Green's Functions

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1 Introduction

From the Wikipedia article on George Green: *George Green (14 July 1793 31 May 1841) was a British mathematical physicist, who wrote* **An Essay on the Application of Mathematical Analysis to the Theories of**

Electricity and Magnetism *(Green, 1828). The essay introduced several important concepts, among them a theorem similar to the modern Green's theorem, the idea of potential functions as currently used in physics, and the concept of what are now called Green's functions. George Green was the first person to create a mathematical theory of electricity and magnetism and his theory formed the foundation for the work of other scientists such as James Clerk Maxwell.*

Green did this in spite of having no formal education, and of not having any known teacher in these areas. So much for the putative benefits of formal education.

2 Formal Definition of a Green's Function

If L is a linear differential operator, Green's function is a solution of

$$LG(x, s) = \delta(x - s),$$

where δ is the delta function, which is zero everywhere except at the origin where it is a magical "infinity" so that its integral equals one.

We use x here as a general point in space, and not necessarily an x coordinate. Suppose we wish to find a solution to the equation

$$Lu(x) = f(x).$$

We have

$$\int LG(x, s)f(s)ds = \int \delta(x - s)f(s)ds = f(x).$$

We can remove L from under the integral sign, because L involves x only, and we are integrating with respect to s. Then

$$L \int G(x, s)f(s)ds = f(x).$$

If we let

$$u(x) = \int G(x, s)f(s)ds,$$

then u is a solution to the problem

$$Lu(x) = f(x).$$

An example of a linear operator is the Laplacian operator

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

3 Sturm-Liouville Theory

Consider the operator

$$L = \frac{d}{dx}(p(x)\frac{d}{dx}) + q(x),$$

and a boundary condition operator D given by

$$(Du)_1 = \alpha_1 u'(0) + \beta_1 u(0)$$

$$(Du)_2 = \alpha_2 u'(\ell) + \beta_2 u(\ell).$$

Let $f(x)$ be a continuous function in the interval $[0, \ell]$. Then there is a unique solution to the above boundary value problem

$$Lu = f,$$

$$Du = 0,$$

given by

$$u(x) = \int_0^\ell f(s)G(x,s)ds,$$

where $G(x, s)$ is a greens function satisfying the following conditions:

(1) $G(x, s)$ is continuous in x and s, for x not equal to s.
(2) $LG(x, s)) = 0$, for s not zero,
(3) $DG(x, s) = 0$.
(4) There is a jump in the derivative of G at (s, s) given by $1/p(s)$.
(5) $G(x, s) = G(s, x)$.

This is related to the Fredholm theory of integral equations.

4 Laplaces Equation

From the divergence theorem one can obtain Green's theorem

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)d^3x = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot dS$$

The Green's function here has the defining property

$$LG(x, x') = \nabla^2 G(x, x') = \delta(x - x').$$

Let

$$\psi = G$$

Then Green's theorem becomes

$$\int_V (\phi(x')\delta(x-x') - G(x, x')\nabla^2\phi(x'))d^3x' = \int_S (\phi(x')\nabla' G(x, x') - G(x, x')\nabla'\phi(x')) \cdot dS'$$

or

$$\phi(x) = \int_V G(x, x')\nabla^2\phi(x'))d^3x' + \int_S (\phi(x')\nabla' G(x, x') - G(x, x')\nabla'\phi(x')) \cdot dS'$$

$$\phi(x) = \int_V G(x, x')\rho(x'))d^3x' + \int_S (\phi(x')\nabla' G(x, x') - G(x, x')\nabla'\phi(x')) \cdot dS'$$

Think of $\phi(x)$ as electric potential, and $\rho(x)$ as charge density, and

$$\nabla'\phi(x')) \cdot dS'$$

as the normal component of the electric field. If there is no boundary and the region is all of three space this equation becomes

$$\phi(x) = \int_V G(x, x')\rho(x'))d^3x'$$

And a Green's function is

$$G(x, x') = \frac{1}{|x - x'|}$$

Indeed the Laplacian of this is zero for x not equal to x'. So for example at $x' = (0, 0, 0)$

$$G(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}},$$

so that the Laplacian of G is

$$\nabla^2 G = -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2) = 0$$

And we get the usual expression for the potential of a charge distribution at x

$$\phi(x) = \int_V \frac{\rho(x')}{|x-x'|} d^3x'$$

George Green (1793 to 1841), having essentially no formal education published these ideas in 1828 in a work titled **On the Application of Mathematical Analysis to the Theories of Electricity and Magnetism**.

5 Generalized Functions and Distributions

Above we have used the delta function $\delta(x)$ which is zero everywhere except at zero, where it is infinite so that its integral is 1. Now this makes no sense in the usual definitions of analysis and of the integral. So we must introduce distributions, also called generalized functions.

A distribution is a linear operator L defined on a space of test functions ϕ which are smooth and have compact support. This means that outside of a bounded finite region they are zero. So an integral operator with kernel f defines such an operator.

$$(T_f, \phi) = \int f(x)\phi(x)dx.$$

Here T_f is a linear operator operating on a test function ϕ. Now consider the linear operator corresponding to the derivative f' of f.

$$(T_{f'}, \phi) = \int f'(x)\phi(x)dx.$$

Applying integration by parts, we can write this as

$$(T_{f'}, \phi) = \int f'(x)\phi(x)dx = \int f(x)\phi'(x)dx = (T_f, \phi')$$

This is our general definition of the derivative of any distribution whether it is defined by as an integral or not.

So for example the delta function is defined by

$$(T_\delta, \phi) = \phi(0).$$

This is certainly a linear operator.

6 Test Functions

For the concept of distributions to make rigorous mathematical sense it is necessary to specify exactly what the test functions are and the topology on which they live.

7 Fundamental Solutions

The **Fundamental Solution**, is a modern development related to the Green's function method used in the modern theories of partial differential equations and boundary value problems.

8 Integral Equations

The theories of the Fredholm and the Volterra integral equations are related to Green's functions theories. Differential equations can often be transformed into integral equations.

9 Fredholm Theory

The Fredholm theory gives a condition, called the Fredholm alternative, which relates to the possible solutions of an integral equation.

10 Bibliography

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[2]Weinberger H. F. , **A First Course in Partial Differential Equations**, Blaisdell, 1965.
[3] Friedman Avner **Partial Differential Equations** Holt, Rinehart, and Winston, 1969. (This is an advanced book which introduces the **Fundamental Solution**, which is a modern development related to the Green's function method).
[4] Widom Harold **Lectures on Integral Equations** 1969, Van Nostrand.

(This book develops theories of the Fredholm and Volterra integral equations using techniques of Banach and Hilbert space. Chapter 5 develops the technique of finding a Green's function for the Sturm-Lioville differential equation using the space of orthogonal eigenfunctions of the operator.)
[5] Treves Francois **Topological Vector Spaces, Distributions and Kernels** Academic Press, 1967.
[6] Lighthill M. J. **An Introduction to Fourier Analysis and Generalized Functions** Cambridge University Press, 1964. (This book introduces generalized functions as limits of ordinary functions.)