

# Fourier Analysis

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## 1 Fourier Series

The sequence of trigonometric functions

$$\{\phi_n\} = \{1/\sqrt{T}, \sqrt{2/T} \cos(\omega t), \sqrt{2/T} \sin(\omega t), \sqrt{2/T} \cos(2\omega t), \sqrt{2/T} \sin(2\omega t), \dots, \\ \dots, \sqrt{2/T} \cos(n\omega t), \sqrt{2/T} \sin(n\omega t), \dots\}$$

of period

$$T = \frac{2\pi}{\omega},$$

form an orthonormal sequence on any interval of length  $T$ .

The inner product of two functions  $f$  and  $g$  is defined as

$$(f, g) = \int_0^T f(t)g(t)dt.$$

The trigonometric functions are orthonormal in the sense that

$$(\phi_n, \phi_n) = 1,$$

and if  $n \neq m$  then

$$(\phi_n, \phi_m) = 0.$$

This orthonormal property of the sequence of trigonometric functions follows from a few facts, two of which are

$$\int_0^T \cos(n\omega t)^2 dt = \int_0^T \sin(n\omega t)^2 dt$$

and

$$\int_0^T (\cos(n\omega t)^2 + \sin(n\omega t)^2) dt = \int_0^T dt = T.$$

So we see that either of the above integrals is equal to  $T/2$ . Continuing with the facts, we have

$$\int_0^T \cos(n\omega t) \sin(m\omega t) dt = 0.$$

And when  $n$  is not equal to  $m$  then

$$\int_0^T \cos(n\omega t) \cos(m\omega t) dt = 0,$$

and again

$$\int_0^T \sin(n\omega t) \sin(m\omega t) dt = 0.$$

These properties follow easily by replacing the trigonometric functions by their complex exponential equivalents. The coefficients of a Fourier series

$$f(t) = a_0/2 + a_1 \cos(\omega t) + b_1 \sin(\omega t) + a_2 \cos(2\omega t) + b_2 \sin(2\omega t) + \dots,$$

are determined by computing the inner product of  $f$  with each of the orthonormal functions. Thus if we write

$$f = \alpha_1 \phi_1 + \alpha_2 \phi_2 + \dots + \alpha_n \phi_n + \dots,$$

then

$$(f, \phi_n) = \alpha_n (\phi_n, \phi_n) = \alpha_n.$$

Therefore, for the Fourier series, we have

$$a_n = \frac{2}{T} \int_0^T f(t) \cos(n\omega t) dt.$$

and

$$b_n = \frac{2}{T} \int_0^T f(t) \sin(n\omega t) dt.$$

## 2 Convergence

There exists a continuous function on a closed interval whose Fourier series is nowhere convergent.

However, there are conditions that guarantee that a Fourier series converges. For example: If a function is Lebesgue integrable, and at a point  $x$ , both continuous from the right, and continuous from the left, then the series converges to the mean of the left and right function limits at  $x$ . (see Goldberg **The Elements of Real Analysis**.)

## 3 The $L^2$ Theory of Fourier Series

See Goldberg **The Elements of Real Analysis**.

## 4 The Fourier Transform

The Fourier transform of the function  $f$  is defined as (Goldberg, **The Fourier Transform**)

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt.$$

By the Fourier integral theorem

$$f(t) = \int_{-\infty}^{\infty} g(\omega)e^{i\omega t} d\omega.$$

**Note.** The transform is defined slightly differently by various authors: **Schaum's Mathematical Handbook**:

$$g(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt.$$

Then the inverse transform is

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega)e^{i\omega t} d\omega.$$

**The Fourier Integral and Its Applications** by Athanasios Papoulis:

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{i\omega t} dt.$$

Then the inverse transform is

$$f(t) = \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} d\omega.$$

**Quantum Mechanics** by Powell and Crasemann:

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt.$$

Then the inverse transform is

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} d\omega.$$

**Digital Signal Processing in Telecommunications** by Kishan Shenoi:

$$g(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\omega t} dt.$$

Then the inverse transform is

$$f(t) = \int_{-\infty}^{\infty} g(\omega) e^{i2\pi\omega t} d\omega.$$

Including the  $2\pi$  factor in the exponential argument eliminates the multiplier in front of both integrals.

## 5 The Discrete Fourier Transform

Suppose  $f$  has its support in the interval  $[0, T]$ . Suppose that we divide this interval into  $N$  equal pieces, where integer  $N$  is even. Let

$$\Delta t = \frac{T}{N}.$$

Define  $t_p = p\Delta t$  for  $p = 0$  to  $N - 1$ . Let the fundamental angular frequency be  $\Delta\omega = \frac{2\pi}{T}$  and define  $\omega_q = q\Delta\omega$ ,  $q = 0, \dots, N - 1$ . A Riemann approximation to  $g$

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt.$$

is

$$G(\omega) = \frac{1}{2\pi} \sum_{p=0}^{N-1} f(t_p) e^{-i\omega t_p} \Delta t.$$

Define

$$G_q = G(\omega_q),$$

and

$$F_p = f(t_p).$$

The mapping

$$\{F_p\} \rightarrow \{G_q\},$$

is called the discrete Fourier Transform. We have

$$t_p \omega_q = \frac{pT}{N} \frac{q2\pi}{T} = \frac{2\pi}{N} pq,$$

so

$$G_q = \frac{T}{2\pi N} \sum_{p=0}^{N-1} F_p (e^{i2\pi/N})^{-pq} = \frac{T}{2\pi N} \sum_{p=0}^{N-1} F_p W^{-pq},$$

where  $W = e^{i2\pi/N}$  is the principal  $N$ th root of unity.

**The Discrete Fourier Integral Theorem.** Let

$$\begin{aligned} G_n &= \frac{T}{2\pi N} \sum_{k=0}^{N-1} F_k e^{-i\omega_n t_k} \\ &= \frac{T}{2\pi N} \sum_{k=0}^{N-1} F_k W^{-nk}, \end{aligned}$$

where  $W = e^{i2\pi/N}$ . Then

$$f(t_j) = F_j = \frac{2\pi}{T} \sum_{n=0}^{N-1} G_n W^{nj}.$$

**Proof.**

$$\begin{aligned} & \frac{2\pi}{T} \sum_{n=0}^{N-1} G_n W^{nj} \\ &= \frac{2\pi}{T} \sum_{n=0}^{N-1} \left( \frac{T}{2\pi N} \sum_{k=0}^{N-1} F_k W^{-nk} \right) W^{nj} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} F_k W^{(j-k)n} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} F_k \sum_{n=0}^{N-1} (W^{j-k})^n \\
&= \frac{1}{N} \sum_{k=0}^{N-1} F_k \delta_j^k N = F_j.
\end{aligned}$$

This follows because when  $j = k$ , then

$$\sum_{n=0}^{N-1} (W^{j-k})^n = N,$$

and when  $j \neq k$ , then

$$\sum_{n=0}^{N-1} (W^{j-k})^n = (1 - (W^{j-k})^N)/(1 - W^{j-k}) = 0.$$

The latter equation follows from the expression for the sum of a geometric series, and from the fact that  $W^{j-k}$  is an  $N$ th root of unity.

## 6 The Matlab FFT

The Matlab function `fft()`, computes essentially

$$G_q^{ML} = \sum_{p=0}^{N-1} F_p (e^{i2\pi/N})^{-pq},$$

where  $q = 0, \dots, N-1$  and  $p = 0, \dots, N-1$ . This is our discrete fourier transform defined above, without the multiplication factor  $\frac{T}{2\pi N}$ .

$$\begin{aligned}
G_q &= \frac{T}{2\pi N} \sum_{p=0}^{N-1} F_p (e^{i2\pi/N})^{-pq} \\
&= \frac{T}{2\pi N} G_q^{ML}.
\end{aligned}$$

Because MatLab does not allow 0 indices, it actually computes

$$G_q^{ML} = \sum_{p=1}^N F_p(e^{i2\pi/N})^{-(p-1)(q-1)}.$$

where  $1 \leq q \leq N$  and  $1 \leq p \leq N$ . So

$$\frac{T}{2\pi N} G_q^{ML}$$

is an approximate value of the continuous Fourier transform

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt.$$

at sample point  $\omega_q$ .

The Fourier Approximation Example on p229, of the book "Mastering Matlab 5," runs without change in Octave. The Matlab fft function computes the same fft as is in my paper foran.tex for the discrete Fourier transform, except for the constant multiplier. In this approximation example the same constant multiplier is applied to the Matlab fft output to get the approximation that is graphed. Because the function is real, the magnitude of the transform is symmetric about the  $t=0$  axis, that is, it is an even function. The Nyquist frequency is 1/2 the sampling frequency. So we compute only the lower half frequencies to avoid aliasing.

## 7 The IMSL FFT

The IMSL 1978 Fortran routine fft2c does the same computation as the MatLab function fft described in the previous section:

```

remarks  1.  fft2c computes the fourier transform, x, according
c          to the following formula;
c
c          x(k+1) = sum from j = 0 to n-1 of
c                   a(j+1)*cexp((0.0, (2.0*pi*j*k)/n))
c          for k=0,1,...,n-1 and pi=3.1415...
```

## 8 Fourier Interpolation

**Proposition.** If  $f$  is real, then

$$g(-\omega) = \overline{g(\omega)}.$$

**Corollary** The magnitude of the transform of a real function is symmetric.

**Proposition.** If  $\{F_k\}$  is real, then

$$G_q = \overline{G_{N-q}}$$

**Proof.**

$$\begin{aligned} G_{N-q} &= \frac{T}{2\pi N} \sum_{k=0}^{N-1} F_k W^{-(N-q)k} \\ &= \frac{T}{2\pi N} \sum_{k=0}^{N-1} F_k (W^N)^{-k} W^{qk} \\ &= \frac{T}{2\pi N} \sum_{k=0}^{N-1} F_k W^{qk} \\ &= \overline{G_q}. \end{aligned}$$

**Proposition.** If  $f$  is real and  $N$  is even, then the trigonometric sum

$$\begin{aligned} \tilde{f}(t) &= \frac{a_0}{2} + a_1 \cos(\omega_1 t) + b_1 \sin(\omega_1 t) + \dots \\ &\quad + a_{K-1} \cos(\omega_{K-1} t) + b_{K-1} \sin(\omega_{K-1} t) \\ &\quad + a_K \cos(\omega_K t) + b_K \sin(\omega_K t). \end{aligned}$$

interpolates  $f$  at each  $t_p$ . That is,

$$f(t_p) = F_p = \tilde{f}(t_p).$$

The coefficients in the function  $\tilde{f}$  are defined as

$$\begin{aligned} K &= N/2, \\ a_0 &= \frac{G_0 4\pi}{T}, \end{aligned}$$

and for  $j = 1, \dots, K - 1$ ,

$$a_j = \frac{\Re(G_j)4\pi}{T},$$

$$b_j = \frac{-\Im(G_j)4\pi}{T}.$$

The last coefficients are

$$a_K = \frac{\Re(G_K)2\pi}{T},$$

and

$$b_K = \frac{-\Im(G_K)2\pi}{T}.$$

**Proof.** The proof consists in showing that at each  $t_p$ , the inverse transform  $F(t)$  equals the trigonometric sum.

By the discrete Fourier integral theorem, we have

$$\begin{aligned} f(t_p) &= F_p = \frac{2\pi}{T} \sum_{q=0}^{N-1} G_q W^{pq} \\ &= \frac{2\pi}{T} \sum_{q=0}^{N-1} G_q W^{pq} \\ &= \frac{2\pi}{T} [G_0 + (G_1 W^{1p} + G_{N-1} W^{(N-1)p}) \\ &\quad + (G_2 W^{2p} + G_{N-2} W^{(N-2)p}) \\ &\quad + \dots \\ &\quad + (G_{K-1} W^{(K-1)p} + G_{K+1} W^{(K+1)p}) \\ &\quad + G_K W^{Kp}] \\ &= \frac{2\pi}{T} \left[ G_0 + \sum_{q=1}^{K-1} (G_q W^{pq} + \overline{G_q W^{pq}}) + G_K W^{Kp} \right] \\ &= \frac{2\pi}{T} \left[ G_0 + \sum_{q=1}^{K-1} 2\Re(G_q W^{pq}) + G_K W^{Kp} \right]. \end{aligned}$$

Let

$$G_q = \alpha_q + i\beta_q.$$

We have

$$\begin{aligned}\Re(G_q W^{pq}) &= \Re[(\alpha_q + i\beta_q)(\cos(\omega_q t_p) + i \sin(\omega_q t_p))] \\ &= \alpha_q \cos(\omega_q t_p) - \beta_q \sin(\omega_q t_p) \\ &= \Re(G_q) \cos(\omega_q t_p) - \Im(G_q) \sin(\omega_q t_p).\end{aligned}$$

Now each  $F_p = f(t_p)$  is real. Then

$$G_0 = \frac{T}{2\pi N} \sum_{k=0}^{N-1} F_k$$

is real. Therefore the equation forces  $G_K W^{Kp}$  to be real. So

$$G_K W^{Kp} = \Re(G_K) \cos(\omega_K t_p) - \Im(G_K) \sin(\omega_K t_p).$$

This finishes the proof.

**Note:**  $W^K = -1$ .  $\tilde{f}(t)$  is not equal to

$$F(t) = \frac{2\pi}{T} \sum_{n=0}^{N-1} G_n e^{i\omega_n t},$$

for all  $t$ , because  $F(t)$  may be complex. A program for doing this interpolation is given in the next section.

## 9 A Fourier Interpolation Program

Program **frintrap.ftn** uses the Fast Fourier Transform to compute an interpolating trigonometric sum.

```
c frintrap.ftn modification of fourier.for 3/14/95
  implicit real*8 (a-h,o-z)
  logical iszero
  integer gfile
  complex*16 g(1024)
  dimension iwk(11)
  dimension frq(1025), amod(1025), arg(1025)
  dimension a(1024), b(1024)
  dimension ain(10)
  character*80 cs
  data pi/3.14159265358979d0/
  one=1.
  zero=0.
```

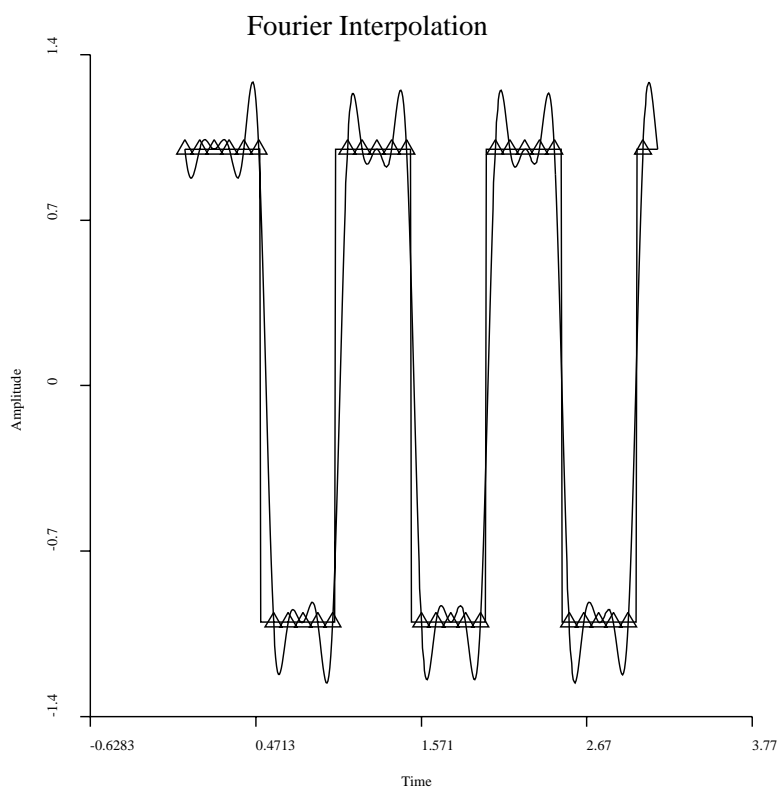


Figure 1: Interpolating a step function with program **frinrp.ftn**.

```

per=1.
m=3
gfile=3
open(gfile,file='p.gi',status='unknown')
1  continue
   cs='(1)Enter period and number of sample points'
   write(*,*)cs(1:lenstr(cs))
   write(*,2)
2  format(
+'(2)List transform modulus.'/
+'(3)List fourier coefficients.'/
+'(4)Plot function and fourier approximation.'/
+'(5)Plot transform modulus.'/
+'(6)Stop.'/
+)
   call readr(0,ain,nr)
   nm=ain(1)
   if(nm.eq.1)go to 9
   if(nm.eq.2)go to 33
   if(nm.eq.3)go to 33
   if(nm.eq.4)go to 33
   if(nm.eq.5)go to 33
   if(nm.eq.6)stop
   go to 1
9  continue
   write(*,*)'Enter period [3.14159]
call readr(0,ain,nr)
if(nr .eq. 0)then
  per=3.1415926535
else
  per=ain(1)
endif
write(*,*)' Number of sample points = 2^m. Enter m [5]
call readr(0,ain,nr)
if(nr .eq. 0)then
  m=5
else
  m=ain(1)
endif
go to 1
33 continue
   n=2**m
   write(*,*)' Sample Points: '
   do 30 i=1,n
     t=(per/n)*(i-1)
     g(i)=cplx(f(t),zero)
     write(*,*)i,g(i)
30  continue
   do 40 i=1,n
     g(i)=conjg(g(i))
40  continue
   call fft2c(g,m,iwk)
   write(*,*)' Transform: '
   do 45 i=1,n
     write(*,*)i,g(i)

```

```

45  continue
    gmax=0.
    do 50 i=1,n
      gm=abs(g(i))
      if (gm.gt.gmax)gmax=gm
50  g(i)=conjg(g(i))*per/(2*pi*n)
      del=gmax*1.e-10
      do 52 i=1,n
        am=abs(g(i))
        if (am .lt. del)g(i)=zero
52  continue
      if (nm.eq.2)go to 55
      if (nm.eq.3)go to 79
      if (nm.eq.4)go to 79
55  k=n+1
      if (nm.eq.2)print 60
60  format(' Freq. (cy/sec)  Modulus  Argument (degrees)')
      amodmx=0.
      do 70 i=1,k
        j=i-(n/2+1)
        l=iabs(j)+1
        frq(i)=j/per
        amod(i)=abs(g(l))
        if (amod(i).gt.amodmx)amodmx=amod(i)
        iszero = amod(i) .eq. zero
        arg(i)=0.
        if (.not. iszero)arg(i)=dimag(log(g(l)))*180./pi
        if (j.lt.0)arg(i)=-arg(i)
        if (nm.eq.2)print 75,frq(i),amod(i),arg(i)
75  format(3(1x,g15.8))
70  continue
      if (nm.eq.2)go to 1
      call xvwpor(gfile,-one,one,-one,one)
      xmn=-frq(k)
      xmx=frq(k)
      ymn=0.
      ymx=amodmx
      call xwindo(gfile,xmn,xmx,ymn,ymx)
      call xmove(gfile,frq(1),amod(1))
      do 77 i=2,k
        call xdraw(gfile,frq(i),amod(i))
77  continue
      go to 1
79  continue
      l=n/2-1
      a0=real(g(1))*2*pi/per
      cmx=abs(a0)
      do 80 j=1,l
        am=real(g(j+1))*4*pi/per
        bm=-dimag(g(j+1))*4*pi/per
        a(j)=am
        b(j)=bm
        if (cmx.lt.am)cmx=am
        if (cmx.lt.bm)cmx=bm
80  continue

```

```

a(l+1)=real(g(l+2))*2*pi/per
b(l+1)=0.
l=l+1
nz=0
del=cmx*1.e-10
do 81 i=1,l
a1=a(i)
b1=b(i)
if(sqrt(a1*a1+b1*b1).lt.del)go to 81
nz=nz+1
a(nz)=a1
b(nz)=b1
frq(nz)=i/per
if(abs(a1).lt.del)a(nz)=0.
if(abs(b1).lt.del)b(nz)=0.
81 continue
print 82
82 format(' nonzero fourier series coefficients')
print 84,a0
84 format(' constant= ',g15.8)
print 90
90 format(' freq. (cy/sec)   cos coeff.   sin coeff.')
```

if(nz.eq.0)go to 110

```

do 100 j=1,nz
fr=frq(j)
print 95,fr,a(j),b(j)
95 format(3(2x,g15.8))
100 continue
110 continue
if(nm.eq.3)go to 1
npts=500
do 135 i=1,npts
t=(i-1)*per/(npts-1)
y=f(t)
if(i.eq.1)ymn=y
if(i.eq.1)ymx=y
if(y.lt.ymn)ymn=y
if(y.gt.ymx)ymx=y
135 continue
xmn=0.
xmx=per
xmarj=(xmx-xmn)*.2
xmn=xmn-xmarj
xmx=xmx+xmarj
ymarj=(ymx-ymn)*.2
ymn=ymn-ymarj
ymx=ymx+ymarj
call xvwpor(gfile,-one,one,-one,one)
call xwindo(gfile,xmn,xmx,ymn,ymx)
call xlincl(gfile,1)
do 140 i=1,npts
t=(i-1)*per/(npts-1)
y=f(t)
if(i .eq. 1)then
call xmove(gfile,t,y)

```

```

        else
            call xdraw(gfile,t,y)
        endif
140 continue
    call xlincl(gfile,2)
c    dw=2*pi/per
    do 160 i=1,npts
        t=(i-1)*per/(npts-1)
        s=a0
        if(nz .gt. 0)then
            do 150 j=1,nz
                w=2.*pi*frq(j)
                s=s+a(j)*cos(w*t)+b(j)*sin(w*t)
150         continue
            endif
        if(i.eq.1)then
            call xmove(gfile,t,s)
        else
            call xdraw(gfile,t,s)
        endif
160 continue
    call xlincl(gfile,3)
    do 170 i=1,n
        t=(per/n)*(i-1)
        ff=f(t)
        ns=1
        size=.02
        call xdsymbo(gfile,ns,t,ff,size)
170 continue
    go to 1
end

c+ fft2c fast fourier transform
    subroutine fft2c (a,m,iwk)
c    imsl routine name - fft2c
c
c-----
c
c    computer          - cdc/single
c
c    latest revision   - january 1, 1978
c
c    purpose           - compute the fast fourier transform of a
c                      - complex valued sequence of length equal to
c                      - a power two
c
c    usage             - call fft2c (a,m,iwk)
c
c    arguments        a   - complex vector of length n, where n=2**m.
c                      - on input a contains the complex valued
c                      - sequence to be transformed.
c                      - on output a is replaced by the
c                      - fourier transform.
c
c                      m   - input exponent to which 2 is raised to
c                      - produce the number of data points, n
c                      - (i.e. n = 2**m).

```

```

c          iwk      - work vector of length m+1.
c
c  precision/hardware - single and double/h32
c                    - single/h36,h48,h60
c
c  reqd. imsl routines - none required
c
c  notation           - information on special notation and
c                    conventions is available in the manual
c                    introduction or through imsl routine uhel
c
c  remarks  1.  fft2c computes the fourier transform, x, according
c              to the following formula;
c
c              x(k+1) = sum from j = 0 to n-1 of
c                    a(j+1)*cexp((0.0,(2.0*pi*j*k)/n))
c              for k=0,1,...,n-1 and pi=3.1415...
c
c              note that x overwrites a on output.
c  2.  fft2c can be used to compute
c
c              x(k+1) = (1/n)*sum from j = 0 to n-1 of
c                    a(j+1)*cexp((0.0,(-2.0*pi*j*k)/n))
c              for k=0,1,...,n-1 and pi=3.1415...
c
c              by performing the following steps;
c
c              do 10 i=1,n
c                a(i) = conjg(a(i))
c  10 continue
c              call fft2c (a,m,iwk)
c              do 20 i=1,n
c                a(i) = conjg(a(i))/n
c  20 continue
c
c  copyright          - 1978 by imsl, inc. all rights reserved.
c
c  warranty           - imsl warrants only that imsl testing has been
c                    applied to this code. no other warranty,
c                    expressed or implied, is applicable.
c
c-----
c
c  implicit real*8 (a-h,o-z)
c
c                    specifications for arguments
c  integer           m,iwk(*)
c  complex*16        a(*)
c
c                    specifications for local variables
c  integer           i,isp,j,jj,jsp,k,k0,k1,k2,k3,kb,kn,mk,mm,mp,n,
1                    n4,n8,n2,lm,nn,jk
c  real              rad,c1,c2,c3,s1,s2,s3,ck,sk,sq,a0,a1,a2,a3,
1                    b0,b1,b2,b3,twopi,temp,
2                    zero,one,z0(2),z1(2),z2(2),z3(2)
c  complex           za0,za1,za2,za3,ak2
c  equivalence       (za0,z0(1)),(za1,z1(1)),(za2,z2(1)),

```

```

1          (za3,z3(1)),(a0,z0(1)),(b0,z0(2)),(a1,z1(1)),
2          (b1,z1(2)),(a2,z2(1)),(b2,z2(2)),(a3,z3(1)),
3          (b3,z3(2))
  data      sq/.70710678118655d0/,
1          sk/.38268343236509d0/,
2          ck/.92387953251129d0/,
3          twopi/6.2831853071796d0/
  data      zero/0.0d0/,one/1.0d0/
c          sq=sqrt2/2,sk=sin(pi/8),ck=cos(pi/8)
c          twopi=2*pi
c          first executable statement

  mp = m+1
  n = 2**m
  iwk(1) = 1
  mm = (m/2)*2
  kn = n+1
c          initialize work vector

  do 5 i=2,mp
    iwk(i) = iwk(i-1)+iwk(i-1)
5  continue
  rad = twopi/n
  mk = m - 4
  kb = 1
  if (mm .eq. m) go to 15
  k2 = kn
  k0 = iwk(mm+1) + kb
10 k2 = k2 - 1
  k0 = k0 - 1
  ak2 = a(k2)
  a(k2) = a(k0) - ak2
  a(k0) = a(k0) + ak2
  if (k0 .gt. kb) go to 10
15 c1 = one
  s1 = zero
  jj = 0
  k = mm - 1
  j = 4
  if (k .ge. 1) go to 30
  go to 70
20 if (iwk(j) .gt. jj) go to 25
  jj = jj - iwk(j)
  j = j-1
  if (iwk(j) .gt. jj) go to 25
  jj = jj - iwk(j)
  j = j - 1
  k = k + 2
  go to 20
25 jj = iwk(j) + jj
  j = 4
30 isp = iwk(k)
  if (jj .eq. 0) go to 40
c          reset trigonometric parameters

  c2 = jj * isp * rad
  c1 = cos(c2)
  s1 = sin(c2)

```

```

35 c2 = c1 * c1 - s1 * s1
   s2 = c1 * (s1 + s1)
   c3 = c2 * c1 - s2 * s1
   s3 = c2 * s1 + s2 * c1
40 jsp = isp + kb
c                                     determine fourier coefficients
c                                     in groups of 4
   do 50 i=1,isp
     k0 = jsp - i
     k1 = k0 + isp
     k2 = k1 + isp
     k3 = k2 + isp
     za0 = a(k0)
     za1 = a(k1)
     za2 = a(k2)
     za3 = a(k3)
     if (s1 .eq. zero) go to 45
     temp = a1
     a1 = a1 * c1 - b1 * s1
     b1 = temp * s1 + b1 * c1
     temp = a2
     a2 = a2 * c2 - b2 * s2
     b2 = temp * s2 + b2 * c2
     temp = a3
     a3 = a3 * c3 - b3 * s3
     b3 = temp * s3 + b3 * c3
45   temp = a0 + a2
     a2 = a0 - a2
     a0 = temp
     temp = a1 + a3
     a3 = a1 - a3
     a1 = temp
     temp = b0 + b2
     b2 = b0 - b2
     b0 = temp
     temp = b1 + b3
     b3 = b1 - b3
     b1 = temp
     a(k0) = cmplx(a0+a1,b0+b1)
     a(k1) = cmplx(a0-a1,b0-b1)
     a(k2) = cmplx(a2-b3,b2+a3)
     a(k3) = cmplx(a2+b3,b2-a3)
50 continue
   if (k .le. 1) go to 55
   k = k - 2
   go to 30
55 kb = k3 + isp
c                                     check for completion of final
c                                     iteration
   if (kn .le. kb) go to 70
   if (j .ne. 1) go to 60
   k = 3
   j = mk
   go to 20
60 j = j - 1

```



```

        a(jk) = ak2
110 j = j + 1
c
c                                     cycle repeated until limiting number
c                                     of changes is achieved
        if (j .le. lm) go to 80
c
9005 return
        end
c+ f signal function
c   function f(t)
c   implicit real*8 (a-h,o-z)
c   data pi/3.14159265358979d0/
c   zero=0.
c   one=1.
c   k=cycles per second
c   ak=1.9098609
c   j=4
c   c=0.
c   f=sin(ak*2.*pi*t)
c   f=f+cos(j*2.*pi*t)+c
c   return
c   end
c
c+ f signal function
c   function f(t)
c   implicit real*8 (a-h,o-z)
c   data pi/3.14159265358979d0/
c   zero=0.
c   one=1.
c   if(t .lt. zero)t=-t
c   m=t
c   tt=t-m
c   f=1.
c   if(tt .gt. one/2.)f=-1.
c   return
c   end
c+ readr read a row of floating point numbers
c   subroutine readr(nf, a, nr)
c   implicit real*8(a-h,o-z)
c   numbers are separated by spaces
c   examples of valid numbers are:
c   12.13 34 45e4 4.78e-6 4e2,5.6D-23,10000.d015
c   nf=file number, 0 for standard input file
c   a=array of returned numbers
c   nr=number of values in returned array,
c   or 0 for empty or blank line,
c   or -1 for end of file on unit nf.
c requires functions val and length
c   dimension a(*)
c   character*200 b
c   character*200 c
c   character*1 d
c   c=' '
c   if(nf.eq.0)then
c     read(*,'(a)',end=99)b

```

```

else
    read(nf, 'a)', end=99) b
endif
nr=0
l=lenstr(b)
if(l.ge.200) then
    write(*,*) ' error in readr subroutine '
    write(*,*) ' record is too long '
endif
do 1 i=1, l
    d=b(i:i)
    if (d.ne.' ') then
        k=lenstr(c)
        if (k.gt.0) then
            c=c(1:k)//d
        else
            c=d
        endif
    endif
    if( (d.eq.' ').or.(i.eq.l)) then
        if (c.ne.' ') then
            nr=nr+1
            call valsub(c,a(nr),ier)
            c=' '
        endif
    endif
1    continue
return
99    nr=-1
return
end

c
c+ lenstr nonblank length of string
function lenstr(s)
c length of the substring of s obtained by deleting all
c trailing blanks from s. thus the length of a string
c containing only blanks will be 0.
character s(*)
lenstr=0
n=len(s)
do 10 i=n, 1, -1
    if(s(i:i).ne.' ') then
        lenstr=i
        return
    endif
10    continue
return
end

c+ valsub converts string to floating point number (r*8)
subroutine valsub(s,v,ier)
implicit real*8(a-h,o-z)
c examples of valid strings are: 12.13 34 45e4 4.78e-6 4E2
c the string is checked for valid characters,
c but the string can still be invalid.
c s-string

```

```

c      v-returned value
c      ier- 0 normal
c          1 if invalid character found, v returned 0
c
c      logical p
c      character s*(*),c*50,t*50,ch*15
c      character z*1
c      data ch/'1234567890+-.eE'/
c      v=0.
c      ier=1
c      l=lenstr(s)
c      if(l.eq.0)return
c      p=.true.
c      do 10 i=1,l
c      z=s(i:i)
c      if((z.eq.'D').or.(z.eq.'d'))then
c          s(i:i)='e'
c      endif
c      p=p.and.(index(ch,s(i:i)).ne.0)
10    continue
c      if(.not.p)return
c      n=index(s,'.')
c      if(n.eq.0)then
c          n=index(s,'e')
c          if(n.eq.0)n=index(s,'E')
c          if(n.eq.0)n=index(s,'d')
c          if(n.eq.0)n=index(s,'D')
c          if(n.eq.0)then
c              s=s(1:l)//'.'
c          else
c              t=s(n:l)
c              s=s(1:(n-1))//'. '//t
c          endif
c          l=l+1
c      endif
c      write(c,'(a30)')s(1:l)
c      read(c,'(g30.23)')v
c      ier=0
c      return
c      end
c+ xszprm write text size parameter (external plot)
c      subroutine xszprm(nfile,h)
c      implicit real*8(a-h,o-z)
c      character s*25,t*26
c      call str(h,s)
c      l=lenstr(s)
c      t(1:1)='z'
c      t(2:(l+1))=s(1:l)
c      write(nfile,'(a)')t(1:(l+1))
c      return
c      end
c+ xangpr write text angle parameter (external plot)
c      subroutine xangpr(nfile,a)
c      implicit real*8(a-h,o-z)
c      character s*25,t*26

```

```

    call str(a,s)
    l=lenstr(s)
    t(1:1)='g'
    t(2:(1+1))=s(1:1)
    write(nfile,'(a)')t(1:(1+1))
    return
end
c+ xdsymbo symbol to external plot file
subroutine xdsymbo(nfile,ns,x,y,size)
implicit real*8(a-h,o-z)
character s*25,t*80
t='s'
n=2
call str(x,s)
l=lenstr(s)
t(n:80)=s
n=n+1+1
call str(y,s)
l=lenstr(s)
t(n:80)=s
n=n+1+1
ans=ns
call str(ans,s)
l=lenstr(s)
t(n:80)=s
n=n+1+1
call str(size,s)
l=lenstr(s)
t(n:80)=s
write(nfile,'(a)')t(1:(n+1-1))
return
end
c+ xlincl line color parameter to external plot file
subroutine xlincl(nfile,nc)
implicit real*8(a-h,o-z)
write(nfile,10)nc
10 format('c',i1)
return
end
c+ xdarc arc into external plot file
subroutine xdarc(nfile,xc,yc,r,a1,a2,npts)
implicit real*8(a-h,o-z)
c nfile-unit number for output file
c xc,yc-arc center
c r-arc radius
c a1,a2-start and end angles
c npts-number of points in arc
character s*25,t*80
t='a'
n=2
call str(xc,s)
l=lenstr(s)
t(n:80)=s
n=n+1+1
call str(yc,s)

```

```

l=lenstr(s)
t(n:80)=s
n=n+1+1
call str(r,s)
l=lenstr(s)
t(n:80)=s
n=n+1+1
call str(a1,s)
l=lenstr(s)
t(n:80)=s
n=n+1+1
call str(a2,s)
l=lenstr(s)
t(n:80)=s
n=n+1+1
an=npts
call str(an,s)
l=lenstr(s)
t(n:80)=s
write(nfile,'(a)')t(1:(n+1-1))
return
end
c+ xwindo window parameters to external plot file
subroutine xwindo(nfile,xmn,xmx,ymn,ymx)
implicit real*8(a-h,o-z)
character s*25,t*80
t='w'
n=2
call str(xmn,s)
l=lenstr(s)
t(n:(n+1-1))=s
n=n+1+1
call str(xmx,s)
l=lenstr(s)
t(n:(n+1-1))=s
n=n+1+1
call str(ymn,s)
l=lenstr(s)
t(n:(n+1-1))=s
n=n+1+1
call str(ymx,s)
l=lenstr(s)
t(n:(n+1-1))=s
write(nfile,'(a)')t(1:(n+1-1))
call wprn(1,xmn,xmx,ymn,ymx)
return
end
c+ xvwpor viewport parameters to external plot file
subroutine xvwpor(nfile,xmn,xmx,ymn,ymx)
implicit real*8(a-h,o-z)
character s*25,t*80
t='v'
n=2
call str(xmn,s)
l=lenstr(s)

```

```

t(n:80)=s
n=n+1+1
call str(xmx,s)
l=lenstr(s)
t(n:80)=s
n=n+1+1
call str(ymn,s)
l=lenstr(s)
t(n:80)=s
n=n+1+1
call str(ymx,s)
l=lenstr(s)
t(n:80)=s
write(nfile,'(a)')t(1:(n+1-1))
call vprm(1,xmn,xmx,ymn,ymx)
return
end
c+ xdraw draw parameters to external plot file
subroutine xdraw(nfile,x,y)
implicit real*8(a-h,o-z)
character s*25,t*80
t='d'
n=2
call str(x,s)
l=lenstr(s)
t(n:80)=s
n=n+1+1
call str(y,s)
l=lenstr(s)
t(n:80)=s
write(nfile,'(a)')t(1:(n+1-1))
return
end
c
c+ xmove move parameters to external plot file
subroutine xmove(nfile,x,y)
implicit real*8(a-h,o-z)
character s*25,t*80
t='m'
n=2
call str(x,s)
l=lenstr(s)
t(n:80)=s
n=n+1+1
call str(y,s)
l=lenstr(s)
t(n:80)=s
write(nfile,'(a)')t(1:(n+1-1))
return
end
c+ str floating point number to string
subroutine str(x,s)
implicit real*8(a-h,o-z)
character s*25,c*25,b*25,e*25
zero=0.

```

```

if(x.eq.zero)then
  s='0'
  return
endif
write(c,'(g11.4)')x
read(c,'(a25)')b
l=lenstr(b)
do 10 i=1,l
  n1=i
  if(b(i:i).ne.' ')go to 20
10  continue
20  continue
   if(b(n1:n1).eq.'0')n1=n1+1
   b=b(n1:l)
   l=l+1-n1
   k=index(b,'E')
   if(k.gt.0)e=b(k:l)
   if(k.gt.0)then
     s=b(1:(k-1))
     k1=index(b,'E+0')
     if(k1.gt.0)then
       e='E'//b((k1+3):l)
     else
       k1=index(b,'E+')
       if(k1.gt.0)e='E'//b((k1+2):l)
     endif
     k1=index(b,'E-0')
     if(k1.gt.0)e='E-'//b((k1+3):l)
     l=k-1
   else
     s=b
   endif
   j=index(s,'.')
   n2=1
   if(j.ne.0)then
     do 30 i=1,l
       n2=l+1-i
       if(s(n2:n2).ne.'0')go to 40
30    continue
   endif
40    continue
   s=s(1:n2)
   if(s(n2:n2).eq.'.')then
     s=s(1:(n2-1))
     n2=n2-1
   endif
   if(k.gt.0)s=s(1:n2)//e
return
end

```

## 10 Finite Fourier Analysis

On a set of uniformly spaced points a function space has a finite Fourier basis. See Emery **Finite Fourier Analysis** .

## 11 Fourier Least Squares

For a general introduction to this topic see: Emery **Least Squares Approximation** (lsq.tex). Suppose we wish to fit a function of the form

$$\sin(\omega t + \phi).$$

We may vary  $\omega$  over a frequency interval and compute the best fit of the function

$$A \cos(\omega t) + B \sin(\omega t).$$

Then

$$\begin{aligned} & A \cos(\omega t) + B \sin(\omega t) \\ &= \sqrt{A^2 + B^2} \left[ \frac{A}{\sqrt{A^2 + B^2}} \cos(\omega t) + \frac{B}{\sqrt{A^2 + B^2}} \sin(\omega t) \right] \\ &= C[\sin \phi \cos \omega t + \cos \phi \sin \omega t] \\ &= C \sin(\omega t + \phi), \end{aligned}$$

where

$$\tan(\phi) = \frac{B}{A}.$$

## 12 Computing The Period of a Sampled Signal

Suppose we have a periodic sampled signal  $\{x_i\}$  consisting of  $n$  values. Suppose it has a period  $T < n/3$ . Suppose the signal has maximum values or peaks at points  $k, k + T, k + 2T$  and so on. Then we can find the period  $T$  by locating these peaks and taking  $T$  to be the average distance between these peaks. First we shall find the maximum of the signal, which we write as  $x_m$ . We let

$$x_1 = .9x_m$$

and

$$x_2 = .95x_m.$$

As we begin searching for a new peak at point  $j$  we let

$$z = x_j$$

Then as we examine subsequent points, we store in  $z$  the largest value found so far. We store the index of the largest point in  $k$ . When a point  $x_i$  is found that has fallen below  $x_1$  (.9 times the largest global value), and when we found a value greater than  $z_2$  (.95 times the largest global value), which we have stored in  $z$ , then we assume that we are descending from a peak at  $z = x_k$ . We store this peak point, and reset  $z$  and  $k$ . Then we look for the next peak. This method is realized in the following C++ function:

```
//c+ peaks finds the peaks of a signal
int peaks(
  dvector& x,
  int& n,
  ivector& k,
  int& np){
  //declaration int peaks(dvector&,int&,ivector&,int&);
  int i;
  int kmxpk;
  double x1;
  double x2;
  double xmx;
  double xmxpk;
  // External Functions:
  double maxd(double,double);
  //c Input:
  //c t    time array
  //c x    signal value array
  //c n    number of elements in the signal sample
  //c Output:
  //c k    array of peaks, indices where peaks occur
  //c np   number of peaks
  //c     find peaks
```

```

//c    get absolute maximum of signal
xmx=x.g(1);
for(i=1;i<=n ;i++){
    xmx=maxd(xmx,x.g(i));
}
x1=.9*xmx;
x2=.95*xmx;
np=0;
kmxpk=1;
xmxpk=x.g(1);
for(i=1;i<=n ;i++){
    if((x.g(i) < x1) && (xmxpk > x2)){
        np = np + 1;
        k.p(np, kmxpk );
        kmxpk=i;
        xmxpk=x.g(i);
    }
    if(x.g(i) > xmxpk){
        xmxpk=x.g(i);
        kmxpk=i;
    }
}
return(0);
}

```

## 13 Finding the Phase and Magnitude of the Fundamental Component of a Sampled Signal

Suppose we have found the period  $T$  of a signal as in the previous section. Then the first component of the signal may be written as

$$c_1 \sin(\omega t + \phi),$$

where

$$\omega = \frac{2\pi}{T},$$

and  $\phi$  is the phase angle. The fourier series for the signal is

$$f(t) = a_0/2 + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t)).$$

Thus (we may assume that  $c_1$  is positive) we have

$$\begin{aligned} c_1 \sin(\omega t + \phi) &= a_1 \cos(\omega t) + b_1 \sin(\omega t) \\ &= \sqrt{a_1^2 + b_1^2} \left[ \frac{a_1}{\sqrt{a_1^2 + b_1^2}} \cos(\omega t) + \frac{b_1}{\sqrt{a_1^2 + b_1^2}} \sin(\omega t) \right] \\ &= \sqrt{a_1^2 + b_1^2} [\sin(\phi) \cos(\omega t) + \cos(\phi) \sin(\omega t)]. \end{aligned}$$

Then the phase  $\phi$  is the argument (or angle of the line from the origin to the point) of  $(\cos(\phi), \sin(\phi)) = (b_1, a_1)$ . That is, in terms of a well known computer function

$$\phi = \text{ATAN2}(a_1, b_1).$$

The magnitude of the first Fourier component is

$$c_1 = \sqrt{a_1^2 + b_1^2}.$$

The fourier coefficients are

$$a_1 = \frac{2}{T} \int_0^T f(t) \cos(\omega t) dt$$

and

$$b_1 = \frac{2}{T} \int_0^T f(t) \sin(\omega t) dt.$$

They may be computed numerically using the trapezoid rule. This is done for the  $b_1$  coefficient in the following C++ function:

```
//c+ sincf sine coefficient of fundamental component of signal
int sincf(
    dvector& f,
    int& n,
    double& p,
    double& b1){
```

```

//declaration int sincf(dvector&,int&,double&,double&);
double dt;
int i;
double omega;
double pi;
double t;
double v;
// External Functions:
// double sin();
//c Input:
//c f    sample signal values evenly spaced on interval [0,p]
//c n    number of sample values
//c p    period of the signal
//c Output:
//c b1   first sine coefficient
//c Method:  trapazoid rule integration of the integral
//c      (2/p) \int{0 to p} f(t) \sin(\omega t) dt
pi=3.14159265358979;
omega=2.*pi/p;
dt=p/(n-1);
b1=0.;
for(i=1;i<=n ;i++){
    t=(i-1)*dt;
    v=f.g(i)*sin(omega*t);
    if((i == 1) || (i == n)){
        v=v/2.;
    }
    b1=b1+v;
}
b1=2.*dt*b1/p;
return(0);
}

```

There is an almost identical function for the cos coefficient.

If we have two signals, we can compute the phase difference between them as the difference of their individual phases.

The technique described here can be used to compute impedance of a

load from voltage and current signals.

The FFT could be used for this calculation.

## 14 Impedance Program

Output of the program **imp.c**:

```
% imp
n= 1024
1 index 146 value          470
2 index 506 value          479
3 index 862 value          414
Period =                   358
nn= 358
Current a1 =                -97.4514
Current b1 =                48.781341
current phase angle =       -63.408784 degrees
current magnitude =         108.97887
Voltage a1 =                -214.00182
Voltage b1 =                -382.58334
Voltage phase angle =       -150.7791 degrees
Voltage magnitude =         438.36833
Impedance phase angle =     -87.370312 degrees
Impedance magnitude =       4.0225075
```

Suppose the sampling rate is  $20mH$ , that is  $20 \times 10^6$  points per second. Then the distance between sample points is

$$\Delta t = \frac{1}{20 \times 10^6} = 50ns.$$

From the program output, the fundamental period in seconds is

$$T = (50)(358) = 17900ns$$

So the fundamental frequency is

$$f = 1/T = 55.8 \times 10^3 H.$$

To compute the actual impedance magnitude in ohms, we need the the current and voltage scale factors. Note that here zero voltage has value 512. The voltage and current signals have been discretized so that they take values between 0 and 1023.

Listing of the program **imp.c**:



```

/* imp.c Impedance from signals from LabWindows data aquisition system*/
/* jim emery 3/21/97 */
#include <stdio.h>
#include <string.h>
#include <stdlib.h>
#include <math.h>
/* #include "matvec.h" */
int da[1024][8];
main(){
    FILE *nf1;
    int n;
    double a[10];
    /*t readfv(FILE*,double*,double*,int*);*/

    nf1=fopen("p.dig","r");
    n=-1;
    while(readr(nf1,a) > 0) {
        n = n + 1;
        da[n][0]=a[0];
        da[n][1]=a[1];
        da[n][2]=a[2];
        da[n][3]=a[3];
        da[n][4]=a[4];
        da[n][5]=a[5];
        da[n][6]=a[6];
        da[n][7]=a[7];
    }
    impedance();
}
int impedance(){
    /*imp.ftn impedance from signals*/
    FILE *out;
    double a[2000];
    double a1;
    double b1;
    double f[2000];
    int i;
    int j;
    int k[100];
    int n;
    int nn;
    int np;
    double omega;
    double phi;
    double pi;
    double s;
    double t[2000];
    double tp;
    double tt;
    double v[2000];
    double c[2000];
    double cm,vm,cphi,vphi;
    double x,y;
    double maxd(double,double);
    int coscf(double*,int,double,double*);

```

```

int sincf(double*,int,double,double*);
int peaks(double*,int,int*,int*);
out=fopen("p.gi","w");
n=1024;
printf("n= %d\n",n);
for(i=0;i<n;i++){
    t[i]=i;
    c[i]=da[i][0];
    v[i]=da[i][5];
}

fprintf(out,"v-1 1 -1 1\n");
fprintf(out,"w0 1024 0 1024\n");
fprintf(out,"c3\n");
for(i=0;i<1024;i++){
    x=t[i];
    y=c[i];
    if(i == 0){
        fprintf(out,"m%15.8g %15.8g\n",x,y);
    }
    else{
        fprintf(out,"d%15.8g %15.8g\n",x,y);
    }
}
fprintf(out,"c4\n");
for(i=0;i<1024;i++){
    x=t[i];
    y=v[i];
    if(i == 0){
        fprintf(out,"m%15.8g %15.8g\n",x,y);
    }
    else{
        fprintf(out,"d%15.8g %15.8g\n",x,y);
    }
}

peaks(c,n,k,&np);
for(i=1;i<=np;i++){
    j=k[i-1];
    printf("%d index %d value %15.8g\n",i,j,v[j-1]);
}
if(np > 1){
    s=0.;
    for(i=1;i<=np-1 ;i++){
        s = s + t[k[i]]-t[k[i-1]];
    }
    tp = s/(np-1);
    printf("Period = %15.8g\n",tp);
}
pi=3.14159265358979;
omega=2.*pi/tp;
nn=tp;
printf("nn= %d \n",nn);
coscf(c,nn,tp,&a1);
printf("Current a1 = %15.8g \n",a1);

```

```

sincf(c,nn,tp,&b1);
printf("Current b1 = %15.8g \n",b1);
cphi = atan2(a1,b1)*180./pi;
printf("current phase angle = %15.8g degrees\n",cphi);
cm=sqrt(a1*a1+b1*b1);
printf("current magnitude = %15.8g \n",cm);
fprintf(out,"c6\n");
for(i=0;i<1024;i++){
    x=t[i];
    y=512. + a1*cos(omega*x) + b1*sin(omega*x) ;
    if(i == 0){
        fprintf(out,"m%15.8g %15.8g\n",x,y);
    }
    else{
        fprintf(out,"d%15.8g %15.8g\n",x,y);
    }
}

coscf(v,nn,tp,&a1);
printf("Voltage a1 = %15.8g \n",a1);
sincf(v,nn,tp,&b1);
printf("Voltage b1 = %15.8g \n",b1);
vphi = atan2(a1,b1)*180./pi;
printf("Voltage phase angle = %15.8g degrees\n",vphi);
vm=sqrt(a1*a1+b1*b1);
printf("Voltage magnitude = %15.8g \n",vm);
fprintf(out,"c7\n");
for(i=0;i<1024;i++){
    x=t[i];
    y=512. + a1*cos(omega*x) + b1*sin(omega*x) ;
    if(i == 0){
        fprintf(out,"m%15.8g %15.8g\n",x,y);
    }
    else{
        fprintf(out,"d%15.8g %15.8g\n",x,y);
    }
}

printf("Impedance phase angle = %15.8g degrees\n",vphi-cphi);
vm=sqrt(a1*a1+b1*b1);
printf("Impedance magnitude = %15.8g \n",vm/cm);

return 0;
}
/*c+ readfv read an array of function pairs*/
int readfv(
FILE* nf,
double* x,
double* y,
int* n){
/**/declaration int readfv(FILE*,dvector&,dvector&,int&);*/
double a[2];
int ntmp;
/* External Functions: */
int readr(FILE*,double*);

```

```

/*
//c input:
//c nf      file number
//c Output:
//c x  array of x values
//c y  array of y values
//c n  number of values
//c
//c Note: uses ireadr, which reads numbers separated
//c by spaces, and returns -1 at end of file.
*/
ntmp=0;
while(readr(nf,a) > 0) {
    ntmp = ntmp + 1;
    x[ntmp-1]=a[0];
    y[ntmp-1]=a[1];
}
*n=ntmp;
return(0);
}
/*c+ readr read row of numbers*/
int readr(FILE *fn,double *a){
/*
//declaration int readr(FILE*,double*);
// input:
// fn file pointer
// output:
// a-array of numbers read
// returned value is:
// -1, end of file
// 0, empty line
// n, n is number of values read
// Remarks:
// separate numbers by blanks.
// to read from keyboard use: n=readr(stdin,a).
// modified for c++, 10/29/96
*/
extern double atof(const char *s);
char b[200],c[25],d[2];
int i,l,nr;
strcpy(c,"");
if(fgets(b,200,fn)==NULL){
    nr=-1;
}
else{
    nr=0;
    /* fgets puts newline character into string, so subtract 1*/
    l=strlen(b)-1;
    d[0]=' ';
    for(i=0;i<l;i++){
        d[0]=b[i];
        if(d[0] != ' ')strncat(c,d,1);
        if ((d[0]==' ') || (i==l-1)){
            if(strlen(c) != 0){
                nr=nr+1;
            }
        }
    }
}
}

```

```

        a[nr-1]=atof(c);
        strcpy(c,"");
    }
}
}
return(nr);
}
/*c+ peaks finds the peaks of a signal*/
int peaks(
double* x,
int n,
int* k,
int* np){
/*declaration int peaks(dvector&,int&,ivector&,int&);*/
int i;
int kmxpk;
double x1;
double x2;
double xmx;
double xmxpk;
int nptmp;
/*// External Functions:*/
double maxd(double,double);
/*
//c Input:
//c t    time array
//c x    signal value array
//c n    number of elements in the signal sample
//c Output:
//c k    array of peaks, indices where peaks occur
//c np   number of peaks
//c      find peaks
//c      get absolute maximum of signal
*/
xmx=x[0];
for(i=1;i<=n ;i++){
    xmx=maxd(xmx,x[i-1]);
}
x1=.9*xmx;
x2=.95*xmx;
nptmp=0;
kmxpk=1;
xmxpk=x[0];
for(i=1;i<=n ;i++){
    if((x[i-1] < x1) && (xmxpk > x2)){
        nptmp = nptmp + 1;
        k[nptmp-1] = kmxpk;
        kmxpk=i;
        xmxpk=x[i-1];
    }
    if(x[i-1] > xmxpk){
        xmxpk=x[i-1];
        kmxpk=i;
    }
}

```

```

}
*np=nptmp;
return(0);
}
/*c+ maxd maximum of two numbers (doubles)*/
double maxd(double x,double y){
/* //declaration double maxd(double,double);*/
if( x < y ){
return(y);
}
else{
return(x);
}
}
}
/*c+ coscf cosine coefficient*/
int coscf(
double* f,
int n,
double p,
double* a1){
/* //declaration int coscf(dvector&,int&,double&,double&);*/
double dt;
int i;
double omega;
double pi;
double t;
double v;
double alp;
/*
// External Functions:
// double cos();
//c Input:
//c f    function values evenly spaced on interval [0,p]
//c n    number of function values
//c p    period of the function
//c Output:
//c a1   first cosine coefficient
//c     trapazoid rule integration
//c     (1/2p) \int f(t)*cos(\omega*t) dt
*/
pi=3.14159265358979;
omega=2.*pi/p;
dt=p/(n-1);
alp=0.;
for(i=1;i<=n ;i++){
t=(i-1)*dt;
v=f[i-1]*cos(omega*t);
if((i == 1) || (i == n)){
v=v/2.;
}
alp=alp+v;
}
*a1=2.*dt*alp/p;
return(0);
}

```

```

/*c+ sincf sine coefficient*/
int sincf(
double* f,
int n,
double p,
double* b1){
/* //declaration int sincf(dvector&,int&,double&,double&);*/
double dt;
int i;
double omega;
double pi;
double t;
double v;
double b1p;
/*
// External Functions:
// double sin();
//c Input:
//c f    function values evenly spaced on interval [0,p]
//c n    number of function values
//c p    period of the function
//c Output:
//c b1   first sine coefficient
//c      trapazoid rule integration
//c      (1/2p) \int f(t)*cos(\omega*t) dt
*/
pi=3.14159265358979;
omega=2.*pi/p;
dt=p/(n-1);
b1p=0.;
for(i=1;i<n ;i++){
    t=(i-1)*dt;
    v=f[i-1]*sin(omega*t);
    if((i == 1) || (i == n)){
        v=v/2.;
    }
    b1p=b1p+v;
}
*b1=2.*dt*b1p/p;
return(0);
}

```

## 15 Applications

Fourier Analysis has been applied in a huge number of areas in science. Some of these areas are: Optics and Diffraction Theory, Signal Processing, X-ray Crystallography and Molecular Structure determination, Approximation Theory, Partial Differential Equations, Tomography, Several Areas in Electrical Engineering, Quantum mechanics, The Theory of Heat Conduc-

tion (where it originated), and so on.

## 16 The Fourier Method of Tomography

A parallel beam of radiation passes through a body and is partially scattered. The intensity of the unscattered radiation is measured on the opposite side of the body. Let a quantity of radiation  $dq$  be scattered in distance  $d\eta$ . This will be proportional to distance and to the amount of original radiation  $q$ . Thus the change in radiation intensity is

$$dq = -\lambda q d\eta$$

where  $\lambda$  is a scattering constant. Then

$$\frac{dq}{d\eta} = -\lambda q.$$

Hence

$$q = q_0 e^{-\lambda \eta}.$$

Now suppose the attenuation function varies throughout the body. Let  $\lambda = f(x, y, z)$ . Then

$$q = q_0 e^{-\int f d\eta}.$$

Then the negative logarithm of the intensity ratios is the integral of  $f$  in the direction of the radiation beam:

$$-\ln \frac{q}{q_0} = \int f d\eta.$$

### The Projection Theorem.

Let  $R$  be an orthogonal transformation. Let

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = R \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Let  $f(x_1, x_2, x_3)$  be the attenuation function. The Fourier transform of  $f$  is

$$Ff(k_1, k_2, k_3) = \frac{1}{(2\pi)^{(3/2)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2, x_3) e^{ix \cdot k} dx_1 dx_2 dx_3$$

We shall make a change of variable

$$Ff(k_1, k_2, k_3) = \frac{1}{(2\pi)^{(3/2)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(R^T x') e^{iR^T x' \cdot k} |J| dx'_1 dx'_2 dx'_3,$$

where  $|J|$  is the determinant of the Jacobian of the transformation. Here, because  $R$  is orthogonal,  $J = 1$ . Let  $k' = R^T k$ . Define

$$g(x') = f(R^T x').$$

Then the transform becomes

$$Ff(k) = \frac{1}{(2\pi)^{(3/2)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x') e^{ix' \cdot k'} dx'_1 dx'_2 dx'_3.$$

The measured intensity at the plane perpendicular to the  $x'_3$  axis is

$$p(x'_1, x'_2) = \int_{-\infty}^{\infty} g(x'_1, x'_2, x'_3) dx'_3$$

Then at a  $k$  vector, with a rotation  $R_k$ , where  $k'_3 = 0$ , we have

$$Ff(k) = \frac{1}{(2\pi)^{(3/2)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x'_1, x'_2) e^{i(k'_1 x'_1 + k'_2 x'_2)} dx'_1 dx'_2 = F_2 p(k'_1, k'_2).$$

This is the two dimensional Fourier transform of the projected intensity. Now given any point  $k$  we can find a plane passing through it and we can find a corresponding rotation  $R_k$ , so that  $k'_3 = 0$ . Therefore we can compute the three dimensional transform by computing the two dimensional transform of the projected intensities. Then we can compute  $f$  by inverting the transform. However, in practice we will have only the transform at a set of finite points, so we must interpolate. The Fourier transform will be computed approximately as the finite Fourier transform, which uses only a finite set of points. However, interpolation is still needed because the proper points for computing the inverse transform will not in general correspond to the measured points. In practice a somewhat different method is used that does the interpolation before the projection intensity transform is calculated. This is the method of back projection.

## 17 Wavelets and Fourier Series

The theory of wavelets, concerns a set of functions with local support, that form, an orthogonal bases for a function space, and thus give a Fourier expansion of a function in terms of wavelets. In one sense wavelets are portions of infinite waves, or wave packets. Some classes of wavelets are derived as a scaled, and translated mother wavelet.

Wavelets have applications in signal processing, image processing, and data compression. See Meyer for a good introduction to wavelet theory, application, and history. See **Numerical Recipes in Fortran** , 2nd. edition, for wavelet subroutines.

A multiresolution analysis with wavelets, allows functions to be approximated to various scales.

## 18 Fourier Transforms of Tempered Distributions

See Yosida **Functional Analysis**.

## 19 A Generalization of Fourier Analysis: Abstract Harmonic Analysis

In the abstract formulation, the Fourier series, and the Fourier transform may be treated similarly. See the references.

## 20 Spaces of Complete Orthogonal Functions

Fourier analysis can be extended to families of orthogonal functions defined on  $L_2$  spaces, see Horvath. See Widom for families of complete orthogonal functions generated by Sturm-Liouville boundary value problems.

## 21 Fourier Optics

Fourier transforms can be computed optically, see Goodman as well as other more modern books on Fourier optics.

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