

The Ellipse Engine

James Emery

5/19/2009

Contents

1	The Equation of an Ellipse in Standard Position	1
2	Mechanically Generating an Ellipse With an Ellipse Engine	3
3	References	5

1 The Equation of an Ellipse in Standard Position

An ellipse is the locus of points (x, y) , such that the sum of the distances to two fixed points is a constant. Let the distance between the fixed points be $2c$. These points are called the focal points of the ellipse. Suppose these points have coordinates $(-c, 0)$ and $(c, 0)$, with $c > 0$. Let point (x, y) be a point on the ellipse. Let d_1 be the distance from (x, y) to the first focal point, and d_2 the distance from (x, y) to the second focal point. Then

$$d_1 = \sqrt{(x - c)^2 + y^2},$$

and

$$d_2 = \sqrt{(x + c)^2 + y^2}.$$

Consider the special case of a point on the ellipse, where the point is on the x -axis, with coordinates $(a, 0)$, where $a > c > 0$. The sum of distances to the focal points is $2a$ here, because

$$d_1 + d_2 = (a - c) + ((a - c) + 2c) = 2a.$$

So a point is on the ellipse if and only if

$$d_1 + d_2 = 2a.$$

Now suppose point $(0, b)$ is on the ellipse, where $b > 0$. Then

$$d_1 + d_2 = \sqrt{c^2 + b^2} + \sqrt{c^2 + b^2} = 2\sqrt{c^2 + b^2} = 2a.$$

So

$$c^2 + b^2 = a^2.$$

The numbers a and b are called the semi-axes of the ellipse.

Proposition. The canonical equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Proof. Squaring

$$d_1 = 2a - d_2,$$

we find

$$(x - c)^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2.$$

Then

$$(x - c)^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2,$$

so

$$-4xc = 4a^2 - 4a\sqrt{(x + c)^2 + y^2}.$$

So

$$xc + a^2 = a\sqrt{(x + c)^2 + y^2}.$$

Squaring and dividing by a^2 , we find

$$\frac{x^2c^2 + 2xca^2 + a^4}{a^2} = (x + c)^2 + y^2,$$

$$\frac{x^2c^2}{a^2} + 2xc + a^2 = (x + c)^2 + y^2.$$

$$\frac{x^2c^2}{a^2} + 2xc + a^2 = x^2 + 2xc + c^2 + y^2$$

Then

$$\begin{aligned}\frac{x^2c^2}{a^2} + a^2 &= x^2 + c^2 + y^2 \\ \frac{x^2(a^2 - b^2)}{a^2} + a^2 &= x^2 + (a^2 - b^2) + y^2 \\ \frac{x^2(a^2 - b^2)}{a^2} &= x^2 - b^2 + y^2 \\ x^2 - \frac{x^2b^2}{a^2} &= x^2 - b^2 + y^2 \\ \frac{x^2b^2}{a^2} &= b^2 - y^2.\end{aligned}$$

Dividing by b^2 , we have

$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}.$$

Then we have the desired equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

2 Mechanically Generating an Ellipse With an Ellipse Engine

An ellipse engine is a device with two perpendicular slots, with a sliding block in each slot. A rod is connected with two pivots, where each pivot is connected to and pivots in a separate block. On the end of the rod there is a pen. As the blocks move in the slots, the rod moves and the pen draws out an ellipse. This device is modeled as a line segment with three points on the line. A point on the end represents the position of the pen. The other two points are always located on the axes of the ellipse.

Referring to figure 1, let line PR have length a and let line QR have length b . Then

$$x = a \cos(\theta),$$

and

$$y = b \sin(\theta).$$

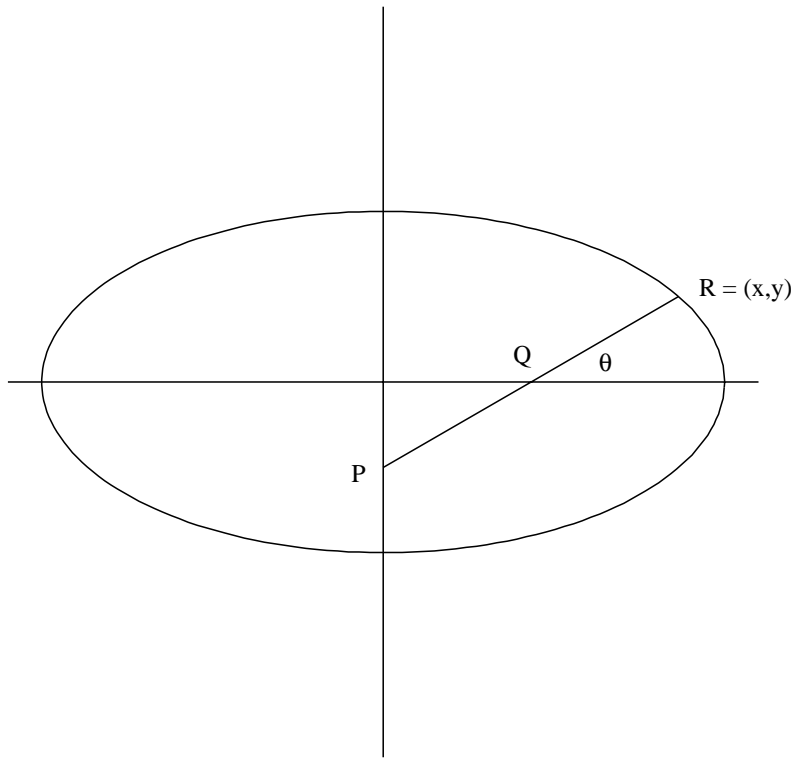


Figure 1: Let the distance between points P and R be a , and the distance between Q and R be b . The line PR makes an angle θ with the x -axis. If the line is rotated where θ ranges from 0 to 2π , while P is always on the y axis, and Q is always on the x axis, then point R traces out an ellipse with major axis a and minor axis b .

Then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2(\theta) + \sin^2(\theta) = 1.$$

Therefore point P traces out a curve with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which is an ellipse with major semi-axis a and minor semi-axis b .

3 References

The following documents and books have material on ellipses. Some of them may be found at

<http://www.stem2.org>

- [1]Emery James D, **Quadric Solids and Computational Geometry**, July 1980, Bendix Kansas City Division BDX-613-2484.
- [2]Emery James D., **The Ellipse**, 2000, (ellipse.tex, ellipse.pdf).
- [3]Emery James D, **Computer Graphics and Geometry**, May 1996, Allied Signal Aerospace, Kansas City.
- [4]Emery James D, **Astronomy and Cosmology**, 2006, (astronomy.tex, astronomy.pdf).
- [5]Emery James D, **Conics, Quadrics, and Projective Space**, 2010, (quadric.tex, quadric.pdf).
- [6]H Karttunen, P. Kroger, H. Oja, M. Poutanen, K. J. Donner; **Fundamental Astronomy**, Springer-Verlag, 1994, Originally a Finnish book.
- [7]Roger R. Bate, Donald D. Mueller, Jerry White, Dover, 1971. **Fundamentals of Astrodynamics**, Dover, 1971.