An Introduction to Laplace's Equation in Physics

Jim Emery

Version 11/25/09

Contents

0.1 Introduction

The general theory of Laplaces equation is called potential theory.

0.2 Gravity

The force on a particle of mass m by a particle of mass M which is located at the origin, is

$$F = -G\frac{nM\hat{r}}{r^2}$$

G is the gravitational constant, r is the distance between the particles and $\hat{r}$ is a unit vector directed from M to m. The mass M establishes a force field, which we call H. We write the force field as

$$H = -G\frac{M\hat{r}}{r^2}.$$

When there is a distribution of mass particles M_i, the force field is the vector sum of the fields of each particle. The force on a particle of mass m by a gravitational field H is

$$F = mH.$$

0.3 Gauss's Law

Referring to the figure, by similarity of the triangles, the area element ds' and the projection of ds, which is $ds\cos(\theta)$, have the ratio

$$\frac{\hat{r}\cdot ds}{ds'} = r^2.$$

So

$$\frac{\hat{r}\cdot ds}{r^2} = ds'.$$

Since ds' is an area element on the unit sphere S', we have

$$\int_S \frac{\hat{r}\cdot ds}{r^2}\int_{S'} ds' = 4\pi.$$

The surface boundary of a volume A, is written ∂A. Therefore

$$\int_{\partial A} H\cdot ds = -\int_{\partial A}\frac{GM\hat{r}\cdot ds}{r^2} = -GM4\pi.$$

For the case of a distributed mass, with density function ρ, each mass element $dm = \rho dv$ produces a contribution to the field dH, where dv is the volume element. Then

$$\int_{\partial A} dH\cdot ds = -G\rho 4\pi dv.$$

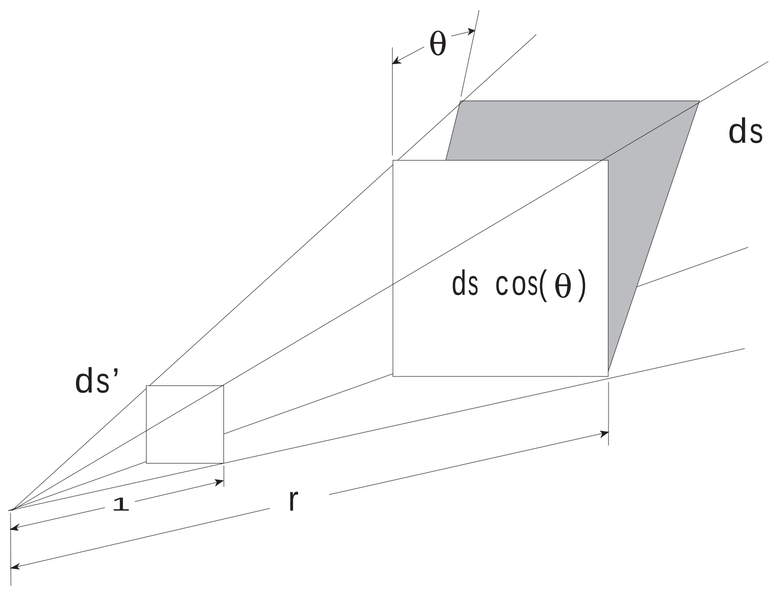

Figure 1: Proof of Gauss' Law.

When we integrate, we get

$$\int_{\partial A} H \cdot ds = -4\pi G \int_A \rho dv.$$

This is the integral form of Gauss's law. It says that the surface integral of the field is equal to $-4\pi G$ times the amount of mass inside the surface. In the next section we shall introduce the concept of the divergence, $\nabla \cdot H$, of a vector field. Then the differential form of Gauss's law is

$$\nabla \cdot H = -4G\pi\rho.$$

0.4 Curl, Divergence, Gradient

The curl of a vector field **F** in cartesian coordinates is

$$\nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{i}$$

$$- \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \mathbf{j}$$

$$\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{k}.$$

Divergence thereom. Stokes theorem. Directional derivative.

0.5 Potential

$$H = -\nabla V$$

$$\nabla \cdot H = -4\pi G\rho$$

Substituting get Laplaces equation

$$\nabla^2 V = 0.$$

0.6 The Heat Equation

Let the heat flux vector be Φ, and the temperature T. The Fourier equation is

$$\Phi = -k\nabla T.$$

Φ has units like calories per second per square meter. We shall use the first law of thermodynamics

$$\Delta Q = \Delta W + \Delta u.$$

Let c be the specific heat with units like calories per Kg per degree. So if m is the mass, the change in internal energy Δu for a change in temperature ΔT is $c\Delta Tm$. Assuming zero work, the heat flow into the volume is

$$\Delta Q = c\Delta Tm.$$

The continuity equation, which says that the rate of increase of internal energy (increase in heat) is equal to the flow of heat through the surface, is

$$\nabla \cdot \Phi = -c\rho \frac{\partial T}{\partial t} = -\frac{\partial \psi}{\partial t}.$$

Substituting for Φ, we have the heat equation

$$\nabla \cdot k \nabla T = c\rho \frac{\partial T}{\partial t}.$$

When k is constant, under steady state conditions,

$$\frac{\partial T}{\partial t} = 0,$$

this reduces to Laplace's equation

$$\nabla^2 T = 0.$$

0.7 Electric Potential

The electric field has zero curl, hence there exists a potential function ϕ such that

$$E = -\nabla \phi.$$

In places where there is no charge we have

$$\nabla \cdot E = 0,$$

Therefore the potential satisfies Laplace's equation

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = -\nabla \cdot E = 0.$$

0.8 Electric Current Flow

For electric current flow, we have Ohm's law,

$$J = \sigma E = \sigma \nabla \phi.$$

This is anolgous to the Fourier law for heat conduction. Also there is a continuity equation for charge conservation, that is the change in charge in a volume is equal to the rate of flow of charge through the bounding surface. This also is completely analogous to the flow of heat and so again the potential ϕ satisfies Laplaces equation. The reciprocal of the conductivity σ is the resistivity, which has units like ohm-meters.

0.9 Diffusion Equation

The equation for diffusion is the same as the heat equation so again we get Laplace's equation in the steady state.

0.10 Analytic and Harmonic Functions

An analytic function satisfies the Cauchy-Riemann equations. Differentiating these two equations we find that the both the real and imaginary parts of the function satisfy Laplace's equation in two dimensions. Such functions are a major source for finding solutions to two dimensional boundary value problems.

0.11 Viscosity

Consider a fluid in a state of laminar flow. Let the rate of change of shear angle be

$$\frac{d\alpha}{dt}.$$

The coefficient of viscosity is defined as

$$\eta = \frac{\sigma_s}{d\alpha/dt},$$

where σ_s is the shear stress. The unit of viscosity is the Poise, which is one dyne second per square cm.

Suppose two nested coaxial cylinders are separated by a fluid. Let the outer cylinder be fixed and the inner cylinder rotated by a torque τ. Let the inner cylinder have radius r_1 and the outer cylinder radius r_2. Let the cylinder be rotating at angular velocity ω. Let the cylinders have height h. Then

$$\tau = r_1 F = r_1 2\pi r_1 h \sigma_s.$$

Now

$$\frac{d\alpha}{dt} = \frac{v}{r_2 - r_1} = \frac{r_1 \omega}{r_2 - r_1}.$$

Then

$$\tau = 2\pi r_1^2 h \eta \frac{d\alpha}{dt} = 2\pi r_1^3 h \eta \frac{\omega}{r_2 - r_1}.$$

If τ and ω are measured, then this equation may be solved for the viscosity. At about 20 degrees C, the viscosity of Benzene is .65 centipoise, and that of mercury is 1.55 cp. At 0 degrees the viscosity of water is 1.79 cp, and at 100 degrees it is .28 cp. The viscosity of heavy oil at 15 degrees is 660 cp. The viscosity of air is 171 μp at ..

0.12 Fluid Mechanics

We shall derive Euler's Equation (Applied Hydro and Aeromechanics). Let the velocity of a fluid be v. It depends upon the time t, and the distance along the streamline s. We write

$$v = e(s, t).$$

Let $s(t)$ be the position of a specific fluid particle. Then its velocity is

$$v = f(t) = e(s(t), t).$$

Its acceleration is

$$\frac{df}{dt} = \frac{\partial e}{\partial s}\frac{ds}{dt} + \frac{\partial e}{\partial t} = \frac{\partial e}{\partial s}v + \frac{\partial e}{\partial t}.$$

Consider a small portion of a stream tube of length δs. The acceleration times the mass is equal to the net force.

$$\frac{df}{dt}\rho dA\delta s = g\rho dA\delta s \sin(\theta) + dA(p - (p + \frac{\partial p}{ds}\delta s),$$

where θ is the angle of incline of the stream tube. Then

$$\frac{\partial e}{\partial s}v + \frac{\partial e}{\partial t} = g\sin(\alpha) - \frac{1}{\rho}\frac{\partial p}{ds}.$$

This is the equation of Euler.

Under steady flow

$$\frac{\partial e}{\partial t} = 0$$

So

$$v\frac{\partial e}{\partial s} = dh - \frac{\partial p}{\partial s}\frac{1}{\rho}.$$

Integrating with time held constant, we find

$$v^2/2 + gh + p/\rho = C,$$

where C is a constant. This is Bernoulli's equation.

There are two approaches to observing fluid flow. One may watch the flow as it passes a fixed position. This is the Eulerian approach. Alternately one may follow a fluid particle, which is the Lagrangean approach. Thus Euler sits on the shore and watches Lagrange go by in a boat. In the Eulerian approach, the velocity at which the fluid flows past a fixed point x at time t is given by a function $e(x, t)$. Let us now look at the problem from the Lagrangian viewpoint. Suppose $(x(t), y(t), z(t))$ is the position of a moving fluid particle. Then the velocity of the particle is

$$v = f(t) = e(x(t), y(t), z(t), t).$$

The acceleration of the particle is

$$\frac{dv}{dt} = \frac{df}{dt} = \frac{\partial e}{\partial t} + \frac{\partial e}{\partial x}\frac{dx}{dt} + \frac{\partial e}{\partial y}\frac{dy}{dt} + \frac{\partial e}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial e}{\partial t} + v \cdot \nabla v.$$

This is called the material derivative of the velocity. The acceleration of the flow is

$$\frac{\partial e}{\partial t}.$$

The introduction of viscosity leads to the Navier-Stokes Equation. See Lass, "Elements of Pure and Applied Mathematics," for a mathematical treatment of the Navier-Stokes Equation.

Viscosity is a consequence of momentum transfer. Consider two trains passing one another at differing velocities. Suppose suitcases are thrown repeatedly back and forth between the trains. As a suitcase is passed from the faster train to the slower, the slower train experiences an impulse and a force and tends to increase its speed. When the opposite happens the fast train experiences a retarding force. No net transfer of mass occurs. Momentum is being exchanged. Viscosity may be considered also as the rate of change of shear strain. The coefficient of viscosity is a proportionality factor. The Navier-Stokes equation is:

$$\frac{\partial w}{\partial t} + \nabla w \cdot w = G - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 w,$$

where G is a body force, such as gravity (See Prandtl and Tietjens, "Hydro and Aeromechanics"). The term

$$\nabla w \cdot w = \nabla v_x \cdot v + \nabla v_y \cdot v + \nabla v_z \cdot v,$$

is nonlinear. The Navier-Stokes equation is a nonlinear partial differential equation, for which there is no general existence theory.

0.13 Velocity Potential and Stream Function

Let

$$w = ui + vj$$

be a two dimensional velocity field. $w(x, y, t)$ is the velocity of a fluid particle at position (x, y) and time t. Suppose the fluid is incompressible, so that its density ρ is constant. The continuity equation is

$$\nabla(\rho w) = \frac{\partial \rho}{\partial t} = 0.$$

Hence

$$\nabla(\rho w) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Now consider the differential form

$$-vdx + udy.$$

We have just shown that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

So that

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}.$$

which means that the differential form is closed and so exact. It is exact so it is the differential of some function f so that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = -vdx + udy.$$

Thus

$$\frac{\partial f}{\partial x} = -v,$$

and

$$\frac{\partial f}{\partial y} = u.$$

We have shown this to be a consequence of the assumed incompressibility. We shall further restrict the fluid to be irrotational, which means it has zero curl. These two assumptions will lead to Laplace's equation. The curl of a vector field A, is defined as

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_x \end{vmatrix}.$$

In two dimensional flow, the z component of velocity is zero, so the curl is

$$\nabla \times w = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) k.$$

Then the assumption of zero curl forces

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}.$$

Combining this with the consequence of incompressibility,

$$\frac{\partial f}{\partial x} = -v,$$

and

$$\frac{\partial f}{\partial y} = u,$$

we get Laplace's equation

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Let us consider why $\nabla \times w$ is called the curl. Stokes theorem is

$$\int_{\partial S} w \cdot d\ell = \int_S (\nabla \times w) \cdot ds,$$

where S is a surface element, and ∂S is the boundary curve of the surface element. Because we have assumed that $\nabla \times w = 0$, the line integral around the surface element is zero, that is

$$\int_{\partial S} w \cdot d\ell = 0.$$

If the fluid were to curl around some point, then the line integral following this circular flow would certainly not be zero. Hence $\nabla \times w = 0$ does imply that such curling of the fluid does not occur. Now given the harmonic function f, one can find a conjugate harmonic function g such that

$$f + ig,$$

is an analytic function. By the Cauchy-Reimann equations

$$w = -\nabla g.$$

Hence g is a velocity potential, because its negative gradient is the velocity. Now ∇f is perpendicular to w, so the lines of constant f are parallel to the velocity w. That is the lines are stream lines. Hence f is called the stream function. If we know the stream function on the boundary of a flow region, then f is a solution to the Laplace boundary value problem. From f, we may compute the conjugate function g, and then the velocity as the negative gradient of g. Assuming steady state flow, we may apply Bernoulli's equation and from the velocity deduce the pressure. In the case of an airfoil we may then deduce the lift by integrating the pressure.

0.14 Solution Methods

There are several ways to solve Laplace's equation and other partial differential equations of mathematical physics. Among the methods are separation of variables, variational methods, and numerical methods such as the finite difference method, and the finite element method.

0.14.1 Separation of Variables

It is frequently possible to write a solution as a product of functions, each function depending on only one variable. Then one finds ordinary differential equations for each function of the product. Then one solves the problem by finding solutions that fits the boundary conditions of the problem.

0.14.2 The Finite Difference Method

By using central difference approximations to the derivatives, one may reduce the problem to a finite difference equation on a grid.

$$\frac{\partial V}{\partial x} = \frac{V(x_n, y_m) - V(x_{n-1}, y_m)}{h},$$

and

$$\frac{\partial^2 V}{\partial x^2} = \left(\frac{\partial V(x_{n+1}, y_m)}{\partial x} - \frac{\partial V(x_{n-1}, y_m)}{\partial x}\right) \Big/ h.$$

Adding a similar expression for

$$\frac{\partial^2 V}{\partial x^2},$$

we get

$$\nabla^2 V = \frac{V(x_{n+1}, y_m) + V(x_{n-1}, y_m) + V(x_n, y_{m-1}) + V(x_n, y_{m+1}) - 4V(x_n, y_m)}{h^2}.$$

Hence setting

$$\nabla^2 V = 0,$$

and solving , we obtain

$$V(x_n, y_m) = \frac{V(x_{n+1}, y_m) + V(x_{n-1}, y_m) + V(x_n, y_{m-1}) + V(x_n, y_{m+1})}{4}.$$

This gives a sparse system of linear equations for the variables $V(x_n, y_m)$. The system is usually solved by iteration.

0.14.3 Variational Methods and the Finite Element Method

0.15 Bibliography