Electromagnetic Theory

James Emery

Last Edit: 5/6/15

Contents

1 Magnetic Poles and Bar Magnets

A bar magnet has a north pole and a south pole. The north pole of a bar magnet is its north seeking pole, that is points to the earth's north. A north pole attracts a south pole. The force on a north pole by a **B** field is in the direction of the **B** field. Since two N poles repel, an N pole is a source of the **B** field. A north pole on a magnetic compass is attracted to the earth's north pole. Therefore the earth being a more or less permanent magnet attracts the north pole of a compass needle. It follows that the earth's north pole is actually a south magnetic pole, and the **B** lines of the earth enter the so called north pole of the earth, which is thus a south magnetic pole. In a bar magnet the north pole is the pole from which **B** lines emerge. The north magnetic pole of a compass needle points in the direction of the magnetic induction field **B**. This is the convention, and it can be quite confusing. Notice however that the north pole of a cylindrical electromagnet is determined by the right hand rule. If you look down the axis of such a magnet with the current circulating in a counterclockwise direction, then the **B** lines are coming toward you making this end a north pole.

2 Stokes' Theorem, The Divergence Theorem

If a surface S has bounding curve ∂S, Stokes' theorem is

$$\int_S \nabla \times \mathbf{A} \cdot \mathbf{n} dS = \int_{\partial S} \mathbf{A} \cdot d\mathbf{r},$$

which allows a surface integral to be evaluated as a line integral around the boundary of the surface. The surface normal is $\mathbf{n}$.

The divergence theorem allows a volume integral to be evaluated as a surface integral. Let V be a volume and ∂V be it enclosing surface. Then

$$\int_V \nabla \cdot \mathbf{A} dv = \int_{\partial V} \mathbf{A} \cdot \mathbf{n} ds.$$

3 Maxwell's Equations

The Maxwell Equations in MKS form are

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{D} = \rho,$$

$$\nabla \cdot \mathbf{B} = 0.$$

E is the electric field vector, and **B** is the magnetic field vector. **J** is the current density, and ρ is the charge density.

We specify the field definitions **D** and **H** and the force on a charged particle. The electric vector field **D** is defined by

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P},$$

where **P** is the electric dipole moment per unit volume in a dielectric material. The number ϵ_0 is called the permittivity of free space. The magnetic field vectors **B** and **H** are related by

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M},$$

where **M** is the magnetic dipole moment per unit volume. The number μ_0 is called the permeability of free space.

The Lorentz force on a charge q is the sum of the electric and magnetic forces

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

4 More About Maxwell's Equations

A vector field is a function defined on a domain of points in space that assigns a vector **V** to each point $p = (x, y, z)$ of the domain.

$$\mathbf{V} = f(p) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit coordinate vectors in the x, y, z directions respectively. For example, if every point in a medium has a velocity, the set of velocity vectors is a vector field. Given a vector field

$$\mathbf{C} = C_x\mathbf{i} + C_y\mathbf{j} + C_z\mathbf{k},$$

the curl of $\mathbf{C}$ is defined to be

$$\nabla \times \mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ C_x & C_y & C_z \end{vmatrix}$$

$$= (\frac{\partial C_z}{\partial y} - \frac{\partial C_y}{\partial z})\mathbf{i} + (\frac{\partial C_x}{\partial z} - \frac{\partial C_z}{\partial x})\mathbf{j} + (\frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y})\mathbf{k}$$

The divergence is defined by

$$\nabla \cdot \mathbf{C} = \frac{\partial C_x}{\partial x} + \frac{\partial C_y}{\partial y} + \frac{\partial C_z}{\partial z}.$$

The gradient of a function f is

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

The Maxwell Equations in MKS form are

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{D} = \rho,$$

$$\nabla \cdot \mathbf{B} = 0.$$

$\mathbf{H}$ and $\mathbf{B}$ are magnetic fields, $\mathbf{E}$ and $\mathbf{D}$ are electric fields, $\mathbf{J}$ is the current density, and ρ is the charge density.

The fields $\mathbf{E}$ and $\mathbf{B}$ may be defined by the forces they exert on a charged particle of charge q. The Lorentz force on a charge q is the sum of the electric and magnetic forces

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Notice that a positive charge q moving perpendicular to an upward $\mathbf{B}$ field is deflected to the right, like the Coriolis force in the northern hemisphere.

When the curl of a vector field is zero,

$$\nabla \times \mathbf{C} = 0,$$

the field is called irrotational. In that case the line integral of the field from a point A to B is independent of the path and there exists a potential function ϕ. So for example in electrostatics where there are no magnetic fields,

$$\nabla \times \mathbf{E} = 0.$$

Then $\mathbf{E}$ is equal to the negative gradient of a potential function ϕ, called the electrical potential.

$$\mathbf{E} = -\nabla\phi.$$

Ampere's Law is given by part of the First Maxwell Equation

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Ampere's Law says that each infinitesimal line element of current flow produces a magnetic field. So for example the current in a loop produces a magnetic field through the loop. The first Maxwell equation is Ampere's Law plus the addition of a displacement current term

$$\frac{\partial \mathbf{D}}{\partial t}.$$

Maxwell showed that this additional term is necessary. So Ampere's Law says that each portion of current flow produces a magnetic field, but more is required. When there is no changing field $\mathbf{D}$, which is a modification of the $\mathbf{E}$ field caused by the presence of electrically polarized materials, then the first Maxwell equation becomes

$$\nabla \times \mathbf{H} = \mathbf{J}.$$

The vector field $\mathbf{D}$ is defined by

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P},$$

where $\mathbf{P}$ is the electric dipole moment per unit volume in a dielectric material. The number ϵ_0 is called the permittivity of free space. The magnetic field vectors $\mathbf{B}$ and $\mathbf{H}$ are related by

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M},$$

where $\mathbf{M}$ is the magnetic dipole moment per unit volume. Circulating currents inside a material give rise to magnetic dipoles, just as separated charges in a material give rise to electric dipoles. For many materials there are linear relationships

$$\mathbf{D} = \epsilon \mathbf{E}$$

and

$$\mathbf{B} = \mu \mathbf{H}.$$

We return to showing how Ampere's law is related to the first Maxwell equation. Ampere's original statement of his law was about the force between two parallel straight wires.

Applying Stoke's Theorem we have

$$\int_C \mathbf{H} \cdot d\mathbf{R} = \int_S \nabla \times \mathbf{H} \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} = i.$$

That is, the line integral of the magnetic intensity $\mathbf{H}$ around a path C equals the amount of current i flowing through the surface S that is bounded by C. This is Ampere's law. Let us remark about the displacement term. Suppose there were no displacement current term. Then if there were a capacitor placed in our wire, there would be current flowing through the wire, but no actual charge flowing between the capacitor plates. Hence, if we let our surface S pass between the capacitor plates then there would be a zero J, and thus a zero current i flowing through the surface. And so our line integral of $\mathbf{H}$ around the magnetic circuit would be zero. So depending on where we place our surface we get zero or not zero for the line integral. This is why the displacement term

$$\frac{\partial \mathbf{D}}{\partial t}$$

must be added to the first Maxwell equation. The displacement current term is nonzero between the capacitor plates.

Faraday's Law of Induction, the second Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

The second Maxwell equation is Faraday's law of induction. Using Stoke's theorem we have

$$\int_C \mathbf{E} \cdot d\mathbf{R} = \int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\frac{\partial \Phi}{\partial t}.$$

That is, the electric potential (MMF) around a circuit C is equal to the rate of magnetic flux change through the circuit.

If the material is soft iron and essentially linear with little hysteresis we may write

$$\mathbf{B} = \mu \mathbf{H},$$

where μ is a constant called the permeability. Such material forms a linear magnetic circuit.

Coulomb's's Law, the third Maxwell equation

$$\nabla \cdot \mathbf{D} = \rho.$$

The third Maxwell equation arises from Coulomb's law, which gives the forces between charges. So suppose we have a small volume of charge located at the origin and a larger spherical volume V surrounding it of radius r. Also assume that we are in free space so that

$$\mathbf{D} = \epsilon_0 \mathbf{E}.$$

We have

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}.$$

Integrating this over the volume V we find

$$\frac{q}{\epsilon_0} = \int_V \nabla \cdot \mathbf{E} dV$$

$$= \int_S \mathbf{E} \cdot d\mathbf{S}$$

$$= E 4 \pi r^2,$$

where we have used the divergence theorem to convert from a volume integral to a surface integral, q is the charge in the small volume, and E is the magnitude of the radial electric field on the spherical surface. So we have the electric field at a distance r^2 from a charge q is given by

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}.$$

This is a form of Coulombs law. The force on a charge q by an electric field E is by definition Eq. Thus given two charges q_1 and q_2, we obtain Coulombs law for the force between two charges

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}.$$

The Absence of Magnetic Monopoles, the fourth Maxwell equation

$$\nabla \cdot \mathbf{B} = 0.$$

The fourth Maxwell equation is has some similarity with the third law provided there is no magnetic charge density, no isolated magnetic charges. So the divergence of the **B** field is zero. Using the divergence theorem to convert a volume integral to a an integral on the bounding surface, we have

$$0 = \int_V \nabla \cdot \mathbf{B} dV = \int_S \mathbf{B} \cdot d\mathbf{S},$$

which means that every flux line entering a volume, leaves the volume. Thus there are no sources of magnetic flux lines, no isolated magnetic poles, and so flux lines form continuous loops.

5 Units and Physical Constants

Magnetic induction is written as **B**. The unit of magnetic induction in the MKS system is now the tesla, which used to be called the weber per square meter. A tesla equals 10^4 gauss, which is the cgs unit of magnetic induction.

The earth's magnetic field is about half a Gauss.

1.5 T strength of a modern neodymium-iron-boron (Nd2Fe14B) rare earth magnet. A coin-sized neodymium magnet can lift more than 9 kg, can pinch skin and erase credit cards.

the strength of a typical refrigerator magnet 5mT

Medical MRI 1.5 to 3 T, experimental 8 T

NMR spectrometer field strength of a 500 MHz NMR spectrometer 11.7 T

strongest (pulsed) magnetic field yet obtained non-destructively in a laboratory 88.9 T

strongest pulsed magnetic field yet obtained in a laboratory, destroying the used equipment, but not the laboratory itself 730 T

$$\mu_0 = 4\pi \times 10^{-7} \text{ tesla} \cdot \text{meter/ampere}$$

$$= 1.256637061435917 \times 10^{-6} \text{ tesla} \cdot \text{meter/ampere.}$$

The magnetic field at the center of a single winding of radius r and carrying a current of i amperes is

$$\mathbf{B} = \frac{\mu_0 i}{2r}.$$

So for example, if the current were 1 ampere with a radius of 3 cm = .03 meters the field would be

$$\mathbf{B} = 2.094 \times 10^{-5}$$

tesla or about .2094 gauss.

The permittivity of free space is

$$\epsilon_0 \approx 8.8541878176\ldots \times 10^{-12}\frac{F}{m}.$$

A Farad is

$$F = \frac{Coulomb}{Volt} = \frac{C}{V},$$

so

$$\epsilon_0 \approx 8.8541878176\ldots \times 10^{-12}\frac{C}{V \cdot m}.$$

The velocity of light in free space is

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}.$$

So

$$\epsilon_0 = \frac{1}{c^2\mu_0} \approx \frac{1}{(9 \times 10^{16})(4\pi \times 10^{-7})} = \frac{1}{36\pi \times 10^9}.$$

6 Coulomb's Law

Let n charges q_i be placed at positions $\mathbf{r'}_i$. Let $\mathbf{a} = \mathbf{r} - \mathbf{r'}$., Then

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i \mathbf{a}_i}{a_i^3}.$$

We have

$$\frac{1}{4\pi\epsilon_0} = 8.987551787388872 \times 10^{-9},$$

which is approximately 9×10^{-9}.

7 Potential

Because

$$\nabla \times \mathbf{E} = 0,$$

a line integral of $\mathbf{E}$ is independent of the path. So there exists a potential ϕ so that

$$\mathbf{E} = -\nabla\phi.$$

We have

$$\phi = \int \mathbf{E} \cdot d\mathbf{l}.$$

For a point charge

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{a}.$$

8 Gauss's Law

Let S be a sphere. Let q be a point charge at the center of S. Then

$$\int_S \mathbf{E} \cdot \mathbf{n} ds = q/\epsilon_0$$

Let S be surrounded by an arbitrary surface G. Integrating the volume bounded by S and G we deduce the integral over G equals the integral over S. The integral of $\mathbf{E}$ over the surface of a volume not containing sources is zero. This follows because in such a volume

$$\nabla \cdot \mathbf{E} = 0$$

We conclude that the integral of a field $\mathbf{E}$ over a surface G, which is due to point charges, is equal to the sum of the point charges contained within the surface, divided by the permittivity of free space.

9 Charge Distribution

Divide a bounded space into small volumes ΔV_j. Let ρ_j be the charge per volume. Let ρ_d be a linear combination of products of characteristic functions of ΔV_j's and ρ_j's. Let the charge distribution (charge density) ρ be a continuous function that approximates this step function. If ρ is continuous, from the divergence theorem and Gauss's law we deduce that

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Now let ρ be a generalized function (i.e. a distribution). Then we define E to be a solution to this differential equation in the distributional sense.

10 Electric Polarization

Two charges of magnitude q and differing signs, which are separated by a vector r, create an electric field. This field depends only on the product of the charge and the separation vector, and the field point location. We let $p = q\mathbf{r}$. p is called the electric dipole moment. We can think of a point dipole moment, where as $\mathbf{r}$ shrinks, the charge q increases proportionately. We may thus consider a vector field P, which is a continuous distribution of point dipoles. P then is a dipole moment density. It is the dipole moment per unit volume. The continuous vector field P is called the polarization. The dipole moment of a volume Δv is

$$P\Delta v.$$

Given a polarized region, we may integrate with respect to the volume to get the electric field at a point due to the polarized region. As in the case of a charge distribution, the electric field will be defined both inside and outside of the polarized region.

The potential due to any localized charge distribution in a volume ΔV may be written as an infinite sum of multipole sources. We retain only monopole and dipole terms. The monopole moment is

$$q = \int_{\Delta V} \rho dv,$$

and the dipole moment is

$$\mathbf{p} = \int_{\Delta V} \mathbf{r} \rho dv.$$

Let $\mathbf{P}$ be the dipole moment per unit volume, which in general is a distribution. The dipole potential due to volume element dv is

$$d\phi = \frac{1}{4\pi\epsilon_0} P \cdot \frac{\mathbf{a}}{a^3} dv =$$

$$\frac{1}{4\pi\epsilon_0} P \cdot (-\nabla f) dv =$$

$$\frac{1}{4\pi\epsilon_0} (f \nabla \cdot P - \nabla \cdot (fP)) dv$$

Integrating over a charged isolated volume V, we get

$$\phi = \int_{\partial V} \frac{\sigma_p}{a} ds + \int_V \frac{\rho_p}{a} dv,$$

where

$$\sigma_p = \mathbf{P} \cdot \mathbf{n}$$

and

$$\rho_p = -\nabla \cdot \mathbf{P}.$$

If we integrate over all space and assume P is zero at infinity, we have

$$\phi = \int_V \frac{\rho_p}{a} dv.$$

Since $\mathbf{P}$ may not be differentiable in the classical sense we take $\mathbf{P}$ to be distribution. We may approximate $\mathbf{P}$ with a smooth function. When we have a finite volume V, we may replace part of the volume integration by surface integration on the boundary of V. This can be done by integrating over a thin shell A that contains the boundary of V. For the thin shell we have

$$\phi_A = \int_A \frac{-\nabla \cdot \mathbf{P}}{a} dv =$$

$$\int_A (-\nabla \cdot (\mathbf{P}/a) + P \cdot \nabla f) dv =$$

$$\int_{\partial A} \frac{-\mathbf{P} \cdot \mathbf{n}}{a} ds + \int_A \mathbf{P} \cdot \nabla f dv$$

If $\mathbf{P}$ is bounded, then the second integral goes to zero as the volume of the thin shell goes to zero. The first integral is over two parallel surfaces, one of which is outside of V, assuming P is zero outside of V, we get back our surface polarization charge density integral.

$$\phi_A = \int_{\partial A} \frac{-\mathbf{P} \cdot \mathbf{n}}{a} ds =$$

$$\int_{\partial V} \frac{\sigma_p}{a} ds$$

11 Electric Displacement

From Gauss's law integrating over a surface S we have

$$\int_S \mathbf{E} \cdot \mathbf{n} ds = (q + q_p)/\epsilon_0$$

$$\int_V \nabla \cdot \mathbf{E} dv = (q + q_p)/\epsilon_0$$

$$q = \int_V \frac{\rho}{a} dv,$$

$$q_p = \int_V \frac{\rho_p}{a} dv,$$

$$= \int_V \frac{-\nabla \cdot \mathbf{P}}{a} dv,$$

It follows that if $\mathbf{D}$ is defined by

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

then

$$\nabla \cdot \mathbf{D} = \rho.$$

12 Electric Susceptibility, Permittivity, and Dielectric Tensors

The polarization can usually be taken to be a linear function of the average applied field. We write the components of the polarization as

$$P_i = \epsilon_0 \chi_{ij} E_j.$$

The tensor χ depends on the material. For an isotropic material it becomes just a constant. The number ϵ_0 is called the permittivity of free space. In terms of the displacement D we have

$$D = \epsilon_0 E + P = \epsilon_0 (I + \chi) E = \epsilon_0 K E$$

where

$$K_{ij} = I + \chi_{ij}.$$

The tensor K is called the dielectric constant, I is the identity matrix. The tensor

$$\epsilon_{ij} = \epsilon_0 K_{ij},$$

is called the permittivity.

13 Energy of a Charge Distribution

Assembling point charges from infinity we find

$$U = \frac{1}{2} \sum_{i=1}^{n} q_i \phi_i.$$

Raising a charge density linearly from zero to full value, we find

$$U = \frac{1}{2} \int \rho(\mathbf{r}) \phi(\mathbf{r}) dv.$$

Using $\nabla \cdot \mathbf{D} = \rho$ and the divergence theorem we find that the energy density is

$$u = \frac{\mathbf{D} \cdot \mathbf{E}}{2}$$

14 Coefficients of Potential

Given n conductors we define p_{ij} to be the potential of conductor i when there is unit charge on conductor j and the other conductors are uncharged.
Proposition. If a potential is multiplied by a constant c, then the charges are multiplied by c.
Proof. Use $E_n = \sigma/\epsilon_0$ and $\nabla\phi = -\mathbf{E}$.

$Q_j p_{ij}$ is the potential on conductor i, when Q_j is the charge on conductor j and the other charges are zero. In the general case, by linear superposition, the potential on conductor i is

$$\phi_i = \sum_{i=1}^{n} p_{ij} Q_j$$

when the charges are Q_j, j=1...n .

The energy of the conductors is

$$U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij} Q_i Q_j.$$

The coefficients of potential are symmetric,

$$p_{ij} = p_{ji}$$

This may be shown by using the expression for the energy of the conductors and taking the differential of the energy. Suppose only the charge Q_1 is nonzero. We get

$$dU = \frac{1}{2}\sum_{j=1}^{n}(p_{1j} + p_{j1})Q_j$$

This is also equal to

$$\phi_1 dQ_1 = \sum_{j=1}^{n} p_{1j} Q_j dQ_1.$$

Equating these two expressions, we find that

$$p_{1j} = p_{j1},$$

and so in general

$$p_{ij} = p_{ji}.$$

The coefficients of potential are positive (Reitz and Milford, 3rd ed., p 121).

Suppose there are only two conductors in a capacitor. The charges are equal, thus

$$C = \frac{1}{p_{11} + p_{22} - 2p_{12}}.$$

When

$$\phi_i = \sum_{i=1}^{n} p_{ij} Q_j$$

is inverted, we get

$$Q_i = \sum_{i=1}^{n} c_{ij} \phi_j.$$

The c_{ij} are called the coefficients of capacitance, and are elements of a symmetric matrix, being the inverse of a symmetric matrix. This follows by the finite spectral theorem. A symmetric matrix can be diagonalized by an orthogonal transformation, that is the eigenvectors of a symmetric matrix are orthogonal.

15 Properties of Harmonic Functions

A harmonic function f is a solution to Laplace's Equation.

$$\nabla^2 f = 0.$$

In electrostatics in a region of zero charge density

$$\nabla \cdot \mathbf{E} = 0$$

Where there are no currents

$$\nabla \times \mathbf{E} = 0,$$

so that a line integral of $\mathbf{E}$ between two points is independent of the path, and so $\mathbf{E}$ is given as the negative gradient of a potential function

$$\mathbf{E} = -\nabla \phi.$$

So substituting this in

$$\nabla \cdot \mathbf{E} = 0$$

we have

$$\nabla^2 \phi = 0.$$

Then the electrical potential is a harmonic function.

A harmonic function satisfies the following properties:

(1) A maxima or minima must occur on a boundary. (2) The average over a spherical surface equals the value at the center.

A study of harmonic functions or potentials is called potential theory. A force proportional to the inverse distance squared from source particles gives rise to potentials and harmonic functions, which satisfy Laplace's equation. Thus besides static electric forces, gravitational sources lead to potentials. For example see the classic book **Potential Theory** by Kellog. The real and imaginary parts of a complex analytic function are harmonic functions in the two dimensions of the complex plane, and so are a source of solutions to two dimensional potential problems.

16 Shielded Conductors

Let uncharged conductors be located inside another conductor. The charge densities of the inside conductors and on the inside surface of the bounding conductor are zero everywhere. Otherwise we may trace a line of flux from a positive charge on a conductor back to a negative charge on the same conductor, possibly travelling through other conductors. The potential drops in some portion of the path and never increases anywhere. This is a contradiction, because we return to the same conductor. Solving the Neumann problem we find the potential constant. It follows that if i and j are two of these conductors and k is an outside conductor, then for shielded conductors

$$p_{ik} = p_{jk}.$$

Proposition. $p_{ij} = p_{ji}$.
Proposition. $p_{ij} > 0$ and $p_{ii} \geq p_{ij}$.

17 Capacitors

Consider two conductors, one shielded by another. Number them 1 and 2. Using Gauss's law, the charges on the matching surfaces are equal. Call this charge Q. If i is not equal to 1 or 2, then $p_{1i} = p_{2i}$. So

$$\Delta\phi = (p_{11} + p_{22} - 2P_{12})Q = \frac{Q}{C}.$$

C is called the capacitance. The energy of a capacitor is

$$U = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} p_{ij} Q_i Q_j$$

$$= \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C\Delta\Phi^2.$$

18 Forces

Suppose the orientations and positions of a set of conductors depends on a set of m generalized coordinates, $u_1, ..., u_m$. Let charge be held fixed. And let the system do work dW. Using the first law of thermodynamics we find

$$dW = -dU$$

Let the ith generalized forces be F_i. We have

$$\sum_{i=1}^{m} F_i dU_i = dW = -\sum_{i=1}^{m} \frac{\partial U}{\partial u_i} dU_i.$$

Thus

$$F_i = -\frac{\partial U}{\partial u_i}.$$

If the potential is held fixed by a battery, then the battery does work $2dU$, and thus

$$F_i = \frac{\partial U}{\partial u_i}.$$

19 Current Density

Let N be the number of charged particles per unit volume. Let each particle have charge q and velocity $\mathbf{v}$. Define the current density

$$\mathbf{J} = Nq\mathbf{v}.$$

Let a surface have normal $\mathbf{n}$. Suppose $\mathbf{J}$ makes an angle θ with $\mathbf{n}$. Consider a tube of flowing charge of cross sectional area da'. In time dt, charge

$$dq = Nqdv = Nqvdtda' = Nqvdtcos(\theta)da$$

crosses the surface. da is the surface area through which the charge flows. Thus

$$\frac{dq}{dt} = \mathbf{J} \cdot \mathbf{n} da.$$

20 The Equation of Continuity

Consider a closed surface bounding a volume where the charge density is ρ. Using the fact that charge is conserved and the divergence theorem, and assuming continuity, we obtain the equation of continuity

$$\nabla \cdot \mathbf{J} + \frac{d\rho}{dt} = 0.$$

21 Ohm's Law

Ohm's law says that the current density $\mathbf{J}$, which is the vector flow of current per unit area, is proportional to the electric field,

$$\mathbf{J} = g\mathbf{E},$$

where g is the conductivity. The resistance of a wire of cross section A and length L, assuming uniform current flow, is

$$R = \frac{V}{I} = \frac{EL}{JA} = \frac{L}{gA}.$$

22 Ohm's Law and Resistance

Ohm's law says that the current density $\mathbf{J}$, the current flow per unit area, is proportional to the electric field. So

$$\mathbf{J} = g\mathbf{E},$$

where g is the conductivity.

For uniform flow in a conductor of cross section A and length L, this becomes

$$\frac{I}{A} = J = gE = g\frac{V}{L},$$

or

$$R = \frac{V}{I} = \frac{L}{gA}.$$

R is called the resistance.

So the resistance of a wire of cross section A and length L is

$$R = \frac{V}{I} = \frac{EL}{JA} = \frac{L}{gA}.$$

So the more well known special form of Ohm's law is

$$V = RI.$$

The conductivity is sometimes written using the letter σ, and the reciprocal of the conductivity, called the resistivity, written as ρ. Then the resistance is given by

$$R = \frac{V}{I} = \frac{EL}{JA} = \frac{L}{\sigma A} = \frac{\rho L}{A}.$$

The unit of resistivity is the ohm meter.

23 Steady Currents

When time derivatives are zero, the equation of continuity becomes

$$\nabla \cdot \mathbf{J} = 0.$$

Using $\mathbf{E} = -\nabla\phi$ we obtain Laplace's equation

$$\nabla^2\phi = 0.$$

Conservation of charge gives the boundary condition between two media

$$g_1\frac{\partial\phi}{\partial n} = g_1\frac{\partial\phi}{\partial n}.$$

24 Magnetic Induction

The magnetic force on a charge q_1 due to a charge q_2 is

$$\mathbf{F}_1 = q_1\mathbf{v}_1 \times (\frac{\mu_0}{4\pi}q_2\mathbf{v}_2 \times \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3})$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the respective velocities and $\mathbf{r}_1$ and $\mathbf{r}_2$ the positions of the charges. We define the magnetic induction field $\mathbf{B}$ by

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

where $\mathbf{B}$ is due to moving charges as in the first equation. The Lorentz force is the sum of the electric and magnetic forces

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

By definition

$$\mu_0 = 4\pi 10^{-7}.$$

We have

$$\mu_0\epsilon_0 == \frac{1}{c^2}$$

Note: ϵ_0 is approximately

$$\frac{1}{36\pi}10^{-9}.$$

25 Biot-Savart Law

The Biot-Savart law gives the field due to a current i flowing in an element of length $d\ell$ as

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{id\ell \times \mathbf{r}}{r^3},$$

where $\mathbf{r}$ is a vector from the current element vector $d\ell$ to the field point. The direction of the magnetic field follows the right hand rule. With the right hand around the current element and the thumb pointing in the current direction, the direction of the magnetic field is given by the direction of the fingers.

Thus integrating around a current a single loop we find at the center of the loop, the magnetic field at the center of a single winding of radius r and carrying a current of i amperes is

$$\mathbf{B} = \frac{\mu_0 i}{2r}.$$

This differential form is equivalent to a current density definition given as

$$\mathbf{B}(\mathbf{r}_1) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}\times(\mathbf{r}_1-\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|^3}dx_2dy_2dz_2.$$

Taking the divergence we find

$$\nabla\cdot\mathbf{B} = \frac{\mu_0}{4\pi}\int_V((\nabla\times\mathbf{J})\cdot\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3} - \mathbf{J}\cdot(\nabla\times\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}))dv_2.$$

The first term is zero because $\mathbf{J}$ is not a function of $\mathbf{r}_1$. The second term is zero because the curl of a gradient is zero. Thus for current sources,

$$\nabla\cdot\mathbf{B} = 0.$$

If monopoles do not exist, this is a general result.

26 The Magnetic Field produced by Various Circuits

Magnetic fields due to electric circuits can often be be computed by using the Biot-Savart Law, or sometimes by the direct use of Maxwell's equations.

26.1 The Field Due to a Straight Infinitely Long Wire

We calculate the field at the point $x = 0, y = d$, where current i flows in the positive x direction. This is the field at a distance d from the wire. According to the right hand rule the field will be in the positive z direction for each differential current element. From the Biot-Savart Law we have

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{id\ell \times \mathbf{r}}{r^3},$$

where $\mathbf{r}$ is a vector from the differential current element to the field point $(0, d$.

The line element is

$$d\ell = dx\mathbf{u}_x,$$

where $\mathbf{u}_x$ is a unit vector in the positive x direction. Hence the angle θ between $d\ell$ and $\mathbf{r}$ for x negative is between 0 and $\pi/2$, whereas for x positive it is greater it ranges from $\pi/2$ to π. For symmetric values x and $-x$ the two angles are supplements so that their sine values are equal. So the integral over negative values equals the integral over positive values. So we can integrate just for x positive to get half the field value. For x positive we have

$$\frac{d\ell \times \mathbf{r}}{r^3} = \frac{\sin(\theta)dx}{r^2}\mathbf{u}_z$$

$$= \frac{\sin(\phi)dx}{r^2}\mathbf{u}_z$$

where $\phi = \pi - \theta$ is the acute angle between $\mathbf{r}$ and the x axis. We have

$$x = \frac{d}{\tan(\phi)}$$

and

$$dx = -d\tan^{-2}(\phi)\sec^2(\phi)d\phi = -\frac{d}{\sin^2(\phi)}d\phi.$$

$$r = d/\sin(\phi).$$

$$\frac{1}{r^2} = \frac{\sin^2(\phi)}{d^2}$$

So

$$\frac{\sin(\phi)dx}{r^2} = -\frac{1}{d}\sin\phi d\phi.$$

Therefore the field is

$$\mathbf{B} = -2\frac{i\mu_0}{d4\pi}\int_{\pi/2}^{0}\sin\phi d\phi\mathbf{u}_z$$

$$= \frac{i\mu_0}{d2\pi}(\cos(0) - \cos(\pi/2))\mathbf{u}_z$$

$$= \frac{\mu_0}{2\pi}\frac{i}{d}\mathbf{u}_z.$$

This can be derived more easily by using Maxwell's equation

$$\nabla \times \mathbf{H} = \mathbf{J}.$$

So consider a circle of radius d around the wire. Integrating the area bounded by the circle we have

$$\int_A \nabla \times \mathbf{H} \cdot d\mathbf{S} = \int_A \mathbf{J} \cdot d\mathbf{S} = i.$$

By symmetry $\mathbf{H}$ is tangent to the circle. So using Stokes's Theorem

$$\int_A \nabla \times \mathbf{H} \cdot d\mathbf{S} = \int_{\partial A} \mathbf{H} \cdot d\ell = d2\pi H,$$

where ∂A is the circular boundary of area A. Hence

$$H = \frac{i}{2\pi d},$$

and so

$$B = \mu_0 H = \frac{\mu_0 i}{2\pi d}.$$

The direction of the field is given by the right hand rule: With the thumb in the direction of the current, the field is in the direction of the fingers curled around the wire.

26.2 The Force Between Two Infinitely Long Parallel Wires

Suppose the wires are separated by a distance d and each carry current i. If the wires are infinitely long then the field B all around the wire by symmetry is constant. The field of wire is as calculated in the previous problem, given by

$$B = \mu_0 H = \frac{\mu_0 i}{2\pi d}.$$

The Lorentz force on an element of length Δx of the second wire is

$$d\mathbf{F} = \Delta q \mathbf{v} \times \mathbf{B},$$

where Δq is the amount of charge on a length of the wire Δx and $\mathbf{v}$ is the velocity of this moving charge. The direction of the force on the second wire is in the direction of

$$\mathbf{v} \times \mathbf{B},$$

because the direction of the current in the second wire is the same as in the first wire. It follows by the right hand rule for cross products, that the force $d\mathbf{F}$ on the second wire is toward the first.

Now $\mathbf{v}$ is perpendicular to $\mathbf{B}$ so we can write

$$\Delta F = \Delta q \frac{\Delta x}{\Delta t} B = \frac{\Delta q}{\Delta t} \Delta x B = i \Delta x B$$

So the force per unit length on the second wire is

$$f = iB = \frac{\mu_0 i^2}{2\pi d},$$

directed toward the first wire. This can be used to define the unit of current, because $\mu_0 = 4\pi \times 10^{-7}$ by definition.

26.3 Field Along the Axis of a Current Loop

Magnitude of the field at distance b from the plane of the loop of radius a

$$B = \frac{\mu_0 i}{2} \frac{a^2}{(a^2 + b^2)^{3/2}}$$

26.4 Long Solenoid

N = number of turns, L length.

At center

$$B = \frac{\mu_0 N i}{L}.$$

At end

$$B = \frac{\mu_0 N i}{2L}.$$

Greek word Solenoid means channel.

27 Torque on a Circuit

$$d\tau = \mathbf{r} \times d\mathbf{F} = \mathbf{r} \times (Id\mathbf{l} \times \mathbf{B})$$

and

$$\tau = I \oint \mathbf{r} \times (d\mathbf{l} \times \mathbf{B}).$$

Define the magnetic moment of the circuit as

$$\mathbf{m} = \frac{I}{2} \oint \mathbf{r} \times d\mathbf{l}.$$

We will prove:
Proposition.

$$\tau = \mathbf{m} \times \mathbf{B}.$$

Lemma 1.

$$\oint \mathbf{r} \cdot d\mathbf{r} = 0$$

and

$$\oint xdx = \oint ydy = \oint ydy = 0.$$

Proof.

$$\oint \mathbf{r} \cdot d\mathbf{r} = \int \nabla \times \mathbf{r} da = 0.$$

Lemma 2.

$$\oint (xdy + ydx) = \oint (xdz + zdx) = \oint (zdy + ydz) = 0.$$

Proof. For example, let

$$\mathbf{U} = y\mathbf{i} + x\mathbf{j}.$$

Then

$$\nabla \times \mathbf{U} = 0,$$

and the result follows from Stokes's Theorem.
Proof of the proposition.

$$\tau = I \oint \mathbf{r} \times (d\mathbf{l} \times \mathbf{B}).$$

$$= I(\oint (\mathbf{r} \cdot \mathbf{B} d\mathbf{r}) - \oint \mathbf{B}(\mathbf{r} \cdot d\mathbf{r}))$$

$$= I \oint d\mathbf{r}(\mathbf{r}\cdot\mathbf{B}).$$

The second integral vanishes by the Lemma. On the other hand

$$\mathbf{m}\times\mathbf{B} = \frac{I}{2}\oint \mathbf{r}\times d\mathbf{l}\times\mathbf{B}$$

$$= \frac{I}{2}(\oint d\mathbf{r}(\mathbf{r}\cdot\mathbf{B}) - (\oint \mathbf{r}(\mathbf{B}\cdot d\mathbf{r}).$$

When one expands these two terms, they are seen to be equal. For example

$$\oint d\mathbf{r}(\mathbf{r}\cdot\mathbf{B}) = \oint (B_y y dx + B_z z dx)\mathbf{i} + \oint (B_x x dy + B_z z dy)\mathbf{j} + \oint (B_x x dz + B_y y dz\mathbf{k}).$$

The equality is seen by using the lemmas. We get

$$\mathbf{m}\times\mathbf{B} = I(\oint d\mathbf{r}(\mathbf{r}\cdot\mathbf{B}) = \tau.$$

28 Amperes' Law

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}.$$

Proof. We take the Curl of the Biot-Savart Law

$$\mathbf{B}(\mathbf{r}_1) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}\times(\mathbf{r}_1-\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|^3} dx_2 dy_2 dz_2.$$

Let

$$\mathbf{G} = \frac{(\mathbf{r}_1-\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|^3}.$$

Then

$$\nabla\times\mathbf{B}(\mathbf{r}_1) = \frac{\mu_0}{4\pi}\int_V \nabla_1\times(\mathbf{J}\times\mathbf{G})dv_2.$$

We use the identity

$$\nabla_1\times(\mathbf{J}\times\mathbf{G}) = (\nabla_1\cdot\mathbf{G})\mathbf{J} - (\nabla_1\cdot\mathbf{J})\mathbf{G} + (\mathbf{G}\cdot\nabla_1)\mathbf{J} - (\mathbf{J}\cdot\nabla_1)\mathbf{G}$$

$$= (\nabla_1\cdot\mathbf{G})\mathbf{J} - (\mathbf{J}\cdot\nabla_1)\mathbf{G}.$$

Terms 2 and 3 are zero because $\mathbf{J}$ is a function of $\mathbf{r}_2$. We have

$$(\mathbf{J}\cdot\nabla_1)\mathbf{G} = -(\mathbf{J}\cdot\nabla_2)\mathbf{G}.$$

So

$$\nabla \times \mathbf{B}(\mathbf{r}_1) = \frac{\mu_0}{4\pi} \int_V ((\nabla_1 \cdot \mathbf{G})\mathbf{J} + (\mathbf{J} \cdot \nabla_2)\mathbf{G}) dv_2.$$

The first term is

$$= \frac{\mu_0}{4\pi} 4\pi \int_V \delta(\mathbf{r}_2 - \mathbf{r}_1)\mathbf{J} dv_2 = \mu_0 \mathbf{J}.$$

The second term is zero. Consider for example the x component. We have

$$\int_v (\mathbf{J} \cdot \nabla_2) G_x dv = \int_v \nabla_2 G_x \cdot \mathbf{J} dv$$

$$= \int_v \nabla_2 \cdot (G_x \mathbf{J}) dv - \int_v G_x \nabla_2 \cdot \mathbf{J} dv$$

$$= \int_{\partial v} G_x \mathbf{J} \cdot \mathbf{n} da = 0.$$

We have assumed that there are no point current sources, i.e. $\nabla \cdot \mathbf{J} = 0$. The last integral is zero because all currents are zero outside of a bounded region contained in V.

29 The Vector Potential

If there are no magnetic monopoles, then

$$\nabla \cdot \mathbf{B} = 0.$$

Then $\mathbf{B}$ is given by the curl of a vector field A

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Then

$$\mu_0 \mathbf{J} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\nabla^2 \mathbf{A},$$

provided we select a gauge so that $\nabla \cdot \mathbf{A} = 0$. The fundamental solution of Poission's equation gives

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}}{|\mathbf{r} - \mathbf{r}'|} dv'.$$

The vector potential for a distant circuit is obtained by using $id\mathbf{r} = \mathbf{J} dv$ and by expanding

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

using the binomial theorem. Keeping linear terms we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{m}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}.$$

This is the potential of a magnetic dipole.

30 Magnetization

The magnetization vector $\mathbf{M}$ is defined to be the magnetic dipole moment per unit volume. We have

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \mathbf{M} \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}dv'$$

$$= \frac{\mu_0}{4\pi}\int \mathbf{M} \times \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|}dv'$$

$$= -\frac{\mu_0}{4\pi}\int \nabla' \times (\mathbf{M}\frac{1}{|\mathbf{r}-\mathbf{r}'|})dv'$$

$$+\frac{\mu_0}{4\pi}\int \frac{\nabla' \times \mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}dv'$$

The first integral can be transformed to a surface integral using the divergence theorem. In general, if $\mathbf{F}$ is an arbitrary constant vector, then

$$\mathbf{F} \cdot \int \nabla \times \mathbf{G} dv$$

$$= -\int \nabla \cdot (\mathbf{F} \times \mathbf{G}) dv$$

$$= -\int (\mathbf{F} \times \mathbf{G}) \cdot \mathbf{n} da$$

$$= -\mathbf{F} \cdot \int \mathbf{G} \times \mathbf{n} da.$$

$\mathbf{F}$ is an arbitrary vector, so we have the general identity

$$\int \nabla \times \mathbf{G} dv = -\int \mathbf{G} \times \mathbf{n} da.$$

So

$$-\frac{\mu_0}{4\pi}\int \nabla' \times (\mathbf{M}\frac{1}{|\mathbf{r}-\mathbf{r}'|})dv'$$

$$= \frac{\mu_0}{4\pi} \int \frac{\mathbf{M} \times \mathbf{n}}{|\mathbf{r} - \mathbf{r}'|} da'.$$

As the bounding surface goes to infinity, where all current sources are zero, the integral goes to zero. Then

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\nabla' \times \mathbf{M}}{|\mathbf{r} - \mathbf{r}'|} dv'$$

It follows in general that

$$\nabla \times \mathbf{M} = \mathbf{J}_m.$$

$\mathbf{J}_m$ is the current density of the magnetic material. If currents are not zero on the surface of a volume, then we have

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}_m}{|\mathbf{r} - \mathbf{r}'|} dv' + \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}_m}{|\mathbf{r} - \mathbf{r}'|} da',$$

where

$$\mathbf{j}_m = \mathbf{M} \times \mathbf{n}.$$

31 Magnetic Intensity

Ampere's law gives

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J}_m + \mathbf{J}) = \mu_0(\nabla \times \mathbf{M} + \mathbf{J}).$$

$\mathbf{J}$ is the free current density. We define a magnetic intensity vector

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}.$$

Then

$$\nabla \times \mathbf{H} = \mathbf{J}.$$

32 Magnetostatics

Suppose the free current density is zero. Then

$$\nabla \times \mathbf{H} = 0$$

so $\mathbf{H}$ is the gradient of a magnetic scalar potential ϕ_m.

$$\mathbf{H} = -\nabla \phi_m.$$

Because

$$\nabla \cdot \mathbf{B} = 0$$

we have

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$$

Thus the magnetic scalar potential ϕ_m satisfies the poisson equation

$$\nabla^2 \phi_m = \rho_m$$

where

$$\rho_m = -\nabla \cdot \mathbf{M}.$$

33 Sources of H

When there are magnetic materials and free currents we have

$$\mathbf{H}(\mathbf{r}_1) = \frac{1}{4\pi} \int_V \frac{\mathbf{J} \times (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} dx_2 dy_2 dz_2 \ - \nabla \phi_m.$$

34 Boundary Conditions

The divergence of $\mathbf{B}$ is zero so the normal component of $\mathbf{B}$ is continuous across a surface separating two media. The boundary conditions on $\mathbf{H}$ are more complex. Let a surface separate material 1 from material 2. Suppose in general there is a surface current. Let $\mathbf{j}$ be the surface current density. This is the current per unit length on the surface. Let $\mathbf{n}_i$ be the surface normal that points into material i. Let C be a rectangular path with a short side δ that tends to zero and a long side h. One long side is in material 1 and the other in material 2. The plane containing C is perpendicular to the original separating surface. Let $\mathbf{t}$ be a unit tangent to c. Let $\mathbf{n}$ be the normal to the plane containing C. Then applying the right hand rule

$$\mathbf{n}_i \times \mathbf{t}_i = \mathbf{n}$$

Neglecting the contribution of the short sides to the line integral we have

$$\oint_c \mathbf{H} \cdot d\mathbf{r} = (\mathbf{H}_1 \cdot \mathbf{t}_1 + \mathbf{H}_2 \cdot \mathbf{t}_2)h = h\delta \mathbf{J} \cdot \mathbf{n}$$

$$= h\delta \mathbf{J}_{||} \cdot \mathbf{n}$$

$$= h\mathbf{j} \cdot \mathbf{n}.$$

$\mathbf{J}_{||}$ is the component of $\mathbf{J}$ parallel to the separating surface. Hence

$$\mathbf{H}_1 \cdot \mathbf{t}_1 + \mathbf{H}_2 \cdot \mathbf{t}_2 = \mathbf{j} \cdot (\mathbf{n}_i \times \mathbf{t}_i) = (\mathbf{j} \times \mathbf{n}_i) \cdot \mathbf{t}_i.$$

We have

$$\mathbf{t}_1 = -\mathbf{t}_2,$$

so when the surface current is zero, the tangential component of $\mathbf{H}$ is continuous across the surface.

35 Magnetic Susceptibility and Permeability

For isotropic and linear materials

$$\mathbf{M} = \chi_m \mathbf{H}.$$

Then

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H} = \mu \mathbf{H}.$$

χ_m is the susceptibility and μ the permeability. Ferromagnetic materials can have a permanent magnetization and the relation between $\mathbf{B}$ and $\mathbf{H}$ depends upon the magnetization history.

36 Magnetic Circuits

A continuous tube of flux Φ forms a magnetic circuit. Let the circuit pass through a coil containing N turns and current i. For a path around the circuit

$$Ni = \oint \mathbf{H} \cdot d\mathbf{r} = \sum\ H_i L_i = \sum \frac{L_i \phi}{\mu_i A_i} = \Phi \sum \Re_i.$$

This equation is an approximation. $\mathbf{H}_i$ is an assumed constant value of $\mathbf{H}$ in the ith piece of the circuit. The reluctance of the ith piece is $\Re_i$. L is the length of the piece and A is the cross sectional area. The magnetomotive force mmf is Ni. We have

$$mmf = \Phi\Re.$$

37 Deriving the Electromagnetic Wave Equation From Maxwell's Equations

We present here a derivation of the electromagnetic wave equation for a simple special case. We assume that the wave is moving in free space, and we derive the equation for the electric field vector.

The Maxwell Equations in MKS form are

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{D} = \rho,$$

$$\nabla \cdot \mathbf{B} = 0.$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P},$$

where $\mathbf{P}$ is the electric dipole moment per unit volume in a dielectric material. The number ϵ_0 is called the permittivity of free space.

The magnetic field vectors $\mathbf{B}$ and $\mathbf{H}$ are related by

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M},$$

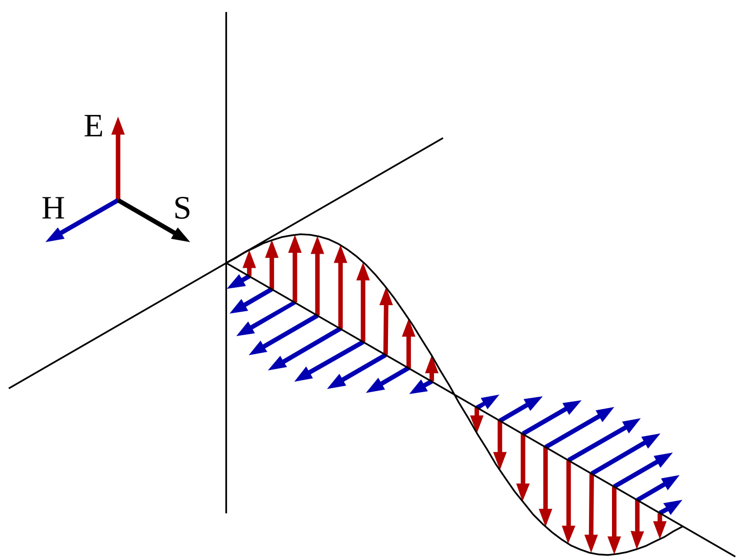

Figure 1: **Electromagnetic Waves**. The electromagnetic wave in free space is a transverse wave consisting of mutually perpendicular vectors, an electric field vector $\mathbf{E}$, and a magnetic field vector $\mathbf{H}$. These vectors in turn are each perpendicular to the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, which defines the direction of the wave and its energy flow.

where $\mathbf{M}$ is the magnetic dipole moment per unit volume. The number μ_0 is called the permeability of free space.

We start with Maxwell's equations for free space, where there is no current density $\mathbf{J}$ or charge sensity ρ, and there are no electric dipole fields or magnetic dipole fields because there is no matter. So in this case

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E}$$

The first Maxwell equation becomes

$$\nabla \times \frac{\mathbf{B}}{\mu_0} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

or

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

The second one is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Taking the curl of this equation we have

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\frac{\partial}{\partial t}(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

We use the identity from vector analysis

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}.$$

Because

$$\nabla \cdot \mathbf{E} = \rho = 0,$$

the first term on the right hand side is zero. Thus we have finally

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Now written out this is

$$\nabla^2 \mathbf{E} = \nabla^2 E_1 \mathbf{i} + \nabla^2 E_2 \mathbf{j} + \nabla^2 E_3 \mathbf{k} = \mu_0 \epsilon_0 (\frac{\partial^2 E_1}{\partial t^2} \mathbf{i} + \frac{\partial^2 E_2}{\partial t^2} \mathbf{j} + \frac{\partial^2 E_3}{\partial t^2} \mathbf{k}).$$

So there are three scalar equations.

$$\frac{\partial^2 E_1}{\partial x^2} + \frac{\partial^2 E_1}{\partial y^2} + \frac{\partial^2 E_1}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_1}{\partial^2 t}$$

$$\frac{\partial^2 E_2}{\partial x^2} + \frac{\partial^2 E_2}{\partial y^2} + \frac{\partial^2 E_2}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_2}{\partial^2 t}$$

$$\frac{\partial^2 E_3}{\partial x^2} + \frac{\partial^2 E_3}{\partial y^2} + \frac{\partial^2 E_3}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_1}{\partial^2 t}$$

Now $E_i = 0$ is clearly a solution to the ith equation. So suppose only E_2 is not sero. Now if E_2 is a function of only x, then the second equation is

$$\frac{\partial^2 E_2}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_2}{\partial^2 t},$$

which is a one dimensional wave equation. And the wave velocity is

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Now

$$\epsilon_0 = 8.85 \times 10^{-12}$$

and

$$\mu_0 = 4\pi \times 10^{-7}.$$

And so

$$v = c$$

the velocity of light.

This demonstrates that light is an electromagnetic wave.

More things can be proven, namely that **B** satisfies the same equation, that the two fields are propagated so that **E H** and the energy propagation vector, the wave direction vector **S** (Poynting vector)are mutually perpendicular. And all the optical properties are explained. That the wave velocity is slower as light passes through matter, explaining light refraction.

38 The Poynting Vector

The Poynting vector **S** gives the direction and magnitude of energy flux of an electromagnetic wave. **E** and **H** are perpendicular to each other and mutually perpendicular to the wave direction and to the direction of energy flow **S**. The Poynting vector is given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

John Henry Poynting (born September 9, 1852, died March 30, 1914) was an English physicist. He was a professor of physics at Mason Science College, which became the University of Birmingham.

39 Electromagnetic Waves: Light and Optics

Light and optics is a theory of electromagnetic waves, and the speed of light in free space is determined by ϵ_0 and $\mu_{\mathbf{0}}$.

40 The Existence of Electromagnetic Waves Imply the Theory of Relativity

Recall that Einstein's original paper is titled, **On the Electrodynamics of Moving Bodies**, which shows that electrodynamics is consistent only if time and space is relative.

41 An Electric Field Can Appear as a Magnetic Field in a Second Relative Coordinate System

Reference: Feynman. Electric fields and magnetic fields are part of the same field theory, which is the basis for the Nobel prize of 1965 awarded to Sin-Itiro Tomonaga, Julian Schwinger, and Richard P. Feynman, for their theory of the electro-weak force.

42 Systems of Units

There are four basic systems of units used in electromagnetic theory, the MKS system, the electrostatic cgs system, the electrodynamic cgs system, and the Gaussian cgs system.

We have been using the rationalized MKS system, whose units agree with the practical system of electrical units based on the Ampere and the Volt.

abampere = 10 amperes

Abampere: definition force per unit length between infinite wires d distance apart.

$$f = \frac{2i^2}{d}.$$

charge statampere (electrostatic system, and gaussian)

$$\text{abcoulomb} = \text{statcoulomb}/c$$

43 Fields Relations and Maxwell's Equations in the Gaussian cgs Units

The Maxwell Equations, in the vacuum case, in Gaussian cgs units are

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho,$$

$$\nabla \cdot \mathbf{B} = 0.$$

Fields:

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$$

$$\mathbf{P} = \chi\mathbf{E}$$

$$\mathbf{D} = K\mathbf{E}$$

Lorentz Force:

$$\mathbf{F} = q\mathbf{E} + \frac{q}{c}\mathbf{v} \times \mathbf{B}$$

References: Purcell Electricity and Magnetism, Kip Fundamentals of electricity and magnetism.

44 The Force Between Infinitely Long Parallel Current Carrying Wires (Gaussian Units)

This can be derived by using Maxwell's equation

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}.$$

So consider a circle of radius d around the wire. Integrating the area bounded by the circle we have

$$\int_A \nabla \times \mathbf{B} \cdot d\mathbf{S} = \int_A \frac{4\pi}{c}\mathbf{J} \cdot d\mathbf{S} = \frac{4\pi}{c}i.$$

By symmetry $\mathbf{B}$ is tangent to the circle. So using Stokes's Theorem

$$\int_A \nabla \times \mathbf{B} \cdot d\mathbf{S} = \int_{\partial A} \mathbf{B} \cdot d\ell = d2\pi H,$$

where ∂A is the circular boundary of area A. Hence

$$B = \frac{4\pi}{c}\frac{i}{2\pi d} = \frac{2i}{cd}.$$

The direction of the field is given by the right hand rule: With the thumb in the direction of the current, the field is in the direction of the fingers curled around the wire.

Suppose the two wires are separated by a distance d and each carry current i. If the wires are infinitely long then the field B all around the wire by symmetry is constant.

The Lorentz force on a charge of q statcoulombs in the Gaussian system is

$$\mathbf{F} = q\mathbf{E} + \frac{q}{c}\mathbf{v} \times \mathbf{B}$$

The Lorentz force on an element of length Δx of the second wire is therefore

$$d\mathbf{F} = \frac{\Delta q}{c}\mathbf{v} \times \mathbf{B},$$

where Δq is the amount of charge on a length of the wire Δx and $\mathbf{v}$ is the velocity of this moving charge. The direction of the force on the second wire is in the direction of

$$\mathbf{v} \times \mathbf{B},$$

because the direction of the current in the second wire is the same as in the first wire. It follows by the right hand rule for cross products, that the force $d\mathbf{F}$ on the second wire is toward the first.

Now $\mathbf{v}$ is perpendicular to $\mathbf{B}$ so we can write

$$\Delta F = \Delta q \frac{\Delta x}{\Delta t} B = \frac{\Delta q}{\Delta t} \Delta x B = i \Delta x B$$

So the force per centimeter on the second wire is

$$f = iB = \frac{1}{c^2} \frac{2i^2}{d}.$$

directed toward the first wire.

The force per centimeter if the current is in abamperes is

$$f = iB = \frac{2i^2}{d}.$$

45 Bibliography and References

[1] Becker Richard, **Electromagnetic Fields and Interactions**, Reprinted Dover, 1982, original German edition **Theory Electrizität**, B. G Teubner, Stuttgart.

[2] Emery James D, **Electromagnetic Theory**, electromagnetictheory.pdf, electromagnetictheory.tex.

[3] Emery James D, **Fourier Analysis**, (fouran.tex, fouran.pdf).

[4] Feynman, Richard, **Feynman's Lectures on Physics**, 3 Volumes.

[5] Jackson John David, **Classical Electrodynamics** Second Edition, John Wiley and Sons, 1975. (Uses Gaussian Units).

[6] Jackson John David, **Classical Electrodynamics**, 3rd Edition, 1998, John Wiley and Sons (Uses the rationalized MKS system).

[7] Kip, Arthur F, **Fundamentals of Electricity and Magnetism**, 2nd edition, 1969.

[8] Loeb Leonard B., **Fundamentals of Electricity and Magnetism**, Dover reprint 1961, from the third edition, John Wiley, 1947.

[9] Maxwell James Clerk, **A Treatise on Electricity and Magnetism**, two volumes, Dover reprint 1954, from the 3rd edition, Clarendon Press, 1891.

[10] Owen George E, **Introduction to Electromagnetic Theory**, Alynn and Bacon, 1963.

[11] Panofsky Wolfgang K. H., Phillips Melba, **Classical Electricity and Magnetism**, 1955, Addison-Wesley.

[12] Purcell Edward M., **Electricity and Magnetism**, Volume 2 of the Berkeley Physics Course of 5 volumes, McGraw-Hill, 1965. (Uses Gaussian units).

[13] Reitz John R, Milford Frederick J, and Cristy Robert W., **Foundations of Electromagnetic Theory**, Addison-Wesley, 3rd edition, 1979.

[14] Stratton Julius Adams, **Electromagnetic Theory**, 1941, McGraw-Hill.

[15] Sommerfeld Arnold, **Electrodynamics**, volume 3 of Sommerfeld's **Lectures on Theoretical Physics**, 1964, Academic Press, Translated by Edward G. Ramberg, Six Volumes, V1 **Mechanics**, V2 **Mechanics of Deformable Bodies**, V3 **Electrodynamics**, V4 **Optics**, V5 **Thermodynamics and Statistical Mechanics**, V6 **Partial Differential Equations of Physics**.