Elasticity

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1 Stress

The stress vector is defined by

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A},$$

where ΔA is a plane area element. Let n be a unit normal to the plane, then

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

The stress tensor is symmetric. This is proved by considering the conditions for static equilibrium of a small volume.

2 Strain

Let u_1, u_2, u_3 be the coordinates of a displacement. Let

$$e_{ij} = (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})/2.$$

See Solkolnikoff, **The Mathematical Theory of Elasticity**, p25, for comments about alternate notation for the strain coefficients. The strain is a cartesian tensor, as one sees by looking at the properties of the gradient and the Jacobian. Let the displacement vector be

$$U(x_1, x_2, x_3) = (u_1(x_1, x_2, x_3), u_2(x_1, x_2, x_3), u_3(x_1, x_2, x_3))$$

at the point (x_1, x_2, x_3). The matrix of the derivative of U with respect to X is called the Jacobian J. It has components

$$J_{ij} = \frac{\partial u_i}{\partial x_j}.$$

The strain is equal to 1/2 of the sum of the Jacobian and its transpose.

$$\frac{1}{2}(J + J^T).$$

Clearly the Jacobian is a tensor, so the strain, being the sum of two tensors is a tensor. Mathematically, a tensor is simply a multilinear functional defined on a differential geometric tangent space or cotangent space (dual space), or on some combination of them. So given a new coordinate system, the new coordinates of the tensor are computed from the coordinate transformation. Physically, a set of quantities might be definable in a given laboratory coordinate system experimentally, or by some calculation. If in any new coordinate system, the measured values, or intrinsically computed values, turn out to be the same as the mathematically transformed values, then the set of quantities is called a tensor. When the coordinates we are dealing with are just the simple cartesian coordinates and with their simple transformations, and if the measured quantities in two coordinate systems agree with the mathematically transformed quantities, according to these special simple cartesian transformations, then the quantities are said to constitute a cartesian tensor. The strain tensor is symmetric by definition, so there are six independent components. The off-diagonal elements are called the shear components. Sometimes an **Engineering Shear Strain** is defined as

$$\gamma_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i},$$

for i not equal to j, which is double the tensor component. Clearly, then doubling the off-diagonal elements, but not the diagonal elements, causes the strain to loose its tensor character. There are two reasons why the engineering shear strains exist in the literature. (1) Love's classical treatise on elasticity used them, and (2) the engineering shear strain equals the shear angle in the case of a pure shear, rather than being equal to one half this angle for the tensorial shear strains.

3 The General Infinitesimal Displacement

See Sommerfeld, **The Mechanics of Deformable Bodies**, or Lass **Elements of Pure and Applied mathematics**. A general displacement consists of, a translation, a rotation (antisymmetrix tensor), and a deformation (symmetric strain tensor). The general displacement is relevant to both elasticity theory and to fluid dynamics.

4 Shear Strain Notation In Various Books

Some books that use engineearing shear strains, rather than the tensor shear strains are: Love, Timoshenko and Goodier, McFarland and Smith and Bernhardt, Soedel, Den Hartog, and most books on strength of materials. The book by Treffitz doubles all components of the strain tensor, thus preserving the tensor character. The classic book by Prescott Applied Elasticity uses such old non-vector notation, that it is not clear what strain system is being used. The book by Lai and Saibel, uses the mathematical tensor definition, but also introduces the engineering strain notation γ_{ij}.

5 Vector Notation For Stress and Strain

The stress and strain tensors are symmetric. In each case there are six independent components. These six components may be stored as a vector. We assign the two tensor indices to a single vector index as follows:

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ \sigma_6 & \sigma_2 & \sigma_4 \\ \sigma_5 & \sigma_4 & \sigma_3 \end{bmatrix}$$

The sequential numbering of vector components is obtained by traveling down the main diagonal of the matrix, then up the last column, and then back along the first row. For the strain case we have:

$$e = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = \begin{bmatrix} e_1 & e_6 & e_5 \\ e_6 & e_2 & e_4 \\ e_5 & e_4 & e_3 \end{bmatrix}$$

6 The Elastic Matrix

The generalized Hook's law in tensor form is

$$\sigma_{ij} = c_{ijkl} e_{kl}.$$

Thus

$$\sigma_{ij} = c_{ij11} e_{11} + c_{ij12} e_{12} + c_{ij13} e_{13}$$
$$+ c_{ij21} e_{21} + c_{ij22} e_{22} + c_{ij23} e_{23}$$

$$+c_{ij31}e_{31} + c_{ij32}e_{32} + c_{ij33}e_{33}$$

$$= c_{ij11}e_1 + (c_{ij12} + c_{ij21})e_6 + (c_{ij13} + c_{ij31})e_5$$

$$c_{ij22}e_2 + (c_{ij23} + c_{ij32})e_4 + c_{ij33}e_3.$$

Also

$$\sigma_{ij} = \sigma_{ji} = c_{ji11}e_{11} + c_{ji12}e_{12} + c_{ji13}e_{13}$$

$$+c_{ji21}e_{21} + c_{ji22}e_{22} + c_{ji23}e_{23}$$

$$+c_{ji31}e_{31} + c_{ji32}e_{32} + c_{ji33}e_{33}$$

$$= c_{ji11}e_1 + (c_{ji12} + c_{ji21})e_6 + (c_{ji13} + c_{ji31})e_5$$

$$c_{ji22}e_2 + (c_{ji23} + c_{ji32})e_4 + c_{ji33}e_3.$$

Because the stress tensor σ is symmetric, and $\sigma_{ij} = \sigma_{ji}$, the elasticity tensor c is symmetric with respect to the first two indices. With respect to the last two indices, we may write the tensor c as a sum of a symmetric and a skew symmetric tensor (Sokolnikoff p59). We can write

$$\sigma_{ij} = c'_{ijkl}e_{kl} + c"_{ijkl}e_{kl}.$$

But because the strain tensor is symmetric, the skew symmetric part of the sum $c"$ vanishes, in the equation, so we might as well take c as its symmetric part, and consider c symmetric in the last two indices also.

Then if p corresponds to ij and q to kl, we may define matrix coefficients c_{pq}.

When $k \neq l$ we define

$$c_{pq} = c_{ijkl} + c_{ijlk} = c_{jikl} + c_{jilk}.$$

When $k = l$, we define

$$c_{pq} = c_{ijkk}.$$

The tensor equation written out is

$$\sigma_p = \sigma_{ij} = c_{ij11}e_{11} + c_{ij22}e_{22} + c_{ij33}e_{33}$$

$$(c_{ij23} + c_{ij32})e_{23} + (c_{ij13} + c_{ij31})e_{13} + (c_{ij12} + c_{ij21})e_{12}$$

$$= c_{p1}e_1 + c_{p2}e_2 + c_{p3}e_3 + c_{p4}e_4 + c_{p5}e_5 + c_{p6}e_6.$$

By considering the inverse coefficients, we also have

$$c_{qp} = c_{kkij}.$$

According to the way we have defined c_{pq}, when $k \neq l$, we have

$$c_{ijkl} = c_{ijlk} = c_{jikl} = c_{jilk} = c_{pq}/2,$$

and when $k = l$, we have

$$c_{ijkk} = c_{jikk} = c_{pq}.$$

The tensor equation becomes the matrix equation

$$\sigma_p = c_{pq} e_q.$$

7 Isotropic Constants

Hooks' law is

$$\sigma = ce,$$

where σ is a six-vector of the stress components, e is a six-vector of the strain components, and c is the 6 by 6 elastic matrix. For an isotropic solid this reduces to the equations:

$$\sigma_1 = (\lambda + 2\mu)e_1 + \lambda e_2 + \lambda e_3$$

$$\sigma_2 = \lambda e_1 + (\lambda + 2\mu)e_2 + \lambda e_3$$

$$\sigma_3 = \lambda e_1 + \lambda e_2 + (\lambda + 2\mu)e_3$$

$$\sigma_4 = 2\mu e_4$$

$$\sigma_5 = 2\mu e_5$$

$$\sigma_6 = 2\mu e_6,$$

where λ and μ are the Lame' constants. Tensor forms of Hooks' law for the isotropic case are

$$\sigma_{ij} = \lambda \delta_{ij}(e_{11} + e_{22} + e_{33}) + 2\mu e_{ij}$$

and

$$e_{ij} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}\delta_{ij}(\sigma_{11} + \sigma_{22} + \sigma_{33}) + \frac{1}{2\mu}\sigma_{ij}.$$

Refer to equation 22.3, of Sokolnikoff. We shall find expressions for the Lame' constants in terms of Young's modulus E and Poisson's ratio ν. Then the tensor equation for strain becomes

$$e_{ij} = -\frac{\nu}{E}\delta_{ij}(\sigma_{11} + \sigma_{22} + \sigma_{33}) + \frac{1}{2\mu}\sigma_{ij}.$$

In matrix form we have

$$\sigma = \begin{bmatrix} (\lambda + 2\mu) & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & (\lambda + 2\mu) & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & (\lambda + 2\mu) & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} e.$$

Suppose a rod is under tensile stress in one direction only. Then the first three equations become:

$$(\lambda + 2\mu)e_1 + \lambda e_2 + \lambda e_3 = \sigma_1$$

$$\lambda e_1 + (\lambda + 2\mu)e_2 + \lambda e_3 = 0$$

$$\lambda e_1 + \lambda e_2 + (\lambda + 2\mu)e_3 = 0$$

Solving these equations, we find the strains to be

$$e_1 = \frac{\sigma_1(\lambda + \mu)}{\mu(3\lambda + 2\mu)}$$

$$e_2 = -\frac{\sigma_1 \lambda}{2\mu(3\lambda + 2\mu)}$$

$$e_3 = -\frac{\sigma_1 \lambda}{2\mu(3\lambda + 2\mu)}$$

Young's modulus is the ratio of tensile stress to tensile strain.

$$E = \frac{\sigma_1}{e_1},$$

Thus

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

Poisson's ratio is the ratio of transverse strain to longitudinal strain,

$$\nu = \frac{-e_2}{e_1},$$

Thus

$$\nu = \frac{\lambda}{2(\lambda + \mu)}.$$

We can solve the equations for Young's modulus and Poisson's ratio to obtain the Lame' constants.

$$\lambda = -\frac{E\nu}{2\nu^2 + \nu - 1} = \frac{E\nu}{(1 - \nu)(1 - 2\nu)}.$$

$$\mu = \frac{E}{2(\nu + 1)}.$$

Then Poisson's ratio is defined by Young's modulus, and the shear modulus μ.

$$\nu = \frac{E}{2\mu} - 1.$$

Let a cube of side x experience a uniform pressure σ. Let Δv be the change in volume. Δv is $(x + dx)^3 - x^3$, which is approximately $3x^2 dx$. Then

$$\Delta v / v = 3dx/x = 3e_1 = e_1 + e_2 + e_3.$$

The bulk modulus k is the ratio of the pressure s to $\Delta v / v$. For the case of uniform pressure the equations become

$$(\lambda + 2\mu)e_1 + \lambda e_2 + \lambda e_3 = \sigma$$

$$\lambda e_1 + (\lambda + 2\mu)e_2 + \lambda e_3 = \sigma$$

$$\lambda e_1 + \lambda e_2 + (\lambda + 2\mu)e_3 = \sigma$$

We find

$$e_1 = e_2 = e_3 = \frac{\sigma}{3\lambda + 2\mu}.$$

Then the bulk modulus is

$$k = \frac{\sigma}{3e_1} = \lambda + \frac{2}{3}\mu.$$

Consider the case of pure shear so that

$$u_1 = \alpha x_2,$$

for a constant α. Then

$$\frac{\partial u_1}{\partial x_2} = \alpha.$$

And so

$$\begin{aligned} e_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \\ &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + 0\right) \\ &= \frac{\alpha}{2}. \end{aligned}$$

Now

$$\frac{\partial u_1}{\partial x_2},$$

is the tangent of the shear angle, which for small shear is nearly equal to the angle itself. Hence the shear strain e_{12} is equal to one half of the shear angle α. We have

$$2\mu e_{12} = \sigma_{12},$$

so

$$\mu = \frac{\sigma_{12}}{2e_{12}} = \frac{\sigma_{12}}{\alpha} = \frac{\sigma_{12}}{\tan(\alpha)}.$$

The Lame' shear modulus μ is also written as g. The engineering shear strain equals the shear angle in the case of a pure shear, rather than being equal to one half this angle for the tensorial shear strain component.

The shear modulus is given as the shear stress divided by the shear angle.

Material	E (10^6 psi)	μ (10^6 psi)	ν
Steel	29.5	11.5	.29
Copper	15.0	5.6	.33
Glass	8	3.2	.25
Spruce	1.5	.08	

For isotropic materials, We have

$$e_i = c'_{ij}\sigma_j,$$

where the inverse elastic matrix is

$$c' = \begin{bmatrix} c'_{11} & c'_{12} & c'_{13} & 0 & 0 & 0 \\ c'_{21} & c'_{22} & c'_{23} & 0 & 0 & 0 \\ c'_{31} & c'_{32} & c'_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c'_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c'_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c'_{66} \end{bmatrix}.$$

$$= \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2\mu} \end{bmatrix}.$$

8 Orthotropic Elasticity Coefficients

An orthotropic problem is one in which the material properties are symmetric with respect to 3 mutually orthogonal planes. (See Sokolnikoff p62.). By applying various symmetry transformations, and invariance, one finds that many of the elastic constants must be zero. The elastic matrix becomes

$$c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}.$$

The inverse matrix is

$$c' = c^{-1} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/(2\mu_{32}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/(2\mu_{31}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/(2\mu_{12}) \end{bmatrix}.$$

The Poisson ratio, ν_{ij}, is the ratio of the negative of the strain in the x_j direction, to the strain in the x_i direction, for a normal stress in the x_i direction. For example, if the only nonzero stress component is σ_1, then

$$\begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/(2\mu_{32}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/(2\mu_{31}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/(2\mu_{12}) \end{bmatrix} \begin{bmatrix} \sigma_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1/E_1 \\ (-\sigma_1\nu_{12})/E_1 \\ (-\sigma_1\nu_{13})/E_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

So that

$$\nu_{12} = \frac{-\varepsilon_2}{\varepsilon_1},$$

and

$$\nu_{13} = \frac{-\varepsilon_3}{\varepsilon_1},$$

By the symmetry of c, we have

$$\nu_{ij}/E_i = \nu_{ji}/E_j.$$

So there are three independent values of Poisson's ratio. We take these to be

$$\nu_{12}, \nu_{13}, \nu_{23}.$$

If the engineering strain vector is used then the shear strains are twice the physics shear strains, so that the 2 multiplying a shear modulus is suppressed.

The engineering shear strain is equal to the shear angle, the physics shear strain is equal to one half the angle.

The notation for shear modulus μ_{ij} depends upon which of the indicial mappings f or g is used in assigning the stress-strain tensor indices to the vector indices.

9 The Thin Beam Equation

Let ρ be the radius of curvature of the neutral axis of the thin beam. Let y be the distance from the neutral axis to a point in the beam cross section. When the beam is bent, a length ℓ changes to a length $\ell + \delta\ell$. The length at the neutral axis does not change. The length ℓ on the neutral axis, and the radius of curvature ρ, are lengths of sides of a triangle. The distance from the neutral axis y and the length change $\delta\ell$, when the deflection is small, are sides of a similar triangle. So using similar triangles, we find

$$\frac{\delta\ell}{y} = \frac{\ell}{\rho}.$$

Then the strain is

$$e = \frac{\delta\ell}{\ell} = \frac{y}{\rho}.$$

The stress is

$$\sigma = E\frac{y}{\rho}.$$

The net force is zero over the cross section, so integrating we get

$$0 = \int \sigma dA = \frac{E}{\rho}\int y dA.$$

The integral is the area centroid. The neutral axis passes through the centroid. The bending moment is given by

$$M = \int y\sigma dA = \frac{E}{\rho}\int y^2 dA = \frac{E}{\rho}I.$$

where I is the area moment of inertia. The curvature is the reciprocal of the radius of curvature and we have

$$\frac{1}{\rho} = \kappa = \frac{d^2y/dx^2}{(1 + (dy/dx)^2)^{3/2}}.$$

When the slope is small the curvature is approximately equal to the second derivative, hence the beam equation is

$$\frac{d^2y}{dx^2} = \kappa = \frac{M}{EI}.$$

When the beam is subjected to point loads , the bending moment is linear between the point loads, hence on each segment of the beam we have an equation of the form

$$\frac{d^2y}{dx^2} = \frac{ax+b}{EI},$$

where a and b are constants depending on the point loads. This integrates to a cubic polynomial. Hence the solution deflection curve is a piecewise cubic polynomial with continuous first and second derivatives, and zero second derivatives at the ends. The deflection curve is a natural cubic spline. A natural cubic spline has zero second derivatives at the ends. The curve becomes straight at the ends.

Let Q be the transverse shear stress in the beam, and let p be the downward pressure per unit length. Then the conditions for equilibrium of a element of the beam of length dx is that the vertical force vanish, so that

$$\frac{\partial Q}{\partial x} + p = 0,$$

and the moment about an edge of the element vanishes so that

$$\frac{\partial M}{\partial x} = Q.$$

Combining these equations we get

$$\frac{\partial^2 M}{\partial x^2} = -p.$$

Substituting above for the moment M, we get

$$\frac{\partial^4 w}{\partial x^4} = \frac{-p}{EI}.$$

Suppose the beam is freely vibrating. We replace the force on the element pdx by a mass times acceleration term

$$\rho A dx \frac{d^2w}{dt^2}.$$

Then the free vibration equation is

$$\frac{\partial^4 w}{\partial x^4} = \frac{-\rho A}{EI}\frac{d^2 w}{dt^2}.$$

We take as a solution

$$w = C\sin(kx)e^{iwt}.$$

The wave number is

$$k = \frac{2\pi}{\lambda},$$

where λ is the wavelength. We find that

$$k^4 = \omega^2\frac{\rho A}{EI},$$

which defines the natural frequency ω as a function of a given wavelength. If two nodes of a standing wave are given, the wave length must fit an integral number of times between these two nodes.

10 The Deflection of a Thin Plate

A reference for this material is **The Analysis of Plates**, McFarland, Smith, and Bernhardt. Let ξ be the distance from the mid plane. Let u be the x displacement, v the y displacement, and w the z displacement. Then the slope angle in the x direction is approximately $\partial w/\partial x$ hence

$$u = -\xi\frac{\partial w}{\partial x} + u_m,$$

where u_m is the displacement of the closest point on the midplane. We assume that the midplane is not distorted or stretched so that

$$\frac{\partial u_m}{\partial y} = 0.$$

Then

$$\frac{\partial u}{\partial y} = -\xi\frac{\partial^2 w}{\partial y \partial x}.$$

Similarly

$$v = -\xi\frac{\partial w}{\partial y} + v_m,$$

and

$$\frac{\partial v}{\partial x} = -\xi \frac{\partial^2 w}{\partial x \partial y}.$$

The strain in the x direction is

$$\epsilon_{11} = -\frac{\Delta s(0) - \Delta s(\xi)}{\Delta s(0)}.$$

Now the element length $\Delta s(\xi)$ is proportional to the radius of curvature, which is

$$\rho_x - \xi,$$

where ρ_x is the radius of curvature of the midplane. Hence

$$\epsilon_{11} = -\frac{\rho_x - (\rho_x - \xi)}{\rho_x} = -\frac{\xi}{\rho_x}.$$

The curvature κ_x is approximately equal to the second derivative, so

$$\frac{1}{\rho_x} = \kappa_x = \frac{\partial^2 w}{\partial x^2}.$$

Therefore

$$\epsilon_{11} = -\xi \frac{\partial^2 w}{\partial x^2}.$$

Similarly

$$\epsilon_{22} = -\xi \frac{\partial^2 w}{\partial y^2}.$$

The shear strain ϵ_{12} is

$$\epsilon_{12} = \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}),$$

which from the above expression for the tangents is

$$\epsilon_{12} = -\xi \frac{\partial^2 w}{\partial x \partial y}.$$

Now introducing the elastic matrix, we may write the stress components in terms of the partial derivatives of the displacement w. From the stress components, we may write the moments and normal forces on a volume element, after integrating over the faces, in terms of the partial derivatives of the displcement w.

The static equilibrium of a rectangular slice of the plate of side dx by dy gives the moment equation

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p,$$

where p is the pressure on top of the plate. Substituting for the moments, we obtain the biharmonic equation

$$\nabla^4 w = -p/D.$$

This is equivalent to two second order equations

$$\nabla^2 M = p,$$

and

$$\nabla^2 w = M/D.$$

$$M = -\frac{M_x + M_y}{1 + \nu},$$

$$D = \frac{h^3 E}{12(1 - \nu^2)},$$

where h is the plate thickness, E is Young's modulus, and ν is Poissons' ratio. In cylindrical coordinates the biharmonic equation becomes the following: (See the special functions document)

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