Inversive Geometry

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Contents

1 Inversion Through a Circle

Let a circle have radius r. Let P be a point in the plane at distance $s = \|P\|$ from the origin. Then the point Q on the line through the origin through P

that is at distance

$$\frac{r^2}{s}$$

from the origin, is called the inverted image of P. Note that the origin goes to infinity and a point on the circle goes to itself. Also point Q has inverted image P. The points P and Q are called inverted pairs and they satisfy

$$\|P\|\|Q\| = r^2.$$

If the radius of the circle is $r = 1$, then the point at distance s from the origin maps to the reciprocal distance $1/s$. When the radius of the circle is not equal to 1, then we get the same kind of reciprocal relationship with the relative distances s/r. So the point at relative distance s/r maps to the point at relative distance

$$\frac{r^2/s}{r} = r/s.$$

2 The Möbius Transformation

Consider the Möbius transformation (linear fractional transformation) of the complex plane given by

$$f(z) = \frac{az+b}{cz+d}$$

A special case is

$$f(z) = \frac{r^2}{z} = \frac{r^2\bar{z}}{z\bar{z}} = \frac{r^2\bar{z}}{|z|^2}$$

This is an inversion through a circle, followed by reflection about the real axis. This is because the distance from the origin to $f(z)$ is $r^2/|z|$. That is

$$|f(z)| = \frac{r^2}{|z|}.$$

Thus a special case of the Möbius transformation is essentially inversion through a circle of radius r.

3 Circles Go To Circles

Proposition If a circle does not pass through the origin, then under inversion it goes to a circle. If it does pass through the origin, it goes to a straight line.

Proof. Without loss of generality consider a circle that lies on the x axis with center $(a, 0)$ and radius ρ. It has equation

$$(x-a)^2 + y^2 = \rho^2.$$

Let $s = \sqrt{x^2+y^2}$. Then the point (x, y) goes to the point (x', y'), where

$$x' = \frac{r^2}{s^2}x = r^2\frac{x}{x^2+y^2}$$

and

$$y' = \frac{r^2}{s^2}y = r^2\frac{y}{x^2+y^2}.$$

Also

$$x = \frac{s^2}{r^2}x'$$

and

$$y = \frac{s^2}{r^2}y'.$$

$$x'^2 + y'^2 = \frac{r^4}{s^4}(x^2+y^2) = \frac{r^4}{s^2}.$$

We have

$$x^2 - 2ax + a^2 + y^2 = \rho^2$$

$$x^2 + y^2 - 2ax = \rho^2 - a^2$$

$$s^2 - 2ax = \rho^2 - a^2$$

$$s^2 - 2a\frac{s^2}{r^2}x' = \rho^2 - a^2$$

$$1 - \frac{2a}{r^2}x' = \frac{\rho^2 - a^2}{s^2} = \frac{\rho^2 - a^2}{r^4}(x'^2 + y'^2)$$

$$\frac{r^4}{(\rho^2 - a^2)} - \frac{2ar^2x'}{(\rho^2 - a^2)} = x'^2 + y'^2$$

$$x'^2 + \frac{2ar^2}{(\rho^2 - a^2)}x' + y'^2 = \frac{r^4}{(\rho^2 - a^2)}$$

$$(x' + \frac{2ar^2}{(\rho^2 - a^2)})^2 - \frac{a^2r^4}{(\rho^2 - a^2)^2} + y'^2 = \frac{r^4}{(\rho^2 - a^2)}$$

$$(x' + \frac{2ar^2}{(\rho^2 - a^2)})^2 + y'^2 = \frac{a^2r^4}{(\rho^2 - a^2)^2} + \frac{r^4}{(\rho^2 - a^2)}$$

$$(x' + \frac{2ar^2}{(\rho^2 - a^2)})^2 + y'^2 = \frac{a^2r^4 + r^4(\rho^2 - a^2)}{(\rho^2 - a^2)^2}$$

$$(x' + \frac{2ar^2}{(\rho^2 - a^2)})^2 + y'^2 = \frac{r^4\rho^2}{(\rho^2 - a^2)^2}.$$

This is the equation of a circle with center on the x axis, and radius equal to

$$\rho' = \sqrt{\frac{r^4\rho^2}{(\rho^2 - a^2)^2}}.$$

The x coordinate of the center is

$$x'_c = -\frac{2ar^2}{(\rho^2 - a^2)}.$$

Notice that if the circle passes through the origin, then $\rho = a$, so the radius goes to infinity, and the circle becomes a straight vertical line.
Proposition Under inversion, a circle that passes through the origin goes to a line.
Proof. Consider a circle of radius a that passes through the origin. Suppose it has center on the x axis. Then it has equation

$$y^2 = x(2a - x)$$

Let $s = \sqrt{x^2 + y^2}$. Then

$$x' = \frac{r^2}{s^2}x = r^2\frac{x}{x^2 + y^2}$$

and

$$y' = \frac{r^2}{s^2}y = r^2\frac{y}{x^2 + y^2}.$$

So

$$x = \frac{s^2}{r^2} x'$$

and

$$y = \frac{s^2}{r^2} y'.$$

We have

$$\frac{s^4}{r^4} y'^2 = \frac{s^2}{r^2}(2a - \frac{s^2}{r^2} x'),$$

$$\frac{s^2}{r^2} y'^2 = x'(2a - \frac{s^2}{r^2} x'),$$

$$\frac{s^2}{r^2}(x'^2 + y'^2) = 2ax',$$

$$\frac{s^2}{r^2}(\frac{r^4}{s^4} x^2 + \frac{r^4}{s^4} y^2) = 2ax',$$

$$\frac{r^2}{s^2}(x^2 + y^2) = 2ax',$$

$$r^2 = 2ax',$$

and finally

$$x' = \frac{r^2}{2a}.$$

This is a vertical line.

4 Inversive Linkage

Consider the linkage shown in Figure 1. This consists of two bars of length s and four bars of length d.

Proposition The linkage shown in Figure 1 performs an inversion through the circle of radius $r = \sqrt{s^2 - d^2}$. Point P is taken to the inverted point Q. So P and Q are inversive pairs, so that $\|P\|\|Q\| = r^2$.

Proof. From the figure we have

$$g^2 = d^2 - f^2$$

and

$$g^2 = s^2 - (\|P\| + f)^2.$$

Then

$$d^2 - f^2 = s^2 - (\|P\| + f)^2 = s^2 - \|P\|^2 - 2\|P\|f - f^2.$$

So

$$d^2 = s^2 - \|P\|^2 - 2\|P\|f.$$

Then

$$\|P\|^2 + 2\|P\|f = s^2 - d^2.$$

Then

$$\|P\| + 2f = \frac{s^2 - d^2}{\|P\|}.$$

And notice that

$$\|Q\| = \|P\| + 2f.$$

Therefore

$$\|Q\| = \|P\| + 2f = \frac{s^2 - d^2}{\|P\|} = \frac{r^2}{\|P\|}.$$

That is

$$\|P\|\|Q\| = r^2.$$

So P and Q are inversive pairs, which was to be proved.

5 Peaucellier-Lipkin Linkage

The Peaucellier-Lipkin Linkage is a solution to the old problem of finding a linkage that generates a perfect straight line. The linkage consists of the inverter linkage given above and a link that constrains an end of the linkage to a circular motion on a circle that passes through the origin. Then necessarily the other end moves in a straight line. See the movie at this Cornell link: http://kmoddl.library.cornell.edu/tutorials/05/

6 A Geometrical Construction of Inversive Pairs

Given a point, one can construct an inverse point in the following way. Refer to Figure 2. Let point P be inside a circle of radius r with center at the

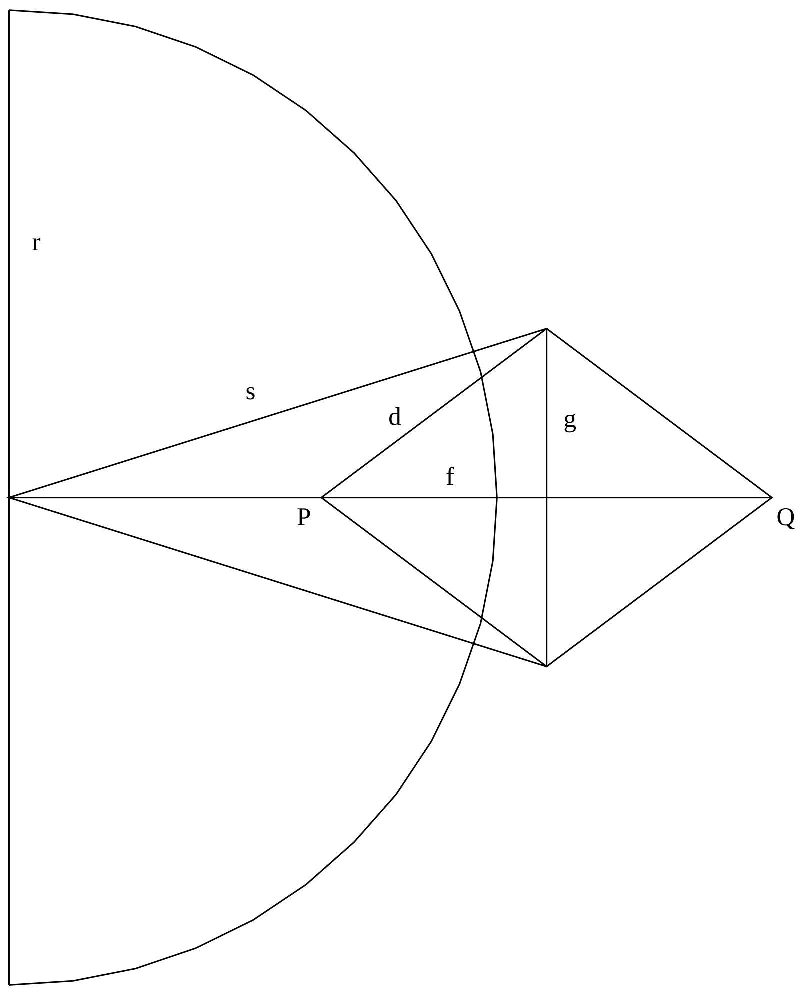

Figure 1: The inversion linkage consists of two bars of length s, and four bars of length d. The circle of inversion has radius $r = \sqrt{s^2 - d^2}$. Points P and Q are inversive pairs, $\|P\|\|Q\| = r^2$.

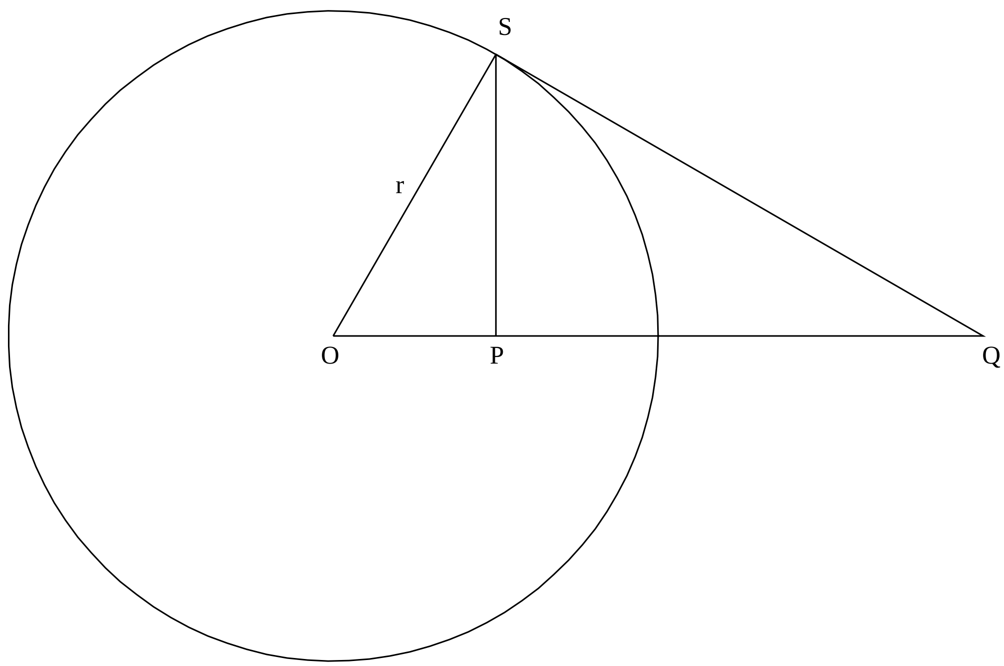

Figure 2: A geometric construction of inversive pairs.

origin O. Draw a line through the origin O and through P. Let a second line be constructed through P

and perpendicular to the first line. Let the second line meet the circle at point S. Draw a tangent to the circle at S. Let it intersect the first line at Q. Draw a radial line of length r from O to S. Then triangle OPS is similar to triangle OSQ. Hence

$$\frac{r}{\|P\|} = \frac{\|Q\|}{r}.$$

Then

$$\|P\|\|Q\| = r^2.$$

So points P and Q are inversive pairs. If external point Q is given, then reverse the construction to find P.

7 Orthogonal Circles

Proposition Suppose a circle is orthogonal to the circle of inversion. Then the circle inverts to itself.
Proof.

Refer to Figure 3. Suppose the circle which is orthogonal to the circle of inversion has radius a. Construct a line through the two circle centers. Let it intersect the second circle at P and Q. We claim that P and Q are inversive pairs. Because circles go to circles. This would imply that the circle inverts to itself. Let O be the origin. Let C_2 be the center of the circle that is inverted. Let I be a point of intersection of the two circles. Then because the two circles are orthogonal, triangle OIC_2 is a right triangle. Hence

$$r^2 + a^2 = (\|P\| + a)^2 = \|P\|^2 + 2\|P\|a + a^2.$$

So

$$r^2 = \|P\|^2 + 2\|P\|a,$$

$$\frac{r^2}{\|P\|} = \|P\| + 2a.$$

From the figure

$$\|Q\| = \|P\| + 2a.$$

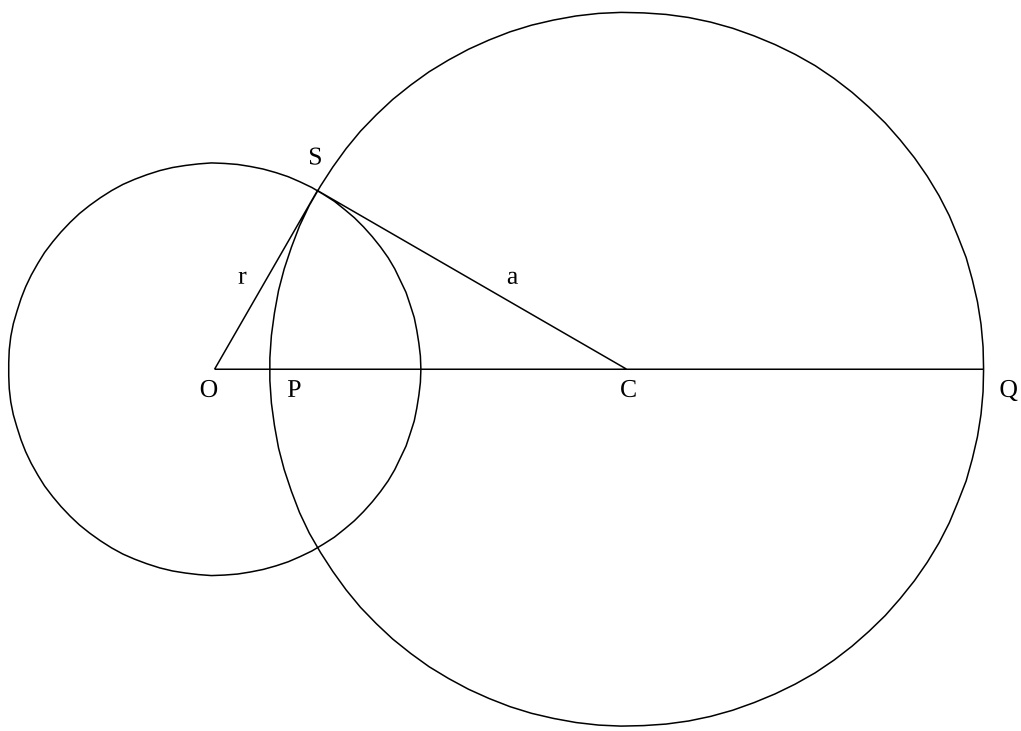

Figure 3: If a circle is orthogonal to the circle of inversion, then the circle inverts to itself. In this Figure, P and Q are inversive pairs, $\|P\|\|Q\| = r^2$. This means that the big circle inverts to itself.

So

$$\frac{r^2}{\|P\|} = \|P\| + 2a = \|Q\|.$$

That is

$$\|P\|\|Q\| = r^2,$$

and so P and Q are inversive pairs. Thus the orthogonal circle goes to itself To show this in another way, one can make use of a theorem of Euclid. Book III, Proposition 36. See the Sir Thomas Heath translation of Euclid, Dover 1956, Volume II, P73.

Theorem . *From an external point of a circle, let a secant line meet the circle in two points, and a tangent line meet the circle at a tangent point. Then the square of the distance to the tangent point equals the product of the distances to the two secant points*

Proof. Refer to Figure 4. Let $a = \|B - A\| = \|D - B\|$, $b == \|A - O\|$, $c = \|C - A\|$, $d = \|C - B\|$, $e = \|E - O\|$, and $f = \|C - O\|$. Then the product of the two secant point distances is

$$b(b + 2a) = b^2 + 2ab.$$

The tangent distance is e^2. We have

$$c^2 = a^2 + d^2,$$

$$f^2 = (a + b)^2 + d^2,$$

$$f^2 = e^2 + c^2,$$

and

$$e^2 + c^2 = (a + b)^2 + d^2 = (a + b)^2 + c^2 - a^2.$$

Then

$$e^2 = (a + b)^2 - a^2 = 2ab + b^2,$$

which was to be proved.

Now we can apply this theorem to show the result on orthogonal circle, namely that if a circle is orthogonal to the circle of inversion, then the circle inverts to itself. So let the external point O be the center of the inversion circle. Let the distance from the external point to the tangent point on the second circle be the radius of the inversion circle r. Let the secant line pass

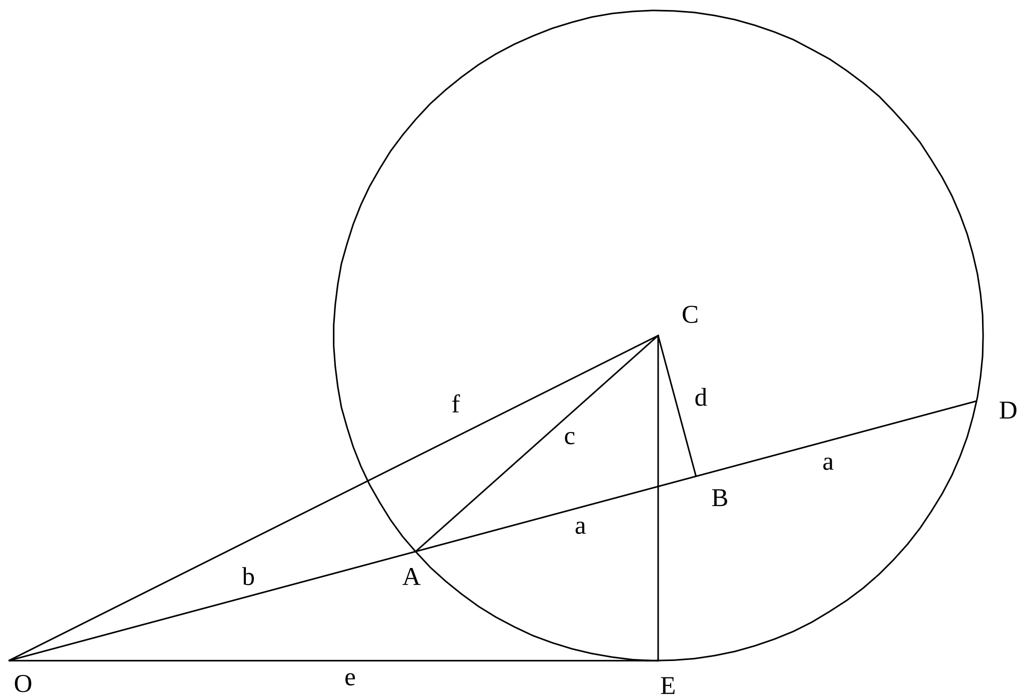

Figure 4: A Theorem of Euclid: From an external point of a circle O, let a secant line meet the circle in two points A and D, and a tangent line meet the circle at a tangent point E. Then the square of the distance to the tangent point equals the product of the distances to the two secant points

through the center of the second circle. Then the two circles are orthogonal. Let P and Q be the intersection of the secant line with the second circle. Then this theorem of Euclid shows that

$$\|P\|\|Q\| = r^2.$$

8 The Riemann Sphere

The Riemann sphere gives a compact representation of the complex plane. Let a sphere of diameter 1 be positioned on the complex plane with the south pole at the origin. Given a point

$$z = x + iy,$$

on the plane, a line through the north pole through this point. cuts the sphere at coordinates x', y', z'. Let

$$r = \sqrt{x^2 + y^2},$$

and

$$r' = \sqrt{x'^2 + y'^2}.$$

Using similar triangles in a plane section normal to the complex plane and through the line, we find that

$$\frac{z'}{r'} = \frac{r}{1},$$

and

$$\frac{1 - z'}{r'} = \frac{1}{r}.$$

Then

$$z' = rr',$$

and

$$1 - z' = \frac{r'}{r}.$$

Eliminating z' we find

$$r' = \frac{r}{r^2 + 1}.$$

Eliminating r' we find

$$z' = \frac{r^2}{1+r^2} = \frac{x^2+y^2}{1+x^2+y^2}.$$

We have

$$(x', y') = \frac{r'}{r}(x, y).$$

And

$$\frac{r'}{r} = 1 - z' = \frac{(1+x^2+y^2) - (x^2+y^2)}{1+x^2+y^2}$$
$$= \frac{1}{1+x^2+y^2}.$$

So

$$x' = \frac{x}{1+x^2+y^2},$$
$$y' = \frac{y}{1+x^2+y^2},$$

and

$$z' = \frac{x^2+y^2}{1+x^2+y^2}.$$

Also

$$x = \frac{x'}{1-z'},$$
$$y = \frac{y'}{1-z'},$$

and

$$r^2 = \frac{z'}{1-z'}.$$

The north pole on the Riemann sphere is the point at infinity. Circles on the sphere map to circles or lines in the plane. The sphere equator maps to the unit circle in the plane.

9 The Möbius Transformation

The Möbius transformation is also called the linear fractional transformation. A Möbius transformation takes the form

$$w = \frac{az+b}{cz+d}.$$

Although it never seems to be mentioned, the Möbius transformation is actually a one dimensional projective transformation in projective space. So given a vector space n+1, the set of lines through the origin is called n dimensional projective space. This is the set of 1 dimensional subspaces. Consider two dimensional complex space. The corresponding projective space is the set of one dimensional subspaces. Each subspace may be represented by a vector pointing in that direction, say (z,w). If we intersect each such subspace with the plane $w = 1$, then we get a point (z',1). This set of points is called the affine plane. A projective transformation is a nonsingular linear transformation of the one dimensional subspaces in the projective space. A point in the projective space has the coordinates say (z,w), these being the coordinates of any vector in the direction of the subspace. These are called homogeneous coordinates. Any multiple of these points is also a homogenious coordinate of of this point. So for example (z/w,1) is such a set of homogenious coordinates. And such a point written in this way lies in the affine plane $w = 1$. Hence a standard way to write the points in the affine plane is to set the second coordinate to one. Now let us apply a projective transformation to the coordinates (z, w), we get $(az + bw, cz + dw)$. Because the projective transformation is to be nonsingular, the determinate must not be zero, that is $ad - bc$ is not zero. If we use homogenious coordinates of the form $(z, 1)$, then this becomes

$$(az + b, cz + d)$$

If we put these coordinates in standard affine form with the second coordinate equal to 1, then we have

$$(\frac{az+b}{cz+d}, 1),$$

and so we get our Möbius transformation

$$\frac{az+b}{cz+d}.$$

This transformation is conformal and circular and has many important properties. See the book by Knopp, **Elements of the Theory of Functions**.

10 Bibliography

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