Joseph Fourier

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Contents

1 Biography

- Joseph Fourier (born March 21, 1768, died May 16, 1830).
- Son of a tailor.
- Educated by monks in a military school.

- Spent much time by himself studying mathematics.
- Rejected by the army for the artillery, not being of noble birth.
- Entered the Benedictine order, pursuing the only other opportunity open to him for an intellectual life(1787-1789).
- Left the monastery at the outbreak of the revolution.
- Appointed by the monks to the principal chair of mathematics at their school in Auxerre.
- Embraced the cause of the French revolution at Auxerre
- Member of the **Citizens Committee of Surveillance**, but exercised such moderation that he was in danger himself of the Terror.
- Chosen in 1794 to teach in the new Normal school.
- Was noticed by Napoleon, when Napoleon attended lectures at the Polytechnic school.
- Had A leading role in Napoleon's scientific expedition to Egypt in 1798.
- As Secretary of the institute of Cairo he instigated the **Description of Egypt** report.
- He returned to France in 1802 and managed various public improvements.
- Submitted a two part memoir for the prize competition of 1812 on the "**Mathematical Theory of the Laws of the Propagation of Heat...**"
- Judges, Laplace, Lagrange, and Legendre, awarded him the price.
- The first part of his memoir was published in 1822 as **Thorie Analytique de la Chaleur**, which contains his theory of trigonometric series, later known as Fourier series.
- On Napoleon's return from Elba in 1814, he had a minor conflict with Napoleon and lost his political status. He was now penniless.

- Later in life he regained somewhat his former status, and scientific reputation, in France.
- Fourier died May 16, 1830.

2 The Heat Equation

Heat flowing in a material has a direction that is perpendicular to the contour lines of equal temperature. Let $\mathbf{F}$ be the heat flux (flow) vector that has the direction of the heat flow, and has the property that if ds is an element of surface area, and $\mathbf{n}$ is a unit surface normal, then the rate of heat flow through the element is the dot product of $\mathbf{F}$ with $\mathbf{n}ds$. That is, the rate of heat flow is

$$\mathbf{F} \cdot \mathbf{n} ds.$$

Let $\mathbf{A}$ be a bounded region with boundary $\partial\mathbf{A}$. The rate of heat flowing into the region is obtained by integrating over the boundary surface

$$\frac{dq}{dt} = -\int_{\partial\mathbf{A}} \mathbf{F} \cdot \mathbf{n} ds$$

Using the divergence theorem this becomes

$$\frac{dq}{dt} = -\int_{\mathbf{A}} \nabla \cdot \mathbf{F} dv,$$

where the divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

Note that the minus sign occurs because $\mathbf{n}$, the normal to the surface, is an outward normal. The heat conduction vector $\mathbf{F}$ is proportional to the gradient of the temperature, thus

$$\mathbf{F} = -k\nabla u,$$

where u is the temperature and k is the conductivity. Then we have

$$\frac{dq}{dt} = \int_{\mathbf{A}} \nabla \cdot (k\nabla u) dv.$$

The rate of change of heat in the body is

$$\frac{dq}{dt} = \int_{\mathbf{A}} c\rho \frac{\partial u}{\partial t} dv.$$

Equating the two expressions for the rate of change of the heat energy in the body, we have

$$\int_{\mathbf{A}} \nabla \cdot (k\nabla u) dv = \int_{\mathbf{A}} c\rho \frac{\partial u}{\partial t} dv.$$

Equating the integrands we get the heat equation:

$$\nabla \cdot (k\nabla u) = c\rho \frac{\partial u}{\partial t}.$$

If we assume that the conductivity is constant, the heat equation becomes

$$\nabla^2 u = K \frac{\partial u}{\partial t},$$

where

$$K = \frac{c\rho}{k},$$

and the Laplacian is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

3 Fourier Series via the Heat Equation

Suppose we have a thin rod of length 1, that at an initial time $t = 0$ has a temperature distribution $h(x)$, where x is the distance along the length of the rod. Suppose the rod is insulated except at its ends. Suppose at time $t = 0$ heat sources at temperature $u = 0$ are placed in contact with the ends. We are to calculate the temperature in the rod over time. The heat equation for this one dimensional problem becomes

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},$$

where we have taken $K = 1$ for simplicity. We solve this problem using the technique of separation of variables. We assume that the solution is given as a product of two functions

$$u(x,t) = f(x)g(t).$$

Then the heat equation becomes

$$f''(x)g(t) = f(x)g'(t)$$

or

$$\frac{f''(x)}{f(x)} = \frac{g'(t)}{g(t)}.$$

But x and t are independent variables, so if the two sides of this equation were not constant then we could say vary x, but not t, and make the two sides not equal. It follows that the two sides are equal to a constant, which we write as $-\lambda$. Then

$$\frac{f''(x)}{f(x)} = \frac{g'(t)}{g(t)} = -\lambda,$$

and we get two ordinary differential equations

$$f''(x) + \lambda f(x) = 0$$

and

$$g'(t) + \lambda g(t) = 0.$$

The number λ is called an eigenvalue, and a corresponding solution is called an eigenfunction. Let $\omega^2 = \lambda$

The solutions to the first equation are

$$f(x) = a\cos(\omega t) + b\sin(\omega t),$$

where a and b are constants. A solution to the second equation is

$$g(t) = exp(-\omega^2 t).$$

On the end $x = 0$, the boundary condition on u gives

$$0 = u(0,t) = f(0)g(t) = (a\cos(\omega 0) + b\sin(\omega 0))exp(-\omega^2 t) = aexp(\omega^2 t).$$

It follows that $a = 0$, because the exponential function is never zero. On the end $x = 1$, the boundary condition on u gives

$$0 = u(1,t) = f(0)g(t) = b\sin(\omega 1))exp(-\omega^2 t),$$

which forces

$$\sin(\omega) = 0,$$

and implies that $\omega = n\pi$, where n is an integer. So for each $n = 1, 2, 3, ...$ we have a solution

$$u_n(x,t) = b_n \sin(n\pi x))exp(-n^2\pi^2 t).$$

But none of these solutions necessarily gives the initial temperature distribution $h(x)$ on the rod. However, the sum of these solutions is also a solution, so let us write

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x))exp(-n^2\pi^2 t).$$

If we can choose the constants $b_n, n = 1, 2, 3, ...$ so that the initial temperature distribution is matched then this will be a solution to our problem. So we require

$$h(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)).$$

Fourier in his book **The Analytical Theory of Heat** introduced the trick of taking the inner product of $\sin(m\pi x)$ with the series in order to compute b_m. It turns out that the functions

$$\phi_n(x) = \sin(n\pi x), n = 1, 2, 3, 4, ...$$

are orthogonal. That is

$$(\phi_n, \phi_m) = \int_0^1 \phi_n(x)\phi_m(x)dx = 0,$$

if $n \neq m$. So to compute b_m we take

$$(h, \phi_m) = \int_0^1 h(x)\phi_m(x)dx$$

$$= \int_0^1 h(x)\sin(m\pi x)dx$$

$$= \sum_{n=1}^{\infty} b_n \int_0^1 \sin(n\pi x)\sin(m\pi x)dx.$$

$$= b_m \int_0^1 \sin^2(m\pi x)dx.$$

This determines b_m. Fourier, in his book, from boundary value problems similar to the one above, made the claim that every function defined on an interval, is given as an infinite sum of sines and cosines. This is essentially true, although it depends on the continuity properties of the function. In honor of Fourier, such trigonometric sums are now called Fourier series.

4 Fourier Series

Given a function $f(t)$ defined on an interval $[0, T]$, it can be expanded in a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\omega t) + b_n \sin(n\omega t)),$$

where

$$\omega = \frac{2\pi}{T},$$

and where

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt, n = 0, 1, 2, 3, ...$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt, n = 1, 2, 3, ...$$

The series converges, depending on the properties of the function f. The term

$$a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

is called the nth harmonic. In the case of a sound wave, the harmonics determine the character of the sound and thus the difference in sound between say a saxophone and a violin, and also explains why an electric guitar sounds like a toothache, and a banjo sounds like several empty beer cans falling down the steps.

The Fourier series also has an exponential form. So expanding the exponential, sine, and cosine functions in a Taylor power series, we have for a complex variable z

$$\exp(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + ...$$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...$$

$$\sin(z) = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{x^7}{7!} + ...$$

Then

$$\sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i},$$

and

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2}.$$

So the fourier series can be written as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp(in\omega t),$$

where

$$c_n = \frac{1}{T}\int_0^T f(t)\exp(-in\omega t)dt.$$

We find that

$$c_n = \frac{a_n - ib_n}{2}, n > 0$$

$$c_n = \frac{a_n + ib_n}{2}, n < 0$$

$$c_0 = \frac{a_0}{2}.$$

Note that if $m \neq -n$, then

$$\int_0^T \exp(in\omega t)\exp(im\omega t)dt$$

$$= \int_0^T \exp(i(n+m)\omega t)dt$$

$$= \left[\frac{1}{i(n+m)\omega}\exp(i(n+m)\omega t)\right]_0^T$$

$$= \frac{1}{i(n+m)\omega}\left[\exp(i(n+m)\omega T) - 1\right]$$

$$= 0,$$

because

$$\exp(i(n+m)\omega T) = \cos((n+m)2\pi) + i\sin((n+m)2\pi) = 1.$$

And if $m = -n$ then

$$\int_0^T \exp(in\omega t)\exp(im\omega t)dt$$

$$= \int_0^T dt = T$$

5 The Fourier Transform

A periodic function f of period T may be expanded in a Fourier series as outlined above. If the period goes to infinity we can give an argument, intuitive, but not really rigorous, that the Fourier series goes to what is called the Fourier transform

The Fourier transform of the function f is defined as (Richard R. Goldberg, **The Fourier Transform**, Cambridge, 1965.)

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt.$$

By the Fourier integral theorem

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{-i\omega t}d\omega.$$

The role of the Fourier series coefficients $\{c_n\}_{-\infty}^{\infty}$ are taken over by the function $\hat{f}(\omega)$. We are said to map the time domain function $f(t)$ to the frequency domain $\hat{f}(\omega)$. We can think of each c_n being a sample point $\hat{f}(\omega_n)$, for a periodic function of very large period T.

6 The Discrete Fourier Transform

Now in the practical use of Fourier series we are not concerned with all of the harmonics in a Fourier expansion, because the frequencies become too high to be of any consequence. So we replace the infinite sum with a finite sum. Say

$$f(t) = \sum_{n=-N}^{N} c_n \exp(in\omega t).$$

We have a transformation of a function $f(t)$ defined in the time domain to a finite set of coefficients of the harmonics

$$f \rightarrow \{c_{-N}, c_{-N+1}, ...c_0, ..., c_{N-1}, c_N\},$$

in the frequency domain. The next step in practical calculation is to replace the function $f(x)$ by a finite set of samples of the function. So in this way our mapping becomes a mapping of discrete points to a set of discrete points. This is called the Discrete Fourier Transform or DFT. This is what is calculated as the Fourier Transform in software libraries. In digital signal processing we calculate the frequency content of the signal or wave using the DFT. In digital image processing we use a two dimensional DFT to compute the frequency content of an image. An algorithm was discovered, or perhaps rediscovered, 30 years ago or so ago, that made these calculations practical on a computer. This algorithm is called the Fast Fourier Transform or FFT. This made possible Computer Aided Tomography, fast determination of molecular structure using X-ray crystallography and so on. The use of the FFT is ubiquitous, and it is hard to find areas in which it is not important.

One should mention that the classical Fourier Transform is a mapping from a function $f(t)$ in the time domain defined on an infinite interval to a frequency function $g(\omega)$ also defined on an infinite interval.

7 Abstract Harmonic Analysis

There is a field in pure mathematics related to problems such as these that is called **Abstract Harmonic Analysis**. In this abstraction, Fourier series, and Fourier transforms, have the same abstract structure.

8 Higher Dimensional Transforms

Higher dimensional transforms are defined in a similar way to the one dimensional transforms, so the role of ωt, or kx in the spacial case, is taken over by the inner product of two vectors $\mathbf{k} \cdot \mathbf{x}$. For example, the two dimensional transform plays a big role in Digital Image Processing.

9 Bibliography and References

[1]Bachman, George, **Elements of Abstract Harmonic Analysis**, Academic Press, 1964.
[2]Barrow, Gordon M, **Physical Chemistry For The Life Sciences**, McGraw-Hill 1974, (2nd Ed. 1984), Johnson County, 541.3 B279, X-Ray Crystal Structure, Proteins, Fourier Analysis, Van Der Wals.
[3]Briggs William L, Henson Van Emden, **The DFT**, (Discrete Fourier Transform), SIAM, 1995.
[4]Chandrakumar Narayanan, **Modern Techniques In High Resolution FT-NMR**, 1987, Linda Hall, Fourier Optics, Nuclear Magnetic Resonance.
[5]Crowther R A, **Procedures For Three-dimensional Reconstruction Of Spherical Viruses By Fourier Synthesis From Electron Micrographs**, Proceedings Of The Royal Society Of London,, B 261, pp231-261.
[6]Elliott Douglas, Rao K Ramamohan, **Fast Transforms, Algorithms, Analysis, Applications**.
[7]Emery James D, **Finite Fourier Analysis**.
[8]Emery, James D, **Least Squares Approximation**, (lsq.tex).
[9]Emery, James D, **Tomography**, (tomog.tex).
[10]Fourier Joseph, **The Theory Of Heat**, Britanica Great Books.
[11]Fourier Joseph, **Theorie Analytique De La Chaleur**, 1822.
[12]Goldberg R R, **Fourier Transforms**, Cambridge, 1965.
[13]Goldberg R R, **Introduction To Real Analysis**.
[14]Goodman Joseph W, **Introduction To Fourier Optics**, Linda 1968 287 Pages.
[15]Horvath John, **Topological Vector Spaces**.
[16]Jardetzky O, Roberts G, **NMR In Molecular Biology**, Linda Hall,, QP519.9.N83j37, 1987, Outline Of History of Nuclear Magnetic Resonance, Fourier Transform,
[16]Katznelson Yitzhak, **An Introduction to Harmonic Analysis**, 1976, Dover.
[17]Korner T W, **Fourier Analysis**, Cambridge, 1988.
[18]Lighthill M J, **Fourier Analysis and Generalized Functions**, Cam-

bridge, 1964.
[19]Loomis L H, **An Introduction to Abstract Harmonic Analysis**, Dover.
[19b]Meyer Yeves **Wavelets: Algorithms and Applications**, SIAM Press, Philadelphi, 1993. Dover.
[20]Papoulis Athanasios, **The Fourier Integral and Its Applications**, McGraw-Hill, 1962.
[20b]Press, Flannery, Teukolsky, Vetterling **Numerical Recipes in C**, Cambridge U. Press, 2nd edition, 1995.
[21]Ramachandran G N, Srinivasan R, **Fourier Methods In Crystallography**, Linda Hall, Qd945.r35, 1970.
[22]Rudin W, **Fourier Transforms On Groups**.
[23]Sederberg Thomas, **Scheduled Fourier Model Morphing**, Siggraph Proceedings, 1992.
[24]Stark Henry, **Applications Of Optical Fourier Transformations**, Ta1632 .a68 1982 Linda.
[25]Stout George H, Jenson Lyle H, **X-Ray Structure Determination**, Linda Hall, Qd945.s8, 1989, Fast Fourier Transform, Hauptman And Karl Technique.
[26]Strang Gilbert, **Wavelet Transform Versus Fourier Transform**, Bulletin Ams V28 N2 April 93, (also Books On Wavelets Reviewed).
[27]Thompson William J, **Computing For Scientists And Engineers**, C Programs, Complex Variables, Much On Fourier Analysis, 1992,, JOCO.
[28]Truesdell C, **The Tragicomical History of Thermodynamics**, Springer-Verlag, 1980, Critique of Fourier's Work on the Theory of Heat.
[29]Walther Adriaan, **The Ray And Wave Theory Of Lenses**, 1995, Fourier Optics, Linda.
[30]Weiner Norbert, **The Fourier Integral and Certain of Its Applications**, Dover.
[31]Wickerhauser Victor Mladen **Adapted Wavelet Analysis from Theory to Software**, A K Peters, 1994
[32]Widder David, **Laplace Transforms**.
[33]Widom Harold, **Lectures on Integral Equations**, D Van Nostrand, 1969, Eigenvectors of Sturm-Liouville Boundary Value Problems as Complete Orthogonal Functions on $L_2[a, b]$.
[34]Wilson Raymond, **Fourier Series And Optical Transforms Techniques In**, 1995 Linda Fourier Optics.

[35]Yosida Kasaku, **Functional Analysis**, Springer-Verlag, Fourier Transforms of Tempered Distributions.
[36]Zemanian A H, **Distribution Theory and Transform Analysis**.
[37]Zigmund Antoni, **Trigonometrical Series**, Dover, 1955.
[38]Fourier Joseph, **The Analytic Theory of Heat**, Translated, with notes, by Alexander Freeman, 1945
[39]Fourier, Jean Baptiste Joseph, baron, 1768-1830. **Thorie analytique de la chaleur.** 1878, LHL QC254 .F59
[40]Fourier, Jean Baptiste Joseph. **Theory of Heat.** Excerpt of the famous book of Fourier in volume 45 of The Great Books of the Western World, Encyclopedia Britannica, Inc, 1952.
[41]MacCluer, Charles R. **Boundary Value Problems and Fourier Expansions.** Dover, Revised edition, 2004.

```
1. The analytical theory of heat / by Joseph Fourier. Translated, with notes,
by Alexander Freeman ... edited for the syndics of the University Press.

Fourier, Jean Baptiste Joseph, baron, 1768-1830. Thorie analytique de la chaleur.

1878.

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2. The analytical theory of heat / Translated, with notes, by Alexander Freeman.

Fourier, Jean Baptiste Joseph, baron, 1768-1830. Thorie analytique de la chaleur.

[1945]

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