomotive force mmf is Ni. We have

$$mmf = \Phi \Re.$$

A large flux path may be treated as a set of parallel paths and the net reluctance can be computed by a technique similar to that of computing parallel resistances.

12 Steady State Alternating Currents And The Concept of Impedance

Consider the RLC circuit with a voltage source. The equation for this circuit consisting of, a resistance R, an inductance L, and a capacitance C in series with an alternating current voltage source v, is

$$L\frac{di}{dt} + Ri + \frac{q}{C} = v,$$

where i is the current in the circuit. Differentiating this equation we get a second order differential equation

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{dv}{dt}.$$

Let

$$i = I_0 \exp j\omega t = I_0(\cos(\omega t) + \sin(\omega t)j),$$

and

$$v = V_0 \exp j\omega t = V_0(\cos(\omega t) + \sin(\omega t)j).$$

We let I_0 and V_0 be complex numbers to allow i and v to be out of phase. Then

$$[-\omega^2 L + Rj\omega + \frac{1}{C}]I_0 \exp j\omega t = V_0 j\omega \exp j\omega t.$$

Then

$$[-\omega^2 L + Rj\omega + \frac{1}{C}]I_0 = V_0 j\omega.$$

Dividing by $j\omega$

$$[-\frac{\omega^2 L}{j\omega} + R + \frac{1}{Cj\omega}]I_0 = V_0.$$

Then

$$[R + (\omega L - \frac{1}{\omega C})j]I_0 = V_0.$$

Then

$$I_0 = \frac{V_0}{Z},$$

where

$$Z = R + (\omega L - \frac{1}{\omega C})j = R + Xj$$

is the impedance. The imaginary part of the impedance X is called the reactance. The inductive reactance is

$$X_L = \omega L,$$

and the capacitive reactance is

$$X_C = -\frac{1}{\omega C}.$$

If

$$I_0 = |I_0| \exp(j\theta_I),$$
$$V_0 = |V_0| \exp(j\theta_V),$$

and

$$Z = |Z| \exp(j\theta_Z,)$$

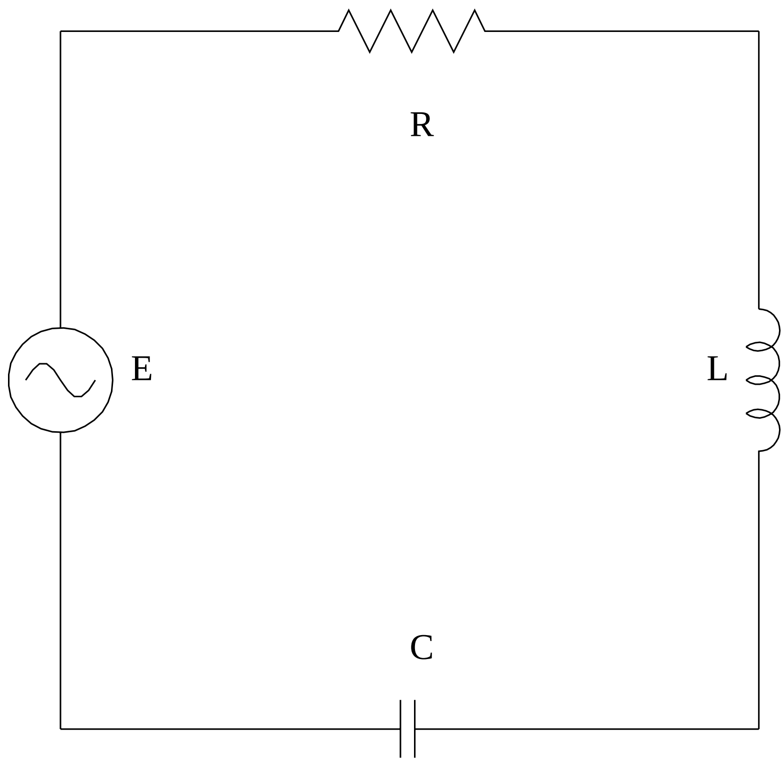

Figure 1: **Basic Impedance.** We find the steady state solution of this RLC circuit, with source EMF E, an AC driving voltage $E = V\exp(\omega t j) = V(\cos(\omega t) + \sin(\omega t) j)$, getting $I = E/Z$, where I, E, Z are respectively the complex current, complex voltage, and the complex impedance.

then

$$i = |I_0| \exp(j(\omega t + \theta_I)),$$

and

$$v = |V_0| \exp(j(\omega t + \theta_V)).$$

Dropping the subscript, we can write complex numbers in boldface and so

$$\mathbf{I} = I \exp((\omega t + \theta_I)j),$$

where $\mathbf{I}$ is the complex current, and I is the magnitude of $\mathbf{I}$. And we can write similar expressions for $\mathbf{V}$ and $\mathbf{Z}$. The complex current $\mathbf{I}$ can be thought of as a vector rotating around at angular velocity ω, and the physical current i the projection of $\mathbf{I}$ to the real axis, that is

$$i = I \cos(\omega t + \theta_I).$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}}.$$

If we are only interested in phase differences between the various rotating complex vectors, and not the actual time dependence, we can omit the ωt in the expression for $\mathbf{I}$ and $\mathbf{V}$, and retain only the magnitudes and the phase angles. So because the vectors rotate at the same frequency, the phase and magnitude relation between them is the same for each time. That is we could let t be some constant say $t = 0$. Because as t varies the same magnitudes and phase relationships are maintained.

So for example consider the voltage across an inductor. We might specify the current to have magnitude 10 and phase angle say 0 degrees. We would specify this phaser in polar notation as a magnitude and an angle

$$\mathbf{I} = 10\angle 0.$$

Now suppose the inductive reactance is 5, so that the impedance is

$$5j,$$

which in terms of magnitude and angle is

$$\mathbf{Z} = 5\angle 90.$$

Then the voltage across the inductor is

$$\mathbf{V} = \mathbf{IZ} = (10\angle 0)(5\angle 90) = (10)(5)\angle(0+90) = 50\angle 90.$$

So the voltage leads the current by 90 degrees. This makes sense because when an alternating current passes through zero, the rate of change of current is a maximum and so the inductive voltage is a maximum. Similarly we can show that for a capacitor the voltage across the capacitor lags the current by 90 degrees.

If the peak value of the current i is I, then the average power dissipated in a resistor R is

$$P = \frac{1}{T}\int_0^T i^2 R dt = \frac{I^2 R}{T}\int_0^T \cos(\omega t)^2 dt$$

$$= \frac{I^2 R}{2} = I_{eff}^2 R,$$

where

$$I_{eff}^2 R = \frac{I^2 R}{2}.$$

So

$$I_{eff} = \frac{I}{\sqrt{2}}.$$

13 The Energy Stored in a Capacitor

The energy stored in a capacitor with total charge Q is

$$E = \frac{1}{C}\frac{Q^2}{2} = \frac{1}{2}CV^2.$$

Indeed, we have

$$q = VC.$$

The change in energy dE in adding charge dq is

$$dE = V dq = \frac{q}{C} dq$$

So

$$E = \int_0^Q \frac{q}{C} dq = \frac{1}{C}\frac{Q^2}{2}.$$

14 The Energy Stored in an Inductor

The energy stored in an inductor is

$$E = \frac{Li^2}{2}.$$

Indeed, the rate of change of energy E in the inductor is the product of the voltage across the inductor and the current flowing across this potential,

$$\frac{dE}{dt} = Vi = L\frac{di}{dt}i.$$

Thus the energy stored in the inductor at time t is

$$E = \int_0^t \frac{dE}{dt}dt = \int_0^t Li\frac{di}{dt}dt = \int_0^i Lidi.$$

$$= L\frac{i^2}{2}.$$

15 Impedance Examples

Suppose we have a circuit consisting of an AC voltage source of voltage V and frequency ω. In series with this source is a resistor R and an inductor of inductance L. The impedance is then

$$Z = R + \omega L\mathbf{j} = R + X_L\mathbf{j},$$

where

$$X_L = \omega L,$$

is the inductive reactance. So the current I is

$$I = \frac{V}{Z}$$

The voltage drop across the resistor is

$$V_R = IR,$$

The voltage drop across the inductor is

$$V_L = IX_L\mathbf{j}$$

and thus is 90 degrees out of phase with V_R and the current I. So the ratio of $|V_L|$ to $|V_R|$ is

$$\gamma = \frac{|V_L|}{|V_R|} = \frac{\omega L}{R}.$$

Thus

$$L = \frac{\gamma R}{\omega}$$

So a knowledge of the voltage ratio γ, and the frequency ω gives the inductance.

Suppose f is 1000 Hz. Then $\omega = 2\pi f = 6280$. If $L = 2mH$ then

$$\omega L = 12.4\Omega.$$

Now we do not know the internal resistance of the voltage source. So in order to measure the inductance by using the voltage drop ratio, resistor R should be large enough so that the internal resistance does not take up all of the voltage drop. So the internal resistance might be on the order of 500Ω. So we might need a resistor of say $10K\Omega$. But then the frequency must be pretty high to make the inductive reactance be a significant part of $10K\Omega$, so that the voltage ratio is significant. So this method of measuring a small inductance might have some difficulty.

16 The Magnetic Induction Field B

The magnetic force on a charge q_1 due to a charge q_2 is

$$\mathbf{F}_1 = q_1\mathbf{v}_1 \times (\frac{\mu_0}{4\pi} q_2\mathbf{v}_2 \times \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3})$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the respective velocities and $\mathbf{r}_1$ and $\mathbf{r}_2$ the positions of the charges. We define the magnetic induction field $\mathbf{B}$ by

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

where **B** is due to moving charges as in the first equation. The Lorentz force is the sum of the electric and magnetic forces

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

By definition

$$\mu_0 = 4\pi 10^{-7}.$$

We have

$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$

Note: ϵ_0 is approximately

$$\frac{1}{36\pi} 10^{-9}.$$

17 Biot-Savart Law

The Biot-Savart law gives the field due to a current i flowing in an element of length $d\ell$ as

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{i d\ell \times \mathbf{r}}{r^3},$$

where $\mathbf{r}$ is a vector from the current element vector $d\ell$ to the field point. The direction of the magnetic field follows the right hand rule. With the right hand around the current element and the thumb pointing in the current direction, the direction of the magnetic field is given by the direction of the fingers.

Thus integrating around a current a single loop we find at the center of the loop, the magnetic field at the center of a single winding of radius r and carrying a current of i amperes is

$$\mathbf{B} = \frac{\mu_0 i}{2r}.$$

This differential form is equivalent to a current density definition given as

$$\mathbf{B}(\mathbf{r}_1) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J} \times (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} dx_2 dy_2 dz_2.$$

Taking the divergence we find

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int_V ((\nabla \times \mathbf{J}) \cdot \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} - \mathbf{J} \cdot (\nabla \times \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3})) dv_2.$$

The first term is zero because $\mathbf{J}$ is not a function of $\mathbf{r}_1$. The second term is zero because the curl of a gradient is zero. Thus for current sources,

$$\nabla \cdot \mathbf{B} = 0.$$

If monopoles do not exist, this is a general result.

18 The Magnetic Field produced by Various Circuits

Magnetic fields due to electric circuits can often be be computed by using the Biot-Savart Law, or sometimes by the direct use of Maxwell's equations.

18.1 The Field Due to a Straight Infinitely Long Wire

We calculate the field at the point $x = 0, y = d$, where current i flows in the positive x direction. This is the field at a distance d from the wire. According to the right hand rule the field will be in the positive z direction for each differential current element. From the Biot-Savart Law we have

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{id\ell \times \mathbf{r}}{r^3},$$

where $\mathbf{r}$ is a vector from the differential current element to the field point $(0, d$.

The line element is

$$d\ell = dx\mathbf{u}_x,$$

where $\mathbf{u}_x$ is a unit vector in the positive x direction. Hence the angle θ between $d\ell$ and $\mathbf{r}$ for x negative is between 0 and $\pi/2$, whereas for x positive it is greater it ranges from $\pi/2$ to π. For symmetric values x and $-x$ the two angles are supplements so that their sine values are equal. So the integral over negative values equals the integral over positive values. So we can integrate just for x positive to get half the field value. For x positive we have

$$\begin{aligned} \frac{d\ell \times \mathbf{r}}{r^3} &= \frac{\sin(\theta)dx}{r^2}\mathbf{u}_z \\ &= \frac{\sin(\phi)dx}{r^2}\mathbf{u}_z \end{aligned}$$

where $\phi = \pi - \theta$ is the acute angle between $\mathbf{r}$ and the x axis. We have

$$x = \frac{d}{\tan(\phi)}$$

and

$$dx = -d\tan^{-2}(\phi)\sec^2(\phi)d\phi = -\frac{d}{\sin^2(\phi)}d\phi.$$

$$r = d/\sin(\phi).$$

$$\frac{1}{r^2} = \frac{\sin^2(\phi)}{d^2}$$

So

$$\frac{\sin(\phi)dx}{r^2} = -\frac{1}{d}\sin\phi d\phi.$$

Therefore the field is

$$\mathbf{B} = -2\frac{i\mu_0}{d4\pi}\int_{\pi/2}^{0}\sin\phi d\phi\mathbf{u}_z$$

$$= \frac{i\mu_0}{d2\pi}(\cos(0) - \cos(\pi/2))\mathbf{u}_z$$

$$= \frac{\mu_0}{2\pi}\frac{i}{d}\mathbf{u}_z.$$

This can be derived more easily by using Maxwell's equation

$$\nabla \times \mathbf{H} = \mathbf{J}.$$

So consider a circle of radius d around the wire. Integrating the area bounded by the circle we have

$$\int_A \nabla \times \mathbf{H} \cdot d\mathbf{S} = \int_A \mathbf{J} \cdot d\mathbf{S} = i.$$

By symmetry $\mathbf{H}$ is tangent to the circle. So using Stokes's Theorem

$$\int_A \nabla \times \mathbf{H} \cdot d\mathbf{S} = \int_{\partial A} \mathbf{H} \cdot d\ell = d2\pi H,$$

where ∂A is the circular boundary of area A. Hence

$$H = \frac{i}{2\pi d},$$

and so

$$B = \mu_0 H = \frac{\mu_0 i}{2\pi d}.$$

18.2 Field Along the Axis of a Current Loop

Magnitude of the field at distance b from the plane of the loop of radius a

$$B = \frac{\mu_0 i}{2} \frac{a^2}{(a^2 + b^2)^{3/2}}$$

18.3 Long Solenoid

N = number of turns, L length.

At center

$$B = \frac{\mu_0 N i}{L}.$$

At end

$$B = \frac{\mu_0 N i}{2L}.$$

Greek word Solenoid means channel.

19 Computing the Inductance of a Circuit

Consider computing the inductance of a toroid of minor radius r_1 and major radius r_2. From above the inductance is

$$L = n^2 \frac{\mu A}{\ell} = n^2 \frac{\mu \pi r_1^2}{\pi (2 r_2)}$$

$$= n^2 \frac{\mu_r \mu_0 r_1^2}{2 r_2}.$$

Suppose $r_1 = 5mm$, $r_2 = 7mm$ $n = 25$ and $\mu_r = 100$. We have $\mu_0 = 4\pi \times 10^{-7}$ Henry per meter. This is $\mu_0 = 4\pi \times 10^{-4}$ Henry per mm. So we calculate

$$L = 14.02 mh.$$

As an example let us compute the inductance of a coil of n circular turns of radius R. We need to compute the flux lines Φ linking the turns. Alternately we need to compute the **B** field, when the current flowing is i, across an area S that cuts all of the flux lines and integrate to get the flux

$$\Phi = n \int_S \mathbf{B} \cdot \mathbf{dS}.$$

A suitable area is the cross sectional area of the coil. We may use the Biot-Savart law to compute **B** at each point of the surface S. We will find that Φ is a linear function of i, so that

$$L = \frac{d\Phi}{di} = \frac{\Phi}{i}.$$

20 The Wheeler Formulas For Inductance

For a circular cylinder of radius r, and length ℓ, an approximate empirical formula for the air core inductance is (Wheelers's formula)

$$L = N^2 r^2 / (9r + 10\ell),$$

where L is in micro Henries, N is the number of turns, and where r and ℓ are in inches.

The reference for this is
Harold A Wheeler, **Simple Formulas for Radio Inductance Coils**, Proceedings of the I.R.E. (Institute of Radio Engineers), October, 1928, pp 1398-1400.

Wheeler was either an engineer or a physicist with the NBS (National Bureau of Standards), which is now called NIST (National Institute for Standards and Technology).

Example. Consider a cylindrical coil 1.5 inches in diameter wound with 110 turns of 28 gauge wire. 28 gauge wire has diameter .0126 inches. So for 110 turns the length of the coil is

$$a = (110)(.0126) = 1.3860$$

inches. The radius is

$$r = 1.5/2 = .75$$

inches.

$$n = 110$$

Then the inductance is

$$L = n^2 r^2/(9(r) + 10(a)) = 330.2402\mu H = .330mH.$$

A rotary tuning capacitor in old AM radios has a maximum capacitance on the order of $C = 350pF$. Hence the resonant frequency with the plates closed would be

$$f = \frac{1}{2\pi}\frac{1}{\sqrt{LC}} = 468kHZ.$$

When the plates are nearly open with $C = 30pF$,

$$f = \frac{1}{2\pi}\frac{1}{\sqrt{LC}} = 1600kHZ.$$

21 Measuring Inductance

There are several ways to measure inductance experimentally. One method uses a function generator and an oscilloscope. So suppose a function generator generates a sinusoidal voltage v_1 and has an internal resistance R. Electrical instruments are designed to have a constant internal resistance of $R = 50\Omega$. Let us connect the function generator to an inductor of inductance L. We measure the voltage across L with an oscilloscope. The inductor has impedance $Z_L = \omega L j$. The current in the circuit is

$$i = \frac{v_1}{Z_L + R}$$

The voltage across L is

$$\begin{aligned} v_2 &= iZ_L \\ &= \frac{v_1}{Z_L + R} Z_L. \end{aligned}$$

This circuit then acts as a voltage divider. As the frequency goes to infinity, Z_L goes to infinity. So

$$\frac{Z_L}{Z_L + R} \to 1.$$

So the voltage v_2 across L goes to the input voltage v_1. Hence as we increase the frequency of the function generator we will see the voltage v_2 reach a steady value of v_1. On the other hand at zero frequency all the voltage is across the resistor R. If we alter the frequency until the magnitude of v_2 equals half the magnitude of v_1, we can determine L. So suppose for frequency ω

$$|v_2| = \frac{|v_1|}{2}.$$

Then

$$\frac{|Z_L|}{|Z_L + R|} = \frac{1}{2}.$$

That is

$$\frac{\omega L}{\sqrt{(\omega L)^2 + R^2}} = \frac{1}{2}.$$

Squaring

$$\frac{(\omega L)^2}{(\omega L)^2 + R^2} = \frac{1}{4}.$$

Then

$$3(\omega L)^2 = R^2.$$

So

$$L = \frac{R}{\omega\sqrt{3}} = \frac{R}{2\pi f\sqrt{3}},$$

where f is the frequency in cycles per second. An example of using this technique is given in the next section.

22 Measuring the Inductance of a Coil Scavenged From A PC Powersupply

This section is copied from the document called **electroniccircuits.tex**, with corresponding pdf file **electroniccircuits.pdf**.

I used the function generator in the cave, at the time CCCKC was located there. I hooked up both the function generator and the Oscilloscope across the inductor (actually one side of a transformer wound on a toroid core of unknown material, probably ferrite). I had some trouble getting the triggering to work and the amplitude set, so that I could obtain a reasonable sin wave. I set the triggering finally to vertical rather than automatic. I set the frequency so that the amplitude had reached a steady value, then adjusted

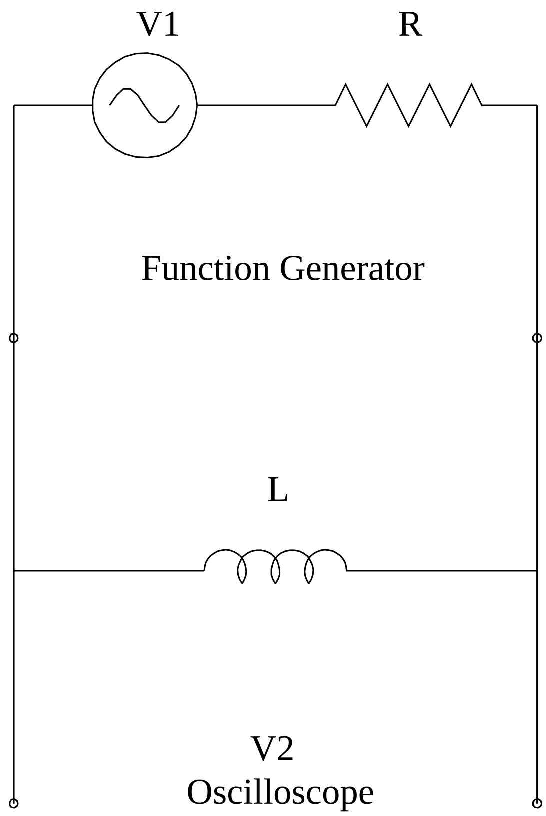

Figure 2: **Measuring Inductance.** The function generator has source voltage V_1, and standard internal resistance $R = 50\Omega$. When the frequency is very large, the voltage V_2, measured at the oscilloscope will equal V_1. Reducing the frequency f until $V_2 = V_1/2$, we find that the inductance is $L = R/(2\pi f\sqrt{3})$.

the function generator amplitude to touch a grating line. Then I reducing the frequency to 550 Hz (5.5 Hz, with 100 multiplier button) to get about half the amplitude of the previous high frequency setting (see the formula in the previous section). For f= 550 Hz I got an inductance of about 8.353 mH. The winding appears to have about 25 turns. Chris wanted an inductance of about 2mH in our circuit. Thus the number of windings should be reduced to about

$$n_2 = \sqrt{2./8.353} n_1.$$

which is about half of 25. This is because self inductance is proportional to the square of the number of turns.

Chris measured his coil to have a inductance of $1.84mh$ using

$$L = \frac{R}{2\pi f \sqrt{3}},$$

where $R = 50\Omega$. So he must have measured a frequency of about

$$f = \frac{R}{2\pi L \sqrt{3}} = 2.50 \times 10^3 Hz$$

Program frequency1.m

```
format long
R= 50.
L = 1.84e-3
f= R/(2.* pi * L * sqrt(3.))
```

I used Octave, Matlab, and Python for calculations. The file names may have been **L.m**, **inductor.m** and **L.py**.

23 Admittance

The admittance is the reciprocal of the impedance. Notice that as ω varies from 0 to ∞, Z is a line in the complex plane parallel to the imaginary axis, and it does not pass through zero. An inversion mapping, $Z \to 1/Z$, being a special case of a linear fractional transformation (also called a projective transformation or a bilinear mapping), maps circles to circles (where a straight line is a special circle of infinite radius). The line maps to a circle on the complex sphere. It passes through the north pole, but not the south pole. On the sphere the mapping $Z \to 1/Z$ corresponds to a 180 degree rotation of the sphere about a horizontal axis so that the image of the circle on the sphere no longer passes through the north pole. Hence the plane image must be a true circle. So the admittance lies on a circle provided the impedance does not pass through zero, that is, provided R is not zero. Details about such mappings are contained in many books on complex analysis. See for example: **Elements of the Theory of Functions** by K. Knopf.

In the case of a circuit network we have

$$Mi = \frac{dv}{dt},$$

where M is a matrix with second order differential operator terms, i is a current vector and v is a voltage vector. The same technique applies. The steady state solution is found by inverting the impedance matrix.

24 Series Resonance

For a series RLC circuit, we found above that the impedance is

$$Z = R + (L\omega j - j/(\omega C))$$

At resonance the impedance is minimized when the inductive reactance $X_L = L\omega$ equals the capacitive reactance $X_C = 1/(\omega C)$. Thus the resonant frequency is obtained from

$$L\omega = \frac{1}{C\omega}$$

That is

$$\omega^2 = \frac{1}{LC}$$

and

$$\omega = \frac{1}{\sqrt{LC}}.$$

At resonance the impedance in a minimum equal to only the resistance R. This is like mechanical resonance where pushing the child in a swing at a resonant frequency, causes the child and the swing to go higher and higher.

25 Parallel Resonance

Consider an inductance with impedance Z_L and a capacitance with impedance Z_C connected in parallel. Then the parallel impedance is, like parallel resistance,

$$Z = \frac{Z_L Z_C}{Z_L + Z_C}.$$

With frequency ω, we have

$$Z_L = \omega L j$$

and

$$Z_C = \frac{-j}{\omega C}$$

$$Z_L Z_C = \frac{L}{C}$$

$$Z_L + Z_C = (\omega L - 1/(\omega C))j$$

So

$$Z = \frac{j\omega L}{1 - \omega^2 LC},$$

which is ∞ as $\omega \to 1/\sqrt{LC}$. This is called parallel resonance. However, an inductor always has some resistance so Z will reach some maximum value. Then the voltage across Z is a maximum. If this parallel circuit is between an antenna and ground, then we will see a maximum voltage when the frequency has this value. By varying C, we can change this maximum frequency to get a maximum voltage at a frequency of a desired radio station. Whereas other frequencies give a low voltage, because for a high frequency Z_C is very small and for a low frequency Z_L is very small, giving low voltage drops across Z for untuned frequencies. Adding a diode or crystal for rectification and a capacitor in parallel with the diode, and an earphone, we have a primitive AM radio. The AM radio band is a frequency f from 535 kilohertz to 1,700 kilohertz, where the frequency in cycles per second or Hertz is

$$f = \frac{\omega}{2\pi}.$$

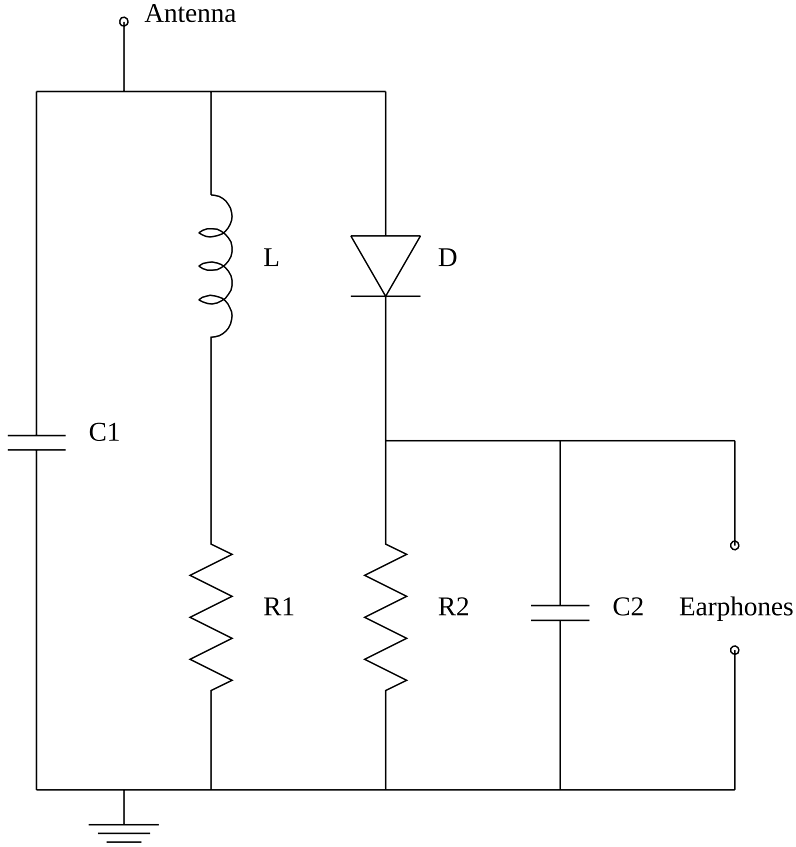

Figure 3: **AM Radio.** The radio is tuned by adjusting capacitor C_1 to obtain parallel resonance. The Diode filters away the bottom half of the AM modulated RF frequency signal leaving direct current pulses, which charge capacitor C_2, between the pulses, filling in the pulses to get the AM audio envelope signal.

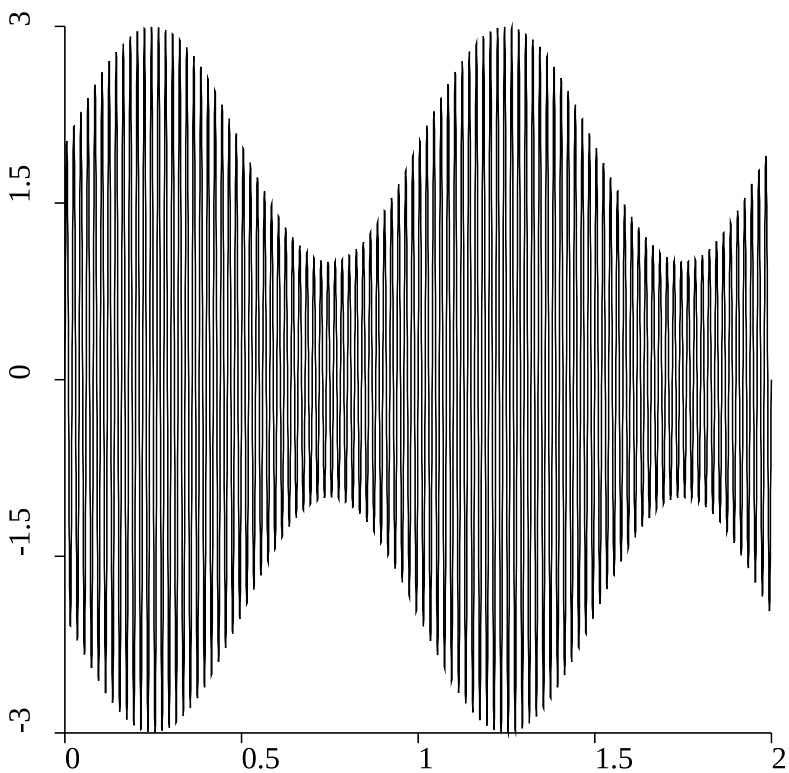

Figure 4: **AM Modulation.** An audio signal $A = 2 + \sin(\omega t)$ modulates the amplitude of a higher frequency RF signal $A \sin(n\omega t)$. The audio signal becomes the envelope. To extract the audio, the RF signal is detected with a diode giving only the positive half of the curve. This signal is filtered with a capacitor, filling in the gaps reproducing the offset audio signal.

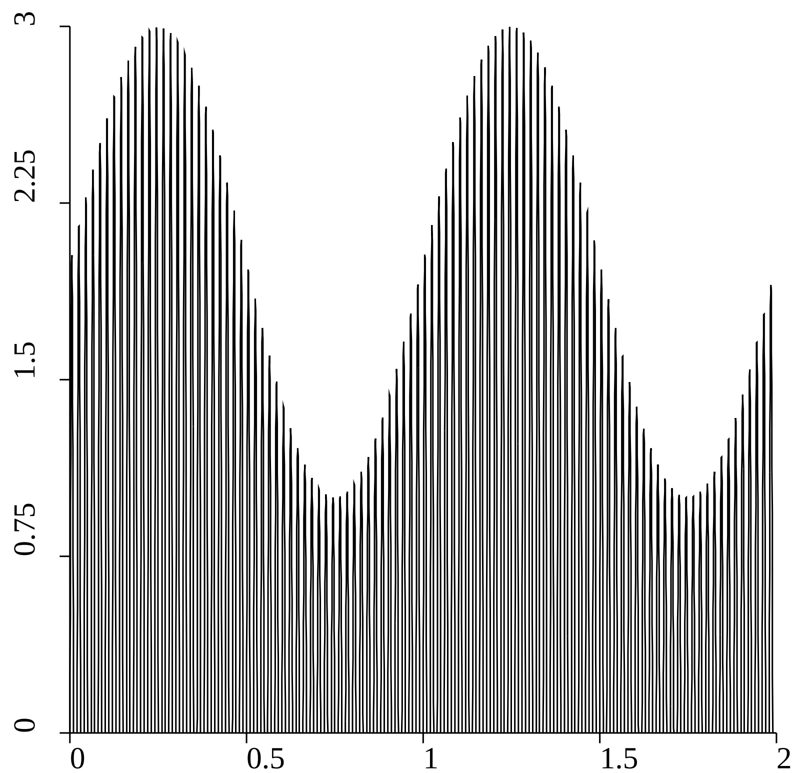

Figure 5: **AM Detection.** An audio signal $A = 2 + \sin(\omega t)$ modulates the amplitude of a higher frequency RF signal $A \sin(n\omega t)$. The audio signal becomes the envelope. To extract the audio, the RF signal is detected with a diode giving only the positive half of the curve. This signal is filtered with a capacitor, filling in the gaps reproducing the offset audio signal.

With a resistor R in series with the inductor,

$$|Z| = \sqrt{\frac{R^2 + (\omega L)^2}{(1 - \omega^2 LC)^2 + (\omega RC)^2}}$$

One can show by setting the derivative of $|Z^2|$ with respect to ω to zero, that the tuning frequency should be

$$\omega = \sqrt{-R^2/L^2 + (1/LC)\sqrt{1 + 2R^2C/L}},$$

which is an approximation to

$$\omega = \frac{1}{\sqrt{LC}},$$

for small R.

26 Amplitude Modulation

An audio signal $A = 2 + \cos(\omega t)$ modulates the amplitude of a higher frequency RF signal $A\cos(n\omega t)$. The audio signal becomes the envelope. See the figure **AM Modulation**.

We have

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b),$$

and

$$\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b).$$

Adding these together we get

$$\cos(a)\cos(b) = \frac{1}{2}(\cos(a - b) + \cos(a + b)).$$

Thus consider an RF (Radio Frequency) carrier wave of frequency

$$f_c = \frac{\omega_c}{2\pi}$$

with an amplitude

$$A = 2 + \cos(\omega_a t)$$

is given by

$$A\cos(\omega_c t) = (2 + \cos(\omega_a t))\cos(\omega_c t)$$

Using the above trigonometric identity, we find that this wave is the sum of three waves

$$2\cos(\omega_c t) + \cos((\omega_c - \omega_a)t) + \cos((\omega_c + \omega_a)t).$$

So this wave is the sum of a high frequency carrier wave of frequency f_c, a slightly lower frequency wave of frequency $f_1 = f_c - f_a$, and a slightly higher frequency wave of frequency $f_2 = f_c + f_a$. The audio frequency f_a is much smaller than the radio frequency f_c. f_a will be between 0 and about 10,000 cycles per second. and for the AM radio band f_c is between 540 and 1600 KHz. f_1 and f_2 are the sideband frequencies. The bandwidth is the band of frequencies between $f_1 = f_c - f_m$ and $f_2 = f_c + f_m$, where f_m is the highest audio frequency being broadcast, and is something less than $f_m = 10,000$ in practice for AM.

27 Microwaves and the Magnetron

Microwaves have wavelengths ranging between about 1 mm m and 30 cm. The frequency ν times the wavelength λ gives the velocity of light, about $c = 3 \times 10^8$ m per second. Thus if $\lambda = 1mm = 10^{-3}m$ then the frequency is

$$\nu = \frac{c}{\lambda} = \frac{3 \times 10^8}{10^{-3}} = 3 \times 10^1 1Hz = 300GHz.$$

At about $30cm = 300mm$ the frequency is 1 GHz.

Microwave ovens usually operate at 2.45 Ghz. So the wave length is about 30/2.45, which is 12.249 cm, or about 4.82 inches.

Microwaves are often generated by a magnetron. This is a vacuum tube with a heated filament in the center which emits electrons by the edison effect. A high voltage is applied between the negative cathode and the surrounding positive metal anode. The anode is in the form of a surrounding thick cylinder with cylindrical holes in the direction of the cylinder axis forming cavities. These cavities, like an organ tube have a resonant frequency for microwaves. Parts of the cavities are open toward the cathode in the center of the cylinder. Electrons are emitted from the heated cathode and are accelerated radially toward the surrounding anode and its cavities. There is a strong permanent magnet that creates a magnetic field in the direction of the cylinder axis. This combination of the electric field and the magnetic field make the electrons spiral outward from the heated cathode to the anode and its cavities. This is a consequence of the Lorentz force. These electrons being accelerated emit radiation in the same way that accelerating electrons in an antenna emit radiation. This radiation sets up resonant radiation in the cavities, which passes into a wave guide generating external microwaves, as a source for radar, and microwave ovens.

28 The RLC Transient Solution

The equation for a RLC circuit with a voltage source is

$$L\frac{di}{dt} + Ri + \frac{q}{C} = v.$$

Differentiating we obtain a second order differential equation with constant coefficients

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{dv}{dt}.$$

The general solution consists of a linear combination of two linearly independent solutions of the homogeneous equation and any particular solution. Given a harmonic voltage

$$v = V_0 \exp j\omega t,$$

a complex particular solution is

$$i_c = I_0 \exp j\omega t,$$

where

$$I_0 = V_0/Z,$$

and where I_0, V_0, and the impedance

$$Z = R + j(\omega L - 1/(\omega C))$$

are complex numbers. Either the real part or the imaginary part of this complex solution may be taken as a particular solution i_p. It remains to find two linearly independent solutions of the homogeneous equation

$$L\frac{d^2i}{dt^2}i + R\frac{di}{dt} + \frac{i}{C} = 0.$$

Substituting $i = \exp(jmt)$ into this equation, we get a polynomial in m, which has roots

$$-\frac{R}{2L} \pm \sqrt{(R/2L)^2 - 1/LC}.$$

The natural undamped (R=0) frequency is

$$\omega_n = \sqrt{1/LC}.$$

The critical damping resistance is the value of R that makes the expression

$$(R/2L)^2 - 1/LC$$

zero. Therefore the critical damping resistance is

$$R_c = 2L\omega_n.$$

The critical damping ratio is

$$\zeta = \frac{R}{R_c}.$$

Then we have

$$R = 2\zeta L\omega_n,$$

and

$$\frac{R}{2L} = \zeta\omega_n.$$

The system is underdamped if $\zeta < 1$, overdamped if $\zeta > 1$, and critically damped if $\zeta = 1$. Suppose the equation is underdamped. Then two independent solutions of the homogeneous equation are

$$i_1 = \exp(-\zeta\omega_n t)\sin(\omega_d t),$$

and

$$i_2 = \exp(-\zeta\omega_n t)\cos(\omega_d t),$$

where the angular frequency of damped oscillation is

$$\omega_d = \omega_n \sqrt{1-\zeta^2}.$$

We may take as particular solution $i_p = I_0 \exp(j(\omega t + \phi))$ where I_0 is real.

We have

$$i = A_1 i_1 + A_2 i_2 + i_p$$

and

$$\frac{di}{dt} = A_1 \frac{di_1}{dt} + A_2 \frac{di_2}{dt} + \frac{di_p}{dt}.$$

We wish to find the constants A_1 and A_2 appearing in these two equations.

Suppose we have the initial conditions

$$i(0) = \alpha_1$$

and

$$\frac{di}{dt}(0) = \alpha_2.$$

Then we can solve the equations for the constants A_1 and A_2 in terms of α_1 and α_2, and thus determine the unique solution $i(t)$.

We have

$$i(0) = A_2 + I_0 \exp(j\phi)$$

and

$$\frac{di}{dt}(0) = A_1 \omega_d - A_2 \zeta \omega_n + j I_0 \omega \exp(j\phi)$$

Using the real parts we have

$$A_2 + I_0 \cos(\phi) = \alpha_1$$

and

$$A_1 \omega_d - A_2 \zeta \omega_n - I_0 \omega \sin(\phi) = \alpha_2.$$

So

$$A_2 = \alpha_1 - I_0 \cos(\phi)$$

and

$$A_1 = \frac{\alpha_2 + A_2 \zeta \omega_n + I_0 \omega \sin(\phi)}{\omega_d}$$

The real solution is

$$i(t) = A_1 \exp(-\zeta \omega_n t) \sin(\omega_d t) + A_2 \exp(-\zeta \omega_n t) \cos(\omega_d t) + I_0 \cos(\omega t + \phi).$$

The program **rlc.ftn** plots this solution.

29 A Piezoelectric Transducer Equivalent Circuit

(See also: Emery **Transducer Equivalent Circuits**). In the the RLC circuit described above, suppose we introduce a second capacitor C_2 connected in parallel across the voltage source. The source is assumed active and able to supply a constant voltage for any load up to its rated capacity. The series resistance of the shunt capacitor is assumed so small that a change in voltage induces an essentially instantaneous change in the charge on the capacitor so that we can ignore the transient response. Then if i_2 is the current through the capacitor, we have

$$\frac{i_2}{C_2} = \frac{dv}{dt}.$$

If $v = V_0 \exp(j(\omega t + \phi))$, then we have

$$i_2 = jV_0 \omega C_2 \exp(j(\omega t + \phi)).$$

The real part is

$$i_2 = -V_0 \omega C_2 \sin(\omega t + \phi).$$

The source current i is the sum of the currents through the parallel branches

$$i = i_1 + i_2,$$

where i_1 is the RLC current from the previous section. Thus

$$i(t) = A_1 \exp(-\zeta\omega_n t) \sin(\omega_d t) + A_2 \exp(-\zeta\omega_n t) \cos(\omega_d t) + I_0 \cos(\omega t + \phi) - V_0 \omega \sin(\omega t + \phi).$$

We compute the constants A_1 and A_2 from the initial conditions.
Suppose the initial conditions are

$$i(0) = \alpha_1$$

and

$$\frac{di}{dt}(0) = \alpha_2.$$

Then we have

$$i(0) = A_2 + I_0 \cos(\phi) - V_0 \omega \sin(\phi)$$

and

$$\frac{di}{dt}(0) = A_1\omega_d - A_2\zeta\omega_n - I_0\omega\sin(\phi) - V_0\omega^2\cos(\phi).$$

So

$$A_2 = \alpha_1 - I_0\cos(\phi) + V_0\omega\sin(\phi)$$

and

$$A_1 = \frac{\alpha_2 + A_2\zeta\omega_n + I_0\omega\sin(\phi) + V_0\omega^2\cos(\phi)}{\omega_d}$$

The program **teqcir.ftn** plots this solution.

30 The Ideal Transformer

Two simplified models of a real transformer are the ideal transformer and the perfect transformer. The ideal transformer is characterized by the turns ratio n. The ideal transformer has the simple defining properties relating the input and output voltages and currents:

$$v_1 = nv_2$$

$$i_2 = -ni_1.$$

See Balabanian and Bickart, page 42.

31 The Perfect Transformer

The perfect transformer is characterized by a coupling ratio $k = 1$, where

$$k^2 = \frac{M^2}{L_1L_2}.$$

If $k = 1$ then $M = \sqrt{L_1L_2}$, so that

$$\frac{v_1}{v_2} = \frac{L_1di_1/dt + Mdi_2/dt}{Mdi_1/dt + L_2di_2/dt}$$

$$= \frac{\sqrt{L_1}}{\sqrt{L_2}}\left[\frac{\sqrt{L_1}di_1/dt + \sqrt{L_2}di_2/dt}{\sqrt{L_1}di_1/dt + \sqrt{L_2}di_2/dt}\right]$$

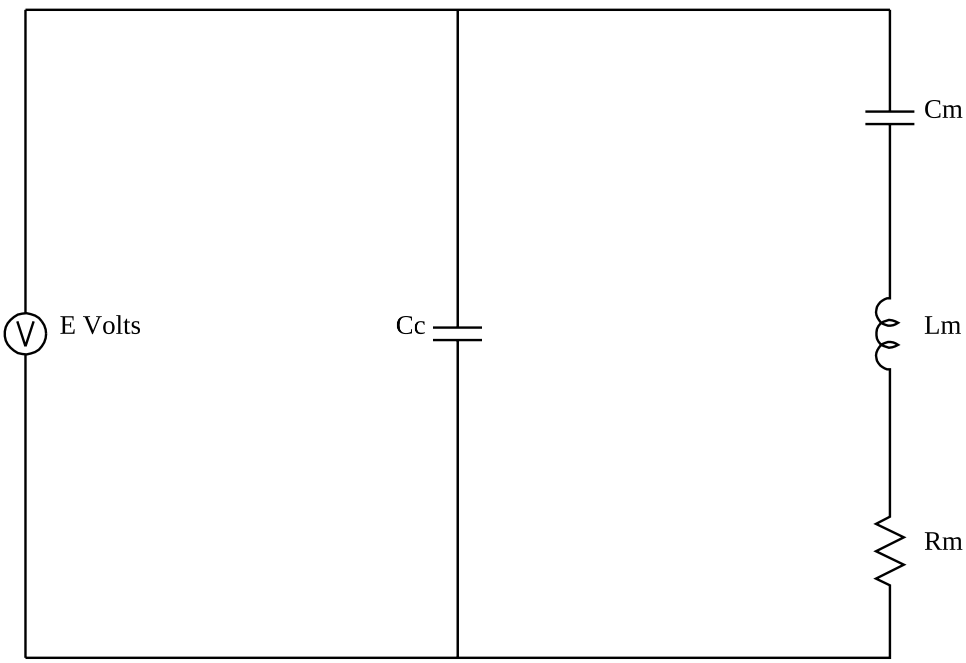

Figure 6: Transducer equivalent circuit, with motional capacitor C_m, and clamped capacitor C_c.

$$= \frac{\sqrt{L_1}}{\sqrt{L_2}}.$$

If the inductance of a coil is proportional to the square of the number of turns then

$$\frac{v_1}{v_2} = n,$$

where n is the turns ratio. The flux through a coil is proportional to the current in each turn, hence to the current times the number of turns. The induced voltage is proportional to the number of turns and to the rate of change of flux. Hence the voltage is proportional to the rate of change of current times the square of the number of turns. Hence the inductance is proportional to the square of the number of turns. Also see the inductance of two inductances in series, (Reitz and Millford). If L_1 and L_2 go to infinity while n is constant, then the perfect transformer becomes an ideal transformer. See Balabanian and Bickart pp. 43-45.

32 Transformer Equivalent Circuits

See notes 1-17-86 and see Millman and Taub, **Pulse and Digital Circuits**, pp 253-257.

33 Superposition in Linear Circuits

33.1 Thevenin's Theorem

Thevenin's theorem is presented in almost all circuit analysis books: *Any two terminals of a network may be replaced by a network consisting of an equivalent series resistor and an equivalent voltage source*

A real proof of the theorem is not often given. For a proof of the theorem, for the general case, see **Linear Network Theory**, Clifford Ferris, p256.

33.2 Norton's Theorem

34 Resonance of an RLC Circuit and The Quality Factor

Reference: Vincent Del Torro, **Principles of Electrical Engineering**, pp. 206-214. Given a series RLC circuit, for resonance we have

$$I = \frac{V}{R + j(\omega L - 1/\omega C)},$$

and

$$\omega L - 1/\omega C = 0.$$

$$\omega = \frac{1}{\sqrt{LC}}.$$

Let

$$I = I_0 \cos(\omega t).$$

The energy in the inductor is

$$e_L = \frac{1}{2} I^2 L = \frac{1}{2} I_0^2 \cos^2(\omega t).$$

The energy stored in the capacitor is

$$\begin{aligned} e_C = \frac{1}{2} C V^2 &= \frac{1}{2} C (I \frac{-j}{\omega C})^2 \\ &= \frac{1}{2} C \frac{I_0^2 \sin^2(\omega t)}{C^2 \omega^2} \\ &= \frac{1}{2} L I_0^2 \sin^2(\omega t). \end{aligned}$$

Hence

$$e_L + e_C = \frac{1}{2} L I_0^2.$$

Let ω_1 and ω_2 be the half-power points, where the power is one half the power at the resonant point ω_0. The magnitude of the impedance at the half power point must be $\sqrt{2}R$. Hence

$$\omega_1 L - \frac{1}{\omega_1 C} = R$$

Then

$$\omega_1 = -\alpha + \sqrt{\alpha^2 + \omega_0^2},$$

where

$$\alpha = \frac{R}{2L}.$$

And

$$\omega_2 = \alpha + \sqrt{\alpha^2 + \omega_0^2}.$$

The bandwidth is

$$\omega_{bw} = \omega_2 - \omega_1 = 2\alpha = \frac{R}{L}.$$

The quality factor is defined as

$$Q_0 = \frac{\omega_0}{\omega_{bw}} = \frac{\omega_0 L}{R} = \frac{X_l}{R}.$$

Then

$$Q_0 = \frac{\omega_0 L}{R} = \frac{\omega_0 L I_0^2/2}{R I_0^2/2}$$
$$= 2\pi \frac{L I_0^2/2}{(R I_0^2/2)/f_0},$$

where f_0 is the resonant frequency. Then the quality factor is 2π times the stored energy, divided by the energy dissipated per cycle.

We may write an alternate expression for Q_0. Since the resonant frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}},$$

we have

$$Q_0 = \frac{\omega_0 L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}.$$

35 Quality Factor at Nonresonance

Let the current in a series RLC circuit be

$$i(t) = I_m e^{j\omega t}.$$

The voltage across the capacitor is

$$v(t) = i\frac{-j}{\omega C} = \frac{I_m}{\omega C}(\sin(\omega t) - j\cos(\omega t))$$

The sum of the energy stored in the inductor and capacitor at time t is

$$W = \frac{1}{2}L\Re(i)^2 + \frac{1}{2}C\Re(v)^2$$

$$= \frac{1}{2}LI_m^2\cos^2(\omega t) + \frac{1}{2}\frac{I_m^2}{\omega^2 C}\sin^2(\omega t).$$

Integrating over period

$$T = \frac{2\pi}{\omega},$$

we get the average stored energy. The average of each of $\cos^2(\omega t)$, and $\sin^2(\omega t)$, is 1/2, so that

$$W_{ave} = \frac{1}{2}I_m^2\frac{1}{2}(L + \frac{1}{\omega^2 C})$$

$$= I_{rms}^2\frac{1}{2}(L + \frac{1}{\omega^2 C})$$

The energy dissipated per cycle in the resistor is

$$W_d = I_{rms}^2 RT = I_{rms}^2 R\frac{2\pi}{\omega}.$$

Then the quality factor at frequency ω is

$$Q(\omega) = \frac{2\pi W_{ave}}{W_d} = \frac{1}{2R}(\omega L + \frac{1}{C\omega}).$$

If we differentiate Q with respect to ω and set the derivative equal to zero, we get

$$L = \frac{1}{\omega^2 C}.$$

So that Q has a local extremum at the resonant frequency

$$\omega = \frac{1}{\sqrt{LC}}.$$

This is a minimum, since Q goes to infinity as ω goes to infinity, and as ω goes to zero.

At Resonance

$$\omega = \omega_r = \frac{1}{\sqrt{LC}},$$

so that

$$Q(\omega_r) = \frac{1}{2R}(\omega_r L + \frac{\omega_r}{C\omega_r^2}).$$

$$= \frac{\omega_r L}{R}.$$

Example. Consider the example of a piezoelectric equivalent circuit from **Mentesana**:

$$Z_r = 7.5\text{ohm}, f_r = 69.624\text{KHz}, Z_a = 450.7\text{ohm}, f_a = 69.942\text{KHz}.$$

Then the parameters for the equivalent mechanical RLC circuit are

$$R = 7.5\text{ohm}.$$

$$L = .0146H.$$

$$C_1 = 3.58 \times 10^{-10}F$$

The so called clamping capacitance is

$$C_2 = 3.91 \times 10^{-8}F$$

```
c qf.ftn print quality factor as a function of frequency
      implicit real*8 (a-h,o-z)
      real*8 l
      f1=20.0e3
      f2=90.0e3
      pi=4.*atan(1.d0)
      n=100
      do i=1,n
       f=(i-1)*(f2-f1)/(n-1) +f1
       omega=f*2.*pi
       q=qf(omega)
```

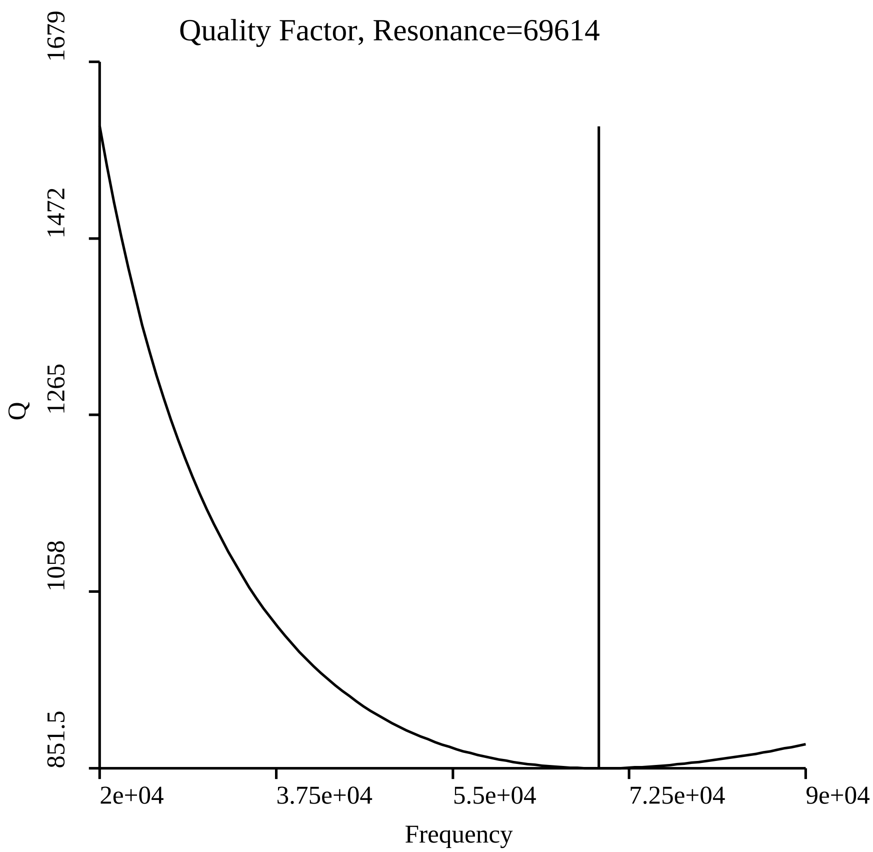

Figure 7: Plot of quality factor as a function of frequency for RLC circuit.

```
       write(*,'(2(g15.8,1x))')f,q
      enddo
      r=7.5
      l=.0146
      c=3.58e-10
      omegar=sqrt(1./(l*c))
      fr=omegar/(2.*pi)
      write(*,'(a,g15.8)')'Resonance frequency = ',fr
      q=omegar*l/r
      write(*,'(a,g15.8)')'Resonance quality factor = ',q
      end
c
      function qf(omega)
      implicit real*8 (a-h,o-z)
      real*8 l
      r=7.5
      l=.0146
      c=3.58e-10
      qf=(omega*l + 1./(c*omega))/(2.*r)
      return
      end
```

36 General Networks

To solve general networks we write down the equations for all the loops in the network. We give a name to the current flowing in each loop. Also we may need the equations for the currents flowing into each node point of the circuit graph. In the case of steady alternating currents, we will get a set of linear algebraic equations with complex coefficients. Which are easily solved using linear algebra.

For transient solutions we will get a similar set of differential equations that are usually solved by approximate numerical techniques.

37 Downloading, Installing, and Running LT Spice

LT Spice is a free version of the circuit analysis program Spice. It has a graphical user interface.

38 A Spice Direct Current Example

We shall place some examples of using Spice in this and in the next two sections.

39 A Spice Alternating Current Example

40 A Spice Transient Example

41 Drawing Circuit Diagrams

The ideal file type for graphics files in a LaTeX document like this is post-script. I wrote a program called **elecschm.ftn** that creates wiring diagrams from an input file. I give these files the filetype and extension "sch," so that a file might be called transistor.sch. The program creates a file of type eg, a file of my design. The program **eg2ps.c** creates a postscript file an eg file.

Circuit diagrams can also be created in drawing programs like Adobe Illustrator, Corel Draw, Freehand, or the open source program Inkscape. These programs can create Postscript files (actually EPS files). However the Postscript is usually ugly and verbose.

LT Spice creates diagrams that can be exported in a few file formats. But Postscript is not one of them. But there are various graphics programs to convert to Postscript. On the other hand various versions of LaTeX can also handle some bitmap file types. I prefer Postscript because file sizes can be tiny.

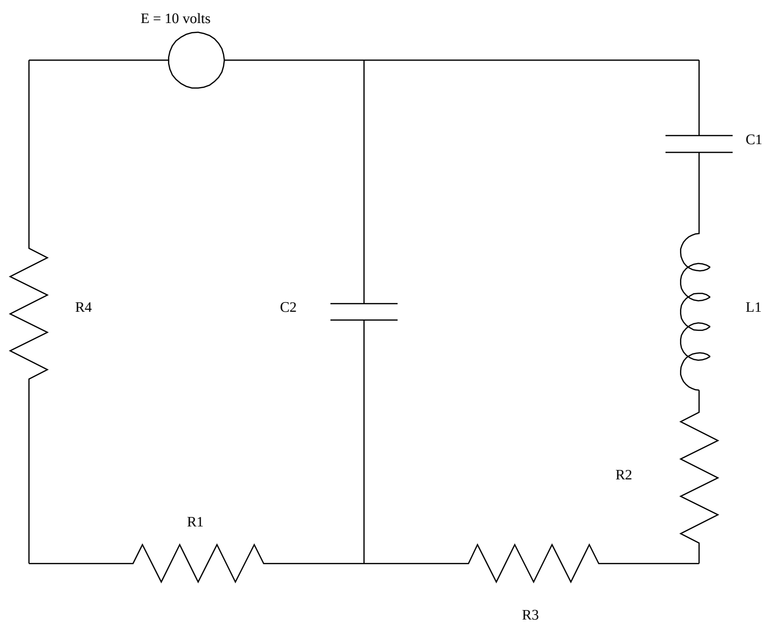

Figure 8: Circuit diagram drawn with program *elecschm.ftn*. Later versions are called *cdiagram.ftn*

42 A Resistor Capacitor Circuit and the Time Constant

Discharging a Capacitor. Consider a capacitor C with charge $q(0)$ at time $t = 0$ connected to resistor R. The voltage drops around the circuit are zero. So

$$R\frac{dq}{dt} + \frac{q}{C} = 0.$$

or

$$\frac{dq}{dt} + \frac{q}{RC} = 0,$$

and so

$$\frac{dq}{q} = -\frac{dt}{RC}$$

Integrating both sides we have

$$\ln(q) = -\frac{t}{RC} + k,$$

where k is a constant of integration. Taking the exponential we have

$$q(t) = \exp(-t/RC)\exp(k).$$

Evaluating at $t = 0$ we see that

$$\exp(k) = q(0),$$

So

$$q(t) = q(0)\exp(-t/RC).$$

and

$$\frac{q(t)}{q(0)} = \exp(-t/RC).$$

$$t_c = RC$$

is called the time constant. So in one time constant

$$\frac{q(t)}{q(0)} = \exp(-tc/RC) = \frac{1}{e} = .3679.$$

So the charge has decayed to about 37 per cent of its initial value. In five time constants it will have decayed to

$$\frac{1}{e^5} = .0067$$

of its initial value. So the capacitor discharges about $100-37 = 63$ percent of its initial charge in one time constant. The energy of the capacitor is stored in the electric field between the capacitor plates. The energy density of the field can be shown to be

$$\frac{\mathbf{E} \cdot \mathbf{D}}{2},$$

and so the total energy of the capacitor could be computed by integrating this expression over the volume between the capacitor plates. As the capacitor discharges this field energy is dissipated as heat in the resistor due to current flow through it.

The RC Circuit. Suppose a circuit consists of a voltage source of magnitude ε in series with a capacitor C and a resistor R. The voltage drops around the circuit give us the equation

$$iR + \frac{q}{C} = \varepsilon.$$

That is

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{\varepsilon}{R}.$$

We can convert the left side to a derivative of a function by multiplying by the integrating factor $e^{t/(RC)}$, and then we are able to integrate to solve our problem. So we have

$$\frac{dq}{dt}e^{t/(RC)} + \frac{q}{RC}e^{t/(RC)} = \frac{\varepsilon}{R}e^{t/(RC)}.$$

$$\frac{d}{dt}(qe^{t/(RC)})' = \frac{\varepsilon}{R}e^{t/(RC)}.$$

Integrating, we have

$$qe^{t/(RC)} = \frac{\varepsilon}{R}\int e^{-t/(RC)}dt = \frac{\varepsilon}{R}(RCe^{t/(RC)} + K),$$

where K is a constant. We have at time $t = 0$

$$q(0) = \frac{\varepsilon}{R}(RC + K),$$

so

$$K = \frac{\varepsilon}{R}q(0) - RC.$$

Then

$$q(t) = \varepsilon C + (q(0) - \varepsilon C)e^{-t/(RC)},$$

is the general solution of the RC circuit.

The capacitor discharges when the source voltage is shorted so that

$$\varepsilon = 0.$$

Then as above we get

$$q(t) = q(0)e^{-t/(RC)}.$$

Charging a Capacitor. Consider that the initial charge on the capacitor is $q(0) = 0$. Then

$$q(t) = \varepsilon C(1 - e^{-t/(RC)}.$$

The final charge reached when $t = \infty$ is $q_f = \varepsilon C$ In one time constant RC the charge on the capacitor is

$$q(\bar{t}) = q_f(1 - 1/e),$$

which is about

$$q_f(1 - .37) = .63q_f.$$

So the time constant is the time it takes to charge the capacitor to about 63 percent of its final value.

So in conclusion, the time constant RC is the time it takes for a charged capacitor C to dissipate about 63 percent of its charge through a resistor R, and on the other hand the time for an initially uncharged capacitor to reach about 63 percent of its final charge.

Example, The frequency of a 555 timer in astable mode The 555 timer in astable mode charges from 1/3 of the supply voltage ε, to 2/3 of the supply voltage through a resistor R_1 for period t_1, then discharges through a different resistor combination of value R_2 from 2/3 of the supply voltage to 1/3 of the supply voltage for period t_2. The period of the oscillation is then

$$t = t_1 + t_2.$$

Consider the time it takes to charge up from voltage $v_0 = \varepsilon/3$ to voltage $v_1 = 2\varepsilon/3$. That is from charge

$$q(0) = \frac{\varepsilon C}{3},$$

to

$$q(t_1) = \frac{2\varepsilon C}{3}.$$

That is

$$\begin{aligned} \frac{2\varepsilon C}{3} &= q(t_1) = \varepsilon C + (q(0) - \varepsilon C)e^{-t/(RC)} \\ &= \varepsilon C + (\frac{\varepsilon C}{3} - \varepsilon C)e^{-t_1/(RC)}. \end{aligned}$$

Thus

$$\frac{2}{3} = 1 + (\frac{1}{3} - 1)e^{-t_1/(RC)}.$$

So

$$-\frac{1}{3} = -\frac{2}{3}e^{-t_1/(RC)}.$$

Then

$$e^{t_1/(RC)} = 2,$$

and

$$t_1 = RC\ln(2).$$

Now let us compute the time it takes discharge the capacitor from voltage v_2 to voltage v_1. The discharge equation has a zero source voltage, that is $\varepsilon = 0$. Let t_2 be the discharge time. We have

$$q(t_2) = q(0)e^{-t_2/(RC)},$$

where

$$q(t_2) = \frac{\varepsilon C}{3},$$

and

$$q(0) = \frac{2\varepsilon C}{3}.$$

So

$$\frac{1}{3} = \frac{2}{3} e^{-t_2/(RC)}.$$

So

$$e^{t_2/(RC)} = 2.$$

So

$$t_2 = RC \ln(2).$$

Now suppose the capacitor charges through resistor R_1, and discharges through resister R_2. Then the period of oscillation is

$$T = t_1 + t_2 = C(R_1 + R_2)\ln(2).$$

The frequency is

$$\nu = \frac{1/\ln(2)}{C(R_1 + R_2)}.$$

We have

$$1/\ln(2) = 1.4427.$$

So

$$\nu = \frac{1.4427}{C(R_1 + R_2)}.$$

Referring to the National Semiconductor data sheet for the LM555, page 7-8, which gives a a circuit for the astable or oscillatory mode of the 555, there is a resistor R_A connecting the positive supply voltage at pin 8 to pin 7. A second resistor R_B connects pin 7 to pin 6. A capacitor C is connected between pin 6 and ground. So the capacitor charges through $R_1 = R_A + R_B$ and discharges back through $R_2 = R_B$. Thus

$$R_1 + R_2 = R_A + 2R_B$$

So the astable frequency is

$$\nu = \frac{1.4427}{C(R_A + 2R_B)},$$

as given in the data sheet.

43 A Resistor Inductor Circuit and the Time Constant

If we replace charge q by current i, this RL circuit has a nearly identical differential equation to the RC circuit. We have

$$L\frac{di}{dt} + iR = V.$$

L replaces R and in turn R replaces $1/C$. So the time constant is L/R. The energy stored in the electric field of the capacitor becomes the energy stored in the magnetic field of the inductor.

44 Designing Printed Circuit Boards, PCB Artist

45 Infinite Resistor Networks

See my notebook for a one dimensional infinite network, March 16, 2012 page 51.
The following paper uses symmetry, and superposition to obtain the result for a D dimensional resistive network.

$$R_{eff} = \frac{V}{I} = 2\frac{R}{M},$$

where R is the resistance of each resistor in the network, M is the connectivity at each node, and R_{eff} is the resistance that would be measured across two adjacent nodes. So for an infinite square lattice where each node connects to 4 resistors and $M = 4$, then $R_{eff} = R/2$ and for a lattice made of equilateral triangles $M = 6$ so $R_{eff} = R/3$.

Impedance between adjacent nodes of infinite uniform D-dimensional resistive lattices Am. J. Phys. 72 (7), July 2004, Peter M. Osterberg and Aziz S. Inan. Department of Electrical Engineering and Computer Science, University of Portland, Portland, Oregon 97203 Received 24 September 2003; accepted 19 December 2003.

`infiniteresistancenetwork.pdf`

Problem Consider a two dimensional network of infinite rectangles of four resistors, each resistor of having a resistance value of one ohm. Compute the resistance measured between two adjacent nodes.

Solution. We use the superposition of two problems. Consider the zero voltage reference at infinity. Let a current source be connected between node A and infinity, with a one amp flow into a node A and out of the ground at infinity. Each node is surrounded by four resistors. By symmetry .25 amps flows through each of the four resistors connected to A. Let B be a neighboring node. Let a second current source be connected between B and infinity so that one amp flows into infinity and out of node B. By superimposing the two current sources, current

$$I_{AB} = .25 + .25 = .5$$

amps flows through R_{AB}, and no current flows into or out of infinity. So the superposition is equivalent to a current source of 1 amp being connected between nodes A and B, with 1 amp going into A and 1 amp coming out of B. The potential across resistor R_{AB} is $V = .5$ volt. So the effective resistance measured between nodes A and B is

$$R = \frac{V}{I} = \frac{.5}{1.0} = .5$$

ohm.

46 Appendix A: The Laplace Transform

46.1 Introduction

One of the primary uses of the Laplace Transform is in the solution of differential equations. So differential equations are mapped to algebraic equations, and often these algebraic equations are often easier to solve than the original equations. This appendix comes from my document titled **The Laplace Transform** with file name **laplacetransform.tex**, which I plan to add to. So see that document for an up to date version.

The Laplace transform maps a function $f(t)$ of a real variable t to a function $Lf(s)$ of a complex variable s. The transform is given by

$$Lf(s) = \int_0^\infty \mathrm{f}(\,t\,)\,\mathrm{e}^{(-s\,t)}\,dt$$

Sometimes we write the transform of a function f by capitalizing. So we write

$$F(s) = Lf(s).$$

The Laplace transform of f in the symbolic computer algebra program Maple is specified as

$$\mathrm{laplace}(\,\mathrm{f}(\,t\,), t, s\,).$$

$f(t)$ is a function of a real variable, but s is a complex variable, so Lf is a complex valued function of a complex variable. Here are a few Laplace transforms.

$$\int_0^\infty \sin(\,t\,)\,\mathrm{e}^{(-s\,t)}\,dt = \frac{1}{s^2+1}$$

$$\int_0^\infty \cos(\,t\,)\,\mathrm{e}^{(-s\,t)}\,dt = \frac{s}{s^2+1}$$

$$\int_0^\infty t^a\,\mathrm{e}^{(-s\,t)}\,dt = \frac{\Gamma(\,a+1\,)}{s^{(\,a+1\,)}}$$

$\Gamma(x)$ is the Gamma function:

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt.$$

$$\lim_{x\to 0}\Gamma(x) = \infty.$$

$$\Gamma(x) = \frac{1}{x}\Gamma(x+1).$$

If n is an integer then

$$\Gamma(n+1) = n!.$$

So if n is an integer,

$$\int_0^\infty t^n e^{-st}dt = \frac{n!}{s^{n+1}}.$$

The Laplace transform of the derivative of a function f is obtained by integrating by parts. We find

$$Lf'(s) = \int_0^\infty \left(\frac{d}{dt}\,\mathrm{f}(\,t\,)\right)\,\mathrm{e}^{(-s\,t)}\,dt = s\int_0^\infty \mathrm{f}(\,t\,)\,\mathrm{e}^{(-s\,t)}\,dt - \mathrm{f}(\,0\,) = sLf - \mathrm{f}(\,0\,)$$

So the transform of a second derivative is

$$Lf'' = sLf' - f'(0) = s(sLf - f(0)) - f'(0) = s^2Lf - sf(0) - f'(0)$$

and so on for higher derivatives.

If $f(t) = A$ is constant then

$$Lf(s) = \int_0^\infty Ae^{-st}dt = \left[-\frac{A}{s}e^{-st}\right]_0^\infty = \frac{A}{s}.$$

Suppose $f(t) = e^{-at}$ then

$$Lf(s) = \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt = \frac{1}{s+a}.$$

Suppose

$$f(t) = \int_0^t g(x)dx.$$

Then $f'(t) = g(t)$, so integrating by parts we have

$$Lf(s) = \int_0^\infty f(t)e^{-st}dt$$

$$= \left[-f(t)\frac{e^{-st}}{s}\right]_0^\infty - \frac{1}{s}\int_0^\infty -e^{-st}f'(t)dt$$

$$= \frac{1}{s}\int_0^\infty e^{-st}g(t)dt$$

$$= \frac{Lg(s)}{s}.$$

We have used

$$u = f(t)$$

$$dv = e^{-st}dt$$

and

$$udv = d(uv) - vdu.$$

Let us compute $L\sin(s)$. Integrating by parts we have

$$L\sin(s) = \int_0^\infty \sin(t)e^{-st}dt$$

$$= \left[-\frac{\sin(t)e^{-st}}{s}\right]_0^\infty + \frac{1}{s}\int_0^\infty \cos(t)e^{-st}dt$$

$$= \frac{1}{s}\int_0^\infty \cos(t)e^{-st}dt$$

$$= \frac{1}{s}L\cos(s).$$

Similarly we compute $L\cos(s)$

$$L\cos(s) = \int_0^\infty \cos(t)e^{-st}dt$$

$$= \left[-\frac{\cos(t)e^{-st}}{s}\right]_0^\infty - \frac{1}{s}\int_0^\infty \sin(t)e^{-st}dt$$

$$= \frac{1 - L\sin(s)}{s}.$$

From above we have

$$L\sin(s) = \frac{1}{s}L\cos(s) = \frac{1}{s}\left[\frac{1 - L\sin(s)}{s}\right] = \frac{1 - L\sin(s)}{s^2}.$$

Solving for $L\sin(s)$, we find

$$L\sin(s) = \frac{1}{s^2+1},$$

and

$$L\cos(s) = sL\sin(s) = \frac{s}{s^2+1}.$$

Let $U(t)$ be the unit step function with step at $t = 0$. The unit step function at t_0 is

$$U_{t_0}(t) = U(t - t_0).$$

Proposition

$$L(U_{t_0}(t)f(t-t_0)) = e^{-st_0}L(f(t)).$$

Proof.

$$\begin{aligned} L(U(t-t_0)f(t-t_0)) &= \int_0^\infty e^{-st}U(t-t_0)f(t-t_0)dt \\ &= \int_{t_0}^\infty e^{-st}f(t-t_0)dt \\ &= \int_0^\infty e^{-s(t+t_0)}f(t)dt \\ &= e^{-st_0}L(f(t)). \end{aligned}$$

Example. Suppose the forcing function on the right side of the following equation is an impulse function at the point t_0. Then

$$x'' + k^2x = \delta(t-t_0)$$

$$Lx(s)(s^2+k^2) = e^{-t_0s}$$

$$\begin{aligned} Lx(s) &= \frac{e^{-t_0s}}{s^2+k^2} = e^{-t_0s}L(\sin(t)) \\ &= L(U(t-t_0)\sin(t-t_0)) \end{aligned}$$

So the solution to the differential equation is

$$x(t) = U_{t_0}\sin(t-t_0),$$

assuming the initial conditions are $x(0)=0, x'(0)=0$.

Example.

$$y'''(t) - y''(t) + y'(t) - y(t) = F(t), y(0) = y'(0) = y''(0) = 0.$$

Applying the Laplace transform, we have

$$L(y(t))(s^3 - s^2 + s - 1) = L(y(t))(s-1)(s^2+1) = L(F(t)).$$

So

$$L(y(t)) = L(F(t))\frac{1}{(s-1)(s^2+1)}.$$

Using partial fractions

$$2\frac{1}{(s-1)(s^2+1)} = \frac{1}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

So

$$2L^{-1}\frac{1}{(s-1)(s^2+1)} = e^t - \cos(t) - \sin(t).$$

Let

$$g(t) = e^t - \cos(t) - \sin(t).$$

Then we have

$$2L(y(t)) = L(F(t))L(g(t)).$$

The Laplace transform of the convolution of two functions is the product of the transforms. Thus

$$2L(y(t)) = L(F * g(t)).$$

So

$$2y(t) = F * g(t) = \int_0^t F(t-\tau)g(\tau)d\tau = \int_0^t F(t-\tau)(e^\tau - \cos(\tau) - \sin(\tau)d\tau.$$

46.2 Bessel Functions

The Bessel function of the first kind of order ν is

$$J_\nu(t) = \sum_{m=0}^{\infty} \frac{(-1)^m t^{\nu+2m}}{2^{\nu+2m} m! \Gamma(\nu+m+1)}.$$

This may also be written as

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-t^2/4)^k}{k! \Gamma(\nu+k+1)}.$$

46.3 Relation to the Fourier Transform

We define the Fourier transform as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt.$$

Some authors define it with a constant multiplier in front. The Fourier inversion theorem is

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega.$$

The double sided Laplace transform is

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt.$$

The single sided definition follows from this if $f(t)$ is zero for $t \leq 0$. Let $s = \phi + i\omega$. Then $F(s)$ is the Fourier transform of $g_\phi(t) = f(t)e^{-\phi t}$, that is

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-\phi t}e^{-i\omega t}dt$$
$$= \hat{g}_\phi(\omega).$$

For more on this see the section on the inversion of the transform.

46.4 Laplace Transform Table

`http://www.vibrationdata.com/math/Laplace_Transforms.pdf`

`or local file:`

`c:/je/pdf/Laplace_Transforms.pdf`

46.5 The Inversion of the Laplace Transform

We define the Fourier transform as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

Some authors define it with a constant multiplier in front. The Fourier inversion theorem is

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega.$$

The double sided Laplace transform is

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt.$$

Let $s = \phi + i\omega$. Then $F(s)$ is the Fourier transform of $g_\phi(t) = f(t)e^{-\phi t}$, that is

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} f(t)e^{-\phi t}e^{-i\omega t}dt \\ &= \hat{g}_\phi(\omega). \end{aligned}$$

Formally applying the Fourier inversion theorem, we have

$$\begin{aligned} f(t)e^{-\phi t} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}_\phi(\omega)e^{i\omega t}d\omega. \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{i\omega t}d\omega. \end{aligned}$$

Then

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{\phi t}e^{i\omega t}d\omega. \\ &= \frac{1}{2\pi i}\int_{C_\phi} F(s)e^{st}ds, \end{aligned}$$

where C_ϕ is the Bromwich contour defined by

$$\{\phi + i\omega : -\infty < \omega < \infty\}.$$

Note that i appears in the expression $2\pi i$ because

$$ds = id\omega.$$

In general we will find that if we define a closed curve consisting of a finite line of length $2R$ on the Bromwich contour, and a semicircle of radius R to the left, then as R goes to infinity, the integral over the semicircle goes to zero, so that the total integral over the curve is equal to the integral on the Bromwich line, which is thus equal to $2\pi i$ times the residues of $F(s)e^{st}$ in the left half-space bounded by the contour. Our inversion expression is therefore equal to the sum of the residues themselves. We get the single sided Laplace transform from the double when $f(t)$ is equal to zero for $t \leq 0$.

Example: Consider
$$F(s) = \frac{1}{s-1},$$
for $\Re(s) > 1$. The residue of $F(s)e^{st}$ is
$$\lim_{s\to 1}(s-1)F(s)e^{st} = e^t.$$
Therefore
$$f(t) = e^t.$$
Example: Consider
$$F(s) = \frac{1}{s^2+1} = \frac{1}{(s-i)(s+i)},$$
for $\Re(s) > 0$. The residues of $F(s)e^{st}$ are
$$\lim_{s\to i}(s-i)F(s)e^{st} = \frac{e^{it}}{2i},$$
and
$$\lim_{s\to -i}(s+i)F(s)e^{st} = \frac{e^{-it}}{-2i},$$
Therefore
$$f(t) = \frac{e^{it} - e^{-it}}{2i} = \sin(t).$$

46.6 The Laplace Transform in Maple

See my documents **maple.tex** and **mapletwelve.tex**, titled **Quintessential Maple V** and **Quintessential Maple XII**. The computer Algebra program Maple, like many software programs changes a bit from time to time, so new documentation is required.

46.7 Solving a Differential Equation With The Laplace Transform Using Maple

This section has been made compatible with Maple 12. We read the following file into Maple:

```
% cat mlaplace
with(invtrans)
de:=diff(y(x),x,x)+2*diff(y(x),x)+y(x) = sin(2*x);
dsolve({de,y(0)=1,D(y)(0)=1},y(x));
laplace(de,x,s);
subs(laplace(y(x),x,s)=G,%);
solve(",G);
subs({D(y)(0)=1,y(0)=1},%);
invlaplace(%,s,x);
```

The above code was pasted into Maple 12. The laplace transform would-not work, until I blundered onto some information that the laplace transform and inverse laplace transform are in the inttrans package that must be loaded. Also the previous expression representation had to be changed to per cent sign from the double quote sign. Maple 12 gives equivalent though different forms for the results calculated by Maple 5, and which are listed here. The session is as follows:

```
> de:=diff(y(x),x,x)+2*diff(y(x),x)+y(x) = sin(2*x);
```

$$de := \left(\frac{\partial^2}{\partial x^2}\mathrm{y}(x)\right) + 2\left(\frac{\partial}{\partial x}\mathrm{y}(x)\right) + \mathrm{y}(x) = \sin(2\,x)$$

```
> dsolve({de,y(0)=1,D(y)(0)=1},y(x));
```

$$\mathrm{y}(x) = -\frac{4}{25}\cos(2\,x) - \frac{3}{25}\sin(2\,x) + \frac{29}{25}\mathrm{e}^{(-x)} + \frac{12}{5}\mathrm{e}^{(-x)}\,x$$

```
> laplace(de,x,s);
```

$$(\mathrm{laplace}(\mathrm{y}(x),x,s)\,s - \mathrm{y}(0))\,s - \mathrm{D}(y)(0) + 2\,\mathrm{laplace}(\mathrm{y}(x),x,s)\,s - 2\,\mathrm{y}(0) + \mathrm{laplace}(\mathrm{y}(x),x,s) = 2\,\frac{1}{s^2+4}$$

```
> subs(laplace(y(x),x,s)=G,%);
```

$$(G\,s - \mathrm{y}(0))\,s - \mathrm{D}(y)(0) + 2\,G\,s - 2\,\mathrm{y}(0) + G = 2\,\frac{1}{s^2+4}$$

```
> solve(%,G);
```

$$-\frac{-s\,\mathrm{y}(0) - \mathrm{D}(y)(0) - 2\,\mathrm{y}(0) - 2\,\dfrac{1}{s^2+4}}{s^2+2\,s+1}$$

```
> subs({D(y)(0)=1,y(0)=1},%);
```

$$-\frac{-s - 3 - 2\,\dfrac{1}{s^2+4}}{s^2+2\,s+1}$$

```
> invlaplace(%,s,x);
```

$$-\frac{4}{25}\cos(2\,x) - \frac{3}{25}\sin(2\,x) + \frac{29}{25}\,\mathrm{e}^{(-x)} + \frac{12}{5}\,\mathrm{e}^{(-x)}\,x$$

The solution using dsolve, and the solution using the Laplace transform method are the same.

46.8 Solving Circuit Problems With the Laplace Transform

Resistor Capacitor Circuit Let a circuit consist of a constant voltage source V be in series with a Resistor R and a capacitor C. The voltage loop equation is

$$Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau + \frac{q_0}{C} = V,$$

where $i(t)$ is the current, and q_0 is the initial charge on the capacitor. We have

$$i(t) + \frac{1}{RC}\int_0^t i(\tau)d\tau + \frac{q_0}{RC} = \frac{V}{R}.$$

Taking the Laplace Transform

$$Li(s) + \frac{1}{RC}\frac{Li(s)}{s} + \frac{q_0}{RC}\frac{1}{s} = \frac{V}{R}\frac{1}{s}.$$

Then

$$Li(s)\left(1 + \frac{1}{RCs}\right) = \frac{CV - q_0}{RCs}$$

and so

$$Li(s) = \frac{CV - q_0}{RCs + 1} = \frac{CV/(RC) - q_0/(RC)}{s + 1/(RC)}$$

Then

$$Li(s) = (V/R - q_0/(RC))\frac{1}{s + 1/(RC)}.$$

So taking the inverse transform

$$i(t) = (V/R - q_0/(RC))e^{-t/(RC)}.$$

To find the charge we integrate

$$q(t) = (V/R - q_0/(RC))\int e^{-t/(RC)}$$

$$= (V/R - q_0/(RC))(-RC)e^{-t/(RC)} + K,$$

where K is a constant. So

$$q(t) = (q_0 - VC)e^{-t/(RC)} + K$$

At zero

$$q_0 = q(0) = (q_0 - VC) + K,$$

so $K = VC$. Finally

$$q(t) = q_0e^{-t/(RC)} + VC(1 - e^{-t/(RC)}).$$

47 Bibliography for Laplace Transforms and Differential Equations

There are a huge number of books on elementary differential equations, so I will list just a couple of representative elementary books. Most books on Advanced Engineering Mathematics have material on Laplace Transforms, as do many Electrical Engineering Books. Most Calculus books have some material on Differential Equations.

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48 General Bibliography

There are pdf files for the tex documents by James Emery listed here. Most of these are at the stem2.org website. So for example to access the pdf file for the document electromagnetictheory.tex with title **Electromagnetic Theory**, use link

`http://stem2.org/je/electromagnetictheory.pdf`

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[3]Balabanian And Bickart, **Linear Network Theory: Analysis, Properties, Design And Synthesis**, 1981, Linda Hall TK454.2 B36.

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[25] Emery James D, **Electronics**

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[26] Emery James D, **Electronic Circuit Examples** This consists of notes and topics related to the the class presented by Cris Wilkson on Analog Electronics at the Cave for CCCKC in 2010, maybe going into 2011. This class was held one night each week, and would sometimes go on until the wee hours of the morning. The cave is an underground limestone mine located at 31st and Mercier in Kansas City Missouri. CCCKC stands for (Cowtown Computer Congress Kansas City). This is a Hacker Space and Maker Space. In the fall of 2011, CCCKC moved to the Hammer Space at 440 E 63rd Street, Kansas City, Missouri.

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49 Index

When viewing this document with a pdf reader, one can search for key words to locate topics. Because this is a dynamic document, it would be difficult to create an accurate index. The key word "figure" can be used to locate the figures.