

# Infinity

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## 1 Cardinal Numbers

Consider the set

$$A = \{a, b, c, d, e, f, g, h\}.$$

The number of elements in a set is called the cardinal number of the set. Any two sets  $A$  and  $B$  have the same cardinal number if there is a one to

one correspondence between the elements of the two sets. This finite set  $A$  above, has cardinal number  $n = 8$ , because it contains 8 elements. The set of all subsets of a set  $A$  is called the power set of  $A$  and is written as  $\mathcal{P}(A)$ . For the finite set  $A$ , the set of all subsets looks like this

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{a, b\}, \dots, \{a, b, c, d, e, f, g, h\}\},$$

where  $\emptyset$  is the empty set. The cardinal number of this set is  $2^8 = 256$ . In fact if a finite set  $A$  has cardinal number  $n$ , then the power set of  $A$ ,  $\mathcal{P}(A)$  has cardinal number  $2^n$ . To see this, let

$$A = \{a_1, a_2, \dots, a_n\}.$$

Let  $k$  be any nonnegative integer less than  $2^n$ . In binary form this number may be represented with  $n$  binary digits, each digit being a 0 or a 1. For example the number  $k = 5$  would be written as

$$0000\dots101 = 0(2^{n-1}) + \dots + 0(2^3) + 1(2^2) + 0(2^1) + 1(2^0).$$

Then such a number  $k$  would map to the subset consisting of the elements of  $A$  that match up with the location of the 1's in the binary representation of  $A$ . Hence for example, if  $k = 5$ , then  $k \rightarrow \{a_1, a_3\}$ . So we see that there is a one to one, onto, mapping of the integers  $0 \leq k < 2^n$  to the subsets of  $A$ . There are  $2^n$  such integers and so  $2^n$  subsets.

The cardinal number of the integers,  $\mathbf{Z}$ , is written as  $\aleph_0$ . Aleph is the first letter of the Hebrew alphabet. This is the smallest infinite cardinal number. Infinite cardinal numbers are called transfinite numbers. The natural numbers are the nonnegative integers. They also have cardinal number  $\aleph_0$ , for we can create the following one to one correspondence from the natural numbers to the integers

$$\{(0, 0), (1, -1), (2, 1), (3, -2), (4, 2), (5, -3), \dots\}.$$

Similarly  $\aleph_0$  is the cardinal number of the positive integers. Sets that have cardinal number  $\aleph_0$  are called countable sets because the elements of the set can be labelled with the counting numbers

$$\{1, 2, 3, 4, 5, 6, \dots\}$$

Notice that there is a one to one correspondence between the positive integers and the even integers. Hence the positive integers are equivalent to a proper subset of themselves. This is characteristic of a transfinite cardinal number. The positive rational numbers are the fractions, each consisting of a positive integer numerator and a positive integer denominator. So let a rational number  $m/n$  be given. We may locate this number in a matrix in row  $m$  and column  $n$ . We can count all of these fractions by repeatedly moving up the matrix diagonals, counting only those fractions that are in lowest terms. It follows that the rationals are also countable. So one might ask, "Are there uncountable numbers?"

Consider the real numbers between zero and 1, assuming that for now that each of these is a number represented by an infinite sequence of decimal digits. These numbers may end in an infinite sequence of zero's. Assume that these numbers are countable. Then the first one may be written as the first row of an infinite matrix. Then the second and so on. Now we claim there is a number that does not occur in this list. For construct a number  $x$  as follows, let the first digit be a digit that is different from the first digit of the first real number in the first row of the matrix. Let the second digit be a digit different from the second digit of the second number in the second row of the matrix. Continue in this way, making the  $n$ th digit of the number  $x$ , different from the  $n$ th digit of the number in the  $n$ th row of the matrix. It follows that the number we have constructed is different from each of the numbers represented in the rows of the matrix. Hence every number has not been counted and it follows that these real numbers are not countable. Hence the cardinal number of the real numbers is greater than  $\aleph_0$ . This number is usually denoted by  $\mathcal{C}$ , and is called the cardinal number of the continuum.

One can show that if  $A$  is a set then the cardinality of the power set  $\mathcal{P}(A)$  is greater than the cardinality of  $A$ . Thus for example

$$\text{Card}(\mathcal{P}(\mathcal{C})) > \text{Card}(\mathcal{C}).$$

Thus there are many orders of infinities.

**Proposition**

$$\text{Card}(\mathcal{P}(\mathcal{Z})) = \mathcal{C}.$$

**Proof.** Roughly, the set of subsets of  $\mathcal{Z}$  is the set of maps of  $\mathcal{Z}$  to the set

$\{0, 1\}$ . Each real number  $x$  in the interval  $(0, 1]$  has the representation

$$x = \sum_{i=1}^{\infty} \frac{c(i)}{2^i},$$

where  $c$  is one of the maps. See Dugundji p 48 for details.

There are various set theory paradoxes. The most famous is the Russell paradox after Bertrand Russell, the philosopher and mathematician.

**Russell Paradox** *Let set  $A$  be the set of all sets that are not members of themselves. So for example the Set  $\{1, 2, 3\}$  is not a member of itself, so is in  $A$ . Thus  $A$  is not empty.  $A$  must either be a member of itself or not a member of itself. If  $A$  is a member of itself, then it is not a member of  $A$ , which is itself. On the other hand if it is not a member of itself, then it is a member of  $A$ , which is itself. These are both contradictions.*

Something is wrong with a set theory that lets sets be defined in that way. To get around this paradox, certain axioms have been devised that forbid such sets. The Zermelo-Fraenkel axioms are one such set of axioms, and are the common ones used in mathematics. The idea around the paradox is to restrict things that can be sets.

The continuum hypothesis is that there is no cardinal number between  $\aleph_0$  and  $\mathcal{C}$ . The generalized continuum hypothesis is that for a transfinite set  $A$ , there is no cardinal number between  $Card(A)$  and  $Card(\mathcal{P}(A))$ . Kurt Godel proved that the generalized continuum hypothesis is consistent with the Zermelo-Fraenkel axioms of set theory. On the other hand Paul J Cohen proved that the negation of the continuum hypotheses is also consistent with the axioms of set theory. It follows that the continuum hypothesis can neither be proved or disproved using the axioms of set theory. Paul Cohen is very famous for having proved this. He received the Fields medal for this in 1966. Paul Cohen died in 2007.

**Note.** Richard Fridshal who was an acquaintance of mine, and who was a CAM-I representative from General Dynamics in San Diego, grew up in Manhattan, and told me that he had Paul Cohen as a classmate at the Stuyvesant High School in New York City (Stuyvesant is a famous public high school devoted to science and mathematics).

## 2 Ordinal Numbers

Ordered sets, are sets upon which are imposed an order. Any two elements of a set are ordered, if  $a$  and  $b$  are in the set, then  $a < b$  or  $b < a$ . Two such sets are equivalent, or of the same order type, if there is a 1-1 correspondence between the set elements that preserves the order. The order type of such equivalent sets is called the ordinal number of the set. Two ordered sets with different order types, can, as regular sets, have the same cardinal number. The ordinal number of the natural numbers is usually written as  $\omega$ . Ordinal numbers can be added so the ordinal number  $\omega + n$  could represent the ordered set of the natural numbers followed by a finite set with  $n$  elements. A well ordered set is a partially ordered set such that every nonempty set has a first element. There is a **Well Ordering Axiom** that says that any set can be well ordered. This is equivalent to any of the famous set axioms: (1) **The Axiom of Choice**, (2) **Zorn's Lemma**, (3) **Zermelo's Theorem**. The concept of transfinite ordinal numbers is much more subtle than is that of cardinal numbers. Infinity is not a single simple concept.

## 3 The Peano Postulates for the Natural Numbers

**Postulate 1** 1 is a natural number.

**Postulate 2** For each  $x$  there is exactly one natural number, called the successor of  $x$  which is denoted by  $x'$ .

**Postulate 3** There is no number whose successor is 1.

**Postulate 4** If  $x' = y'$ , then  $x = y$ .

**Postulate 5 (Induction)** Let there be given a subset  $A$  of natural numbers, with the following properties:

(I) 1 belongs to  $A$ .

(II) If  $x$  belongs to  $A$ , then  $x'$  belongs to  $A$ .

Then  $A$  contains all natural numbers.

A concrete model of the natural numbers, using sets, which satisfies these postulates is:

$$1 = \{\emptyset\}$$

If  $x$  is a natural number, then the successor of  $x$  is

$$x' = \{x, \emptyset\}.$$

Thus

$$1 = \{\emptyset\},$$

$$2 = \{\{\emptyset\}, \emptyset\},$$

$$3 = \{\{\{\emptyset\}, \emptyset\}, \emptyset\},$$

and so on.

## 4 Metric Spaces

A metric space is a pair  $(A, d)$ , where  $A$  is a set, and  $d$  is a distance function, which is defined on each pair of elements of the set. The distance function is called a metric. It must have the following properties:

- (1)  $d(x, x) = 0$
- (2) If  $x$  is not equal to  $y$ , then  $d(x, y) > 0$
- (3)  $d(x, y) = d(y, x)$
- (4)  $d(x, y) \leq d(x, z) + d(z, y)$ , (The triangle Inequality)

Here are some examples of metric spaces:

- (a) The real numbers  $\mathbf{R}$  with  $d(x, y) = |x - y|$ .
- (b) Any set  $A$ , where  $d(x, y) = 0$ , if  $x = y$ , and  $d(x, y) = 1$  otherwise. This is called the discrete metric.
- (c) The set of all continuous functions defined on the interval  $[0, 1]$  with metric defined by

$$d(f, g) = \left[ \int_0^1 (f(x) - g(x))^2 dx \right]^{1/2}.$$

## 5 Complete Metric Spaces

Given a metric space  $(A, d)$ , let  $\{x_n\}_1^\infty$  be an infinite sequence, where each element  $x_k$  is in the set  $A$ . This sequence is said to converge to  $x \in A$  if, given any  $\epsilon > 0$ , there exists an integer  $N$  such that for all  $n > N$

$$d(x_n, x) < \epsilon.$$

Take the set  $\mathbb{Q}$  of all rational numbers. Consider the infinite decimal number

$$.102003000400005000006\dots$$

This defines a sequence of rational numbers as

$$x_1 = .1$$

$$x_2 = .10$$

$$x_3 = .102$$

$$x_4 = .1020$$

$$x_5 = .10200$$

$$x_6 = .102003$$

$$x_7 = .1020030$$

$$x_8 = .10200300$$

$$x_9 = .102003000$$

$$x_{10} = .1020030004$$

and so on. Now intuitively this infinite decimal does not represent a rational number, because the decimals will never repeat. But notice that the elements get ever closer together. So it seems to be converging to something. Sequences that get arbitrarily close as  $n$  goes to infinity are called Cauchy sequences after the French mathematician Augustin Louis Cauchy (b. August 21, 1789 , d. May 23, 1857).

**Definition** A sequence  $\{x_n\}_1^\infty$  is called a Cauchy sequence, if for every  $\epsilon > 0$ , there exist an integer  $N$ , such that if both  $m$  and  $n$  are both greater than  $N$ , then

$$d(x_m, x_n) < \epsilon.$$

If a sequence converges to some value  $x$  then one can show that the sequence is Cauchy. But the opposite is not true. A Cauchy sequence in a metric space need not converge to an element of the metric space. If a metric space does have this property that every Cauchy sequence converges, it is called a complete metric space. From examples such as the one above we see that the rational numbers  $\mathbb{Q}$  are not complete. A Cauchy rational



Using a Cantor like process, we can show that this Cauchy sequence in  $M'$  converges to the diagonal sequence, namely to the equivalence class of the Cauchy sequence

$$\{a_{i,i}\}_{i=1}^{\infty}.$$

The steps that must be proved in this construction are outlined in problem 24 on page 82 of Rudin.

Let  $N$  be the set of all null sequences, that is sequences converging to zero. Two Cauchy sequences are equivalent if their difference is a null sequence. The completion is the quotient space  $C/N$ .  $M$  is embedded into  $C/N$  by mapping any element  $m$  of  $M$  to the constant sequence, where each element in the sequence is equal to  $m$ . To show that  $\bar{M}$  is complete, a Cantor process is used.

Augustin Louis Cauchy (August 21, 1789 – May 23, 1857) Georg Ferdinand Ludwig Philipp Cantor (3 March [O.S. 19 February] 1845[1] – January 6, 1918) was a German mathematician, born in Russia. He is best known as the creator of set theory

## 6 The Real Numbers

The integers form a ring with the operations of addition and multiplication. Every integer  $n$  has an additive inverse,  $m$  such that  $n + m = 0$ . This unique  $m$  is written as  $-n$ . However, in a ring, every element does not necessarily have a multiplicative inverse. The rational numbers are the fractions. They are a field. A field is a ring in which every nonzero element has a multiplicative inverse. That is, given  $x$ , there is a  $y$  such that  $xy = 1$ . An ordered field has an order, so that for any two nonequal numbers  $x$  and  $y$ , either  $x > y$  or  $y > x$ . An ordered field  $F$ , with positive elements  $P$ , is Archimedean ordered if given  $a \in F$  and  $b \in P$ , there exists an integer  $n$  so that  $nb > a$ .

Hewitt and Stromberg, in section 5, give a detailed construction of the real and complex number systems, using the Cauchy sequence completion technique on the rational numbers. This gives a concrete representation of real numbers having the properties described in the statement:

**The real numbers are any Archimedean ordered complete field.**

Among the properties of the real numbers, is that any bounded set of real numbers has a least upper bound. The set of Real numbers contain the

integers and the rational numbers. It is divided into three parts: the negative numbers, zero, and the positive numbers.

The Real numbers may also be constructed using the concept of Dedekind cuts (Julius Wilhelm Richard Dedekind, October 6, 1831 February 12, 1916). Dedekind cuts are treated in Landau.

## 7 Inner Product Space

An inner product is like the dot product in vector analysis. An inner product on a complex vector space  $(f, g)$  has the following properties:

$$(f, g) = \overline{(g, f)}$$

$$(a_1 f_1 + a_2 f_2, g) = a_1 (f_1, g) + a_2 (f_2, g)$$

$$(f, f) \geq 0,$$

and equality only if  $f = 0$ .

The norm of  $f$  is

$$\|f\| = (f, f)^{1/2}.$$

A metric defined by

$$d(f, g) = \|f - g\|,$$

makes an inner product space a metric space.

## 8 Hilbert Space

A Hilbert space is an inner product space, which as a metric space, is complete. Quantum Mechanics was formulated as a theory of operators on Hilbert space by John Von Neumann.

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