The Laplace Transform

Jim Emery

3/20/11

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1 The Laplace Transform

The Laplace transform maps a function $f(t)$ of a real variable to a function of a complex variable $Lf(s)$.

$$Lf(s) = \int_0^\infty \mathrm{f}(\,t\,)\,\mathrm{e}^{(-s\,t)}\,dt$$

Sometimes we write the transform of a function f by capitalizing, so we write

$$F(s) = Lf(s).$$

The Laplace transform of f in Maple is specified as

$$\mathrm{laplace}(\,\mathrm{f}(\,t\,), t, s\,)$$

$f(t)$ is a function of a real variable, but s is a complex variable, so Lf is a complex valued function of a complex variable. Here are a few Laplace transforms.

$$\int_0^\infty \sin(\,t\,)\,\mathrm{e}^{(-s\,t)}\,dt = \frac{1}{s^2+1}$$

$$\int_0^\infty \cos(\,t\,)\,\mathrm{e}^{(-s\,t)}\,dt = \frac{s}{s^2+1}$$

$$\int_0^\infty t^a\,\mathrm{e}^{(-s\,t)}\,dt = \frac{\Gamma(\,a+1\,)}{s^{(\,a+1\,)}}$$

$\Gamma(x)$ is the Gamma function:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

$$\lim_{x \to 0} \Gamma(x) = \infty.$$

$$\Gamma(x) = \frac{1}{x}\Gamma(x+1).$$

If n is an integer then

$$\Gamma(n+1) = n!.$$

So if n is an integer,

$$\int_0^\infty t^n e^{-st} dt = \frac{n!}{s^{n+1}}.$$

The Laplace transform of the derivative of a function f is obtained by integrating by parts. We find

$$Lf'(s) = \int_0^\infty \left(\frac{d}{dt}\,\mathrm{f}(\,t\,)\right)\,\mathrm{e}^{(-s\,t)}\,dt = s\int_0^\infty \mathrm{f}(\,t\,)\,\mathrm{e}^{(-s\,t)}\,dt - \mathrm{f}(\,0\,) = sLf - \mathrm{f}(\,0\,)$$

So the transform of a second derivative is

$$Lf'' = sLf' - f'(0) = s(sLf - f(0)) - f'(0) = s^2 Lf - sf(0) - f'(0)$$

and so on for higher derivatives.
If $f(t) = A$ is constant then

$$Lf(s) = \int_0^\infty Ae^{-st}dt = \left[-\frac{A}{s}e^{-st}\right]_0^\infty = \frac{A}{s}.$$

Suppose

$$f(t) = \int_0^t g(x)dx.$$

Then $f'(t) = g(t)$, so integrating by parts we have

$$Lf(s) = \int_0^\infty f(t)e^{-st}dt$$

$$= \left[-f(t)\frac{e^{-st}}{s}\right]_0^\infty - \frac{1}{s}\int_0^\infty -e^{-st}f'(t)dt$$

$$= \frac{1}{s}\int_0^\infty e^{-st}g(t)dt$$

$$= \frac{Lg(s)}{s}.$$

We have used

$$u = f(t)$$

$$dv = e^{-st}dt$$

and

$$udv = d(uv) - vdu.$$

Let us compute $L\sin(s)$. Integrating by parts we have

$$L\sin(s) = \int_0^\infty \sin(t)e^{-st}dt$$

$$= \left[-\frac{\sin(t)e^{-st}}{s}\right]_0^\infty + \frac{1}{s}\int_0^\infty \cos(t)e^{-st}dt$$

$$= \frac{1}{s}\int_0^\infty \cos(t)e^{-st}dt$$

$$= \frac{1}{s}L\cos(s).$$

Similarly we compute $L\cos(s)$

$$L\cos(s) = \int_0^\infty \cos(t)e^{-st}dt$$

$$= \left[-\frac{\cos(t)e^{-st}}{s}\right]_0^\infty - \frac{1}{s}\int_0^\infty \sin(t)e^{-st}dt$$

$$= \frac{1 - L\sin(s)}{s}.$$

From above we have

$$L\sin(s) = \frac{1}{s}L\cos(s) = \frac{1}{s}\left[\frac{1 - L\sin(s)}{s}\right] = \frac{1 - L\sin(s)}{s^2}.$$

Solving for $L\sin(s)$, we find

$$L\sin(s) = \frac{1}{s^2+1},$$

and

$$L\cos(s) = sL\sin(s) = \frac{s}{s^2+1}.$$

Let $U(t)$ be the unit step function with step at $t = 0$. The unit step function at t_0 is

$$U_{t_0}(t) = U(t - t_0).$$

Proposition

$$L(U_{t_0}(t)f(t - t_0)) = e^{-st_0}L(f(t)).$$

Proof.

$$L(U(t - t_0)f(t - t_0)) = \int_0^\infty e^{-st}U(t - t_0)f(t - t_0)dt$$

$$= \int_{t_0}^\infty e^{-st}f(t - t_0)dt$$

$$= \int_0^\infty e^{-s(t+t_0)}f(t)dt$$

$$= e^{-st_0} L(f(t)).$$

Example. Suppose the forcing function on the right side of the following equation is an impule function at the point t_0. Then

$$x'' + k^2 x = \delta(t - t_0)$$

$$Lx(s)(s^2 + k^2) = e^{-t_0 s}$$

$$Lx(s) = \frac{e^{-t_0 s}}{s^2 + k^2} = e^{-t_0 s} L(\sin(t))$$

$$= L(U(t - t_0)\sin(t - t_0))$$

So the solution to the differential equation is

$$x(t) = U_{t_0} \sin(t - t_0),$$

assuming the initial conditions are $x(0) = 0, x'(0) = 0$.

Example.

$$y'''(t) - y''(t) + y'(t) - y(t) = F(t), y(0) = y'(0) = y''(0) = 0.$$

Applying the Laplace transform, we have

$$L(y(t))(s^3 - s^2 + s - 1) = L(y(t))(s - 1)(s^2 + 1) = L(F(t)).$$

So

$$L(y(t)) = L(F(t)) \frac{1}{(s - 1)(s^2 + 1)}.$$

Using partial fractions

$$2\frac{1}{(s - 1)(s^2 + 1)} = \frac{1}{s - 1} - \frac{s}{s^2 + 1} - \frac{1}{s^2 + 1}$$

So

$$2L^{-1}\frac{1}{(s - 1)(s^2 + 1)} = e^t - \cos(t) - \sin(t).$$

Let

$$g(t) = e^t - \cos(t) - \sin(t).$$

Then we have

$$2L(y(t)) = L(F(t))L(g(t)).$$

The Laplace transform of the convolution of two functions is the product of the transforms. Thus

$$2L(y(t)) = L(F * g(t)).$$

So

$$2y(t) = F * g(t) = \int_0^t F(t-\tau)g(\tau)d\tau = \int_0^t F(t-\tau)(e^\tau - \cos(\tau) - \sin(\tau)d\tau.$$

2 Bessel Functions

The Bessel function of the first kind of order ν is

$$J_\nu(t) = \sum_{m=0}^{\infty} \frac{(-1)^m t^{\nu+2m}}{2^{\nu+2m} m! \Gamma(\nu+m+1)}.$$

This may also be written as

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-t^2/4)^k}{k! \Gamma(\nu+k+1)}.$$

3 Relation to the Fourier Transform

We define the Fourier transform as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt.$$

Some authors define it with a constant multiplier in front. The Fourier inversion theorem is

$$f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega.$$

The double sided Laplace transform is

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt.$$

The single sided definition follows from this if $f(t)$ is zero for $t \leq 0$. Let $s = \phi + i\omega$. Then $F(s)$ is the Fourier transform of $g_\phi(t) = f(t)e^{-\phi t}$, that is

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} f(t) e^{-\phi t} e^{-i\omega t} dt \\ &= \hat{g}_\phi(\omega). \end{aligned}$$

For more on this see the section on the inversion of the transform.

4 Laplace Transform Table

`http://www.vibrationdata.com/math/Laplace_Transforms.pdf`

`or local file:`

`c:/je/pdf/Laplace_Transforms.pdf`

5 The Inversion of the Laplace Transform

We define the Fourier transform as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

Some authors define it with a constant multiplier in front. The Fourier inversion theorem is

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega.$$

The double sided Laplace transform is

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt.$$

Let $s = \phi + i\omega$. Then $F(s)$ is the Fourier transform of $g_\phi(t) = f(t)e^{-\phi t}$, that is

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} f(t)e^{-\phi t}e^{-i\omega t}dt \\ &= \hat{g}_\phi(\omega). \end{aligned}$$

Formally applying the Fourier inversion theorem, we have

$$f(t)e^{-\phi t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}_\phi(\omega)e^{i\omega t}d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{i\omega t}d\omega.$$

Then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{\phi t}e^{i\omega t}d\omega.$$

$$= \frac{1}{2\pi i}\int_{C_\phi} F(s)e^{st}ds,$$

where C_ϕ is the Bromwich contour defined by

$$\{\phi + i\omega : -\infty < \omega < \infty\}.$$

Note that i appears in the expression $2\pi i$ because

$$ds = id\omega.$$

In general we will find that if we define a closed curve consisting of a finite line of length $2R$ on the bromwich contour, and a semicircle of radius R to the left, then as R goes to infinity, the integral over the semicircle goes to zero, so that the total integral over the curve is equal to the integral on the Bromwich line, which is thus equal to $2\pi i$ times the residues of $F(s)e^{st}$ in the left halfspace bounded by the contour. Our inversion expression is therefore equal to the sum of the residues themselves. We get the single sided Laplace transform from the double when $f(t)$ is equal to zero for $t \leq 0$.
Example: Consider

$$F(s) = \frac{1}{s-1},$$

for $\Re(s) > 1$. The residue of $F(s)e^{st}$ is

$$\lim_{s\to 1}(s-1)F(s)e^{st} = e^t.$$

Therefore

$$f(t) = e^t.$$

Example: Consider

$$F(s) = \frac{1}{s^2+1} = \frac{1}{(s-i)(s+i)},$$

for $\Re(s) > 0$. The residues of $F(s)e^{st}$ are

$$\lim_{s\to i}(s-i)F(s)e^{st} = \frac{e^{it}}{2i},$$

and

$$\lim_{s\to -i}(s+i)F(s)e^{st} = \frac{e^{-it}}{-2i},$$

Therefore

$$f(t) = \frac{e^{it} - e^{-it}}{2i} = \sin(t).$$

6 The Laplace Transform in Maple

Import this from the maple.tex document.

7 Solving a Differential Equation With The Laplace Transform Using Maple

This section has been made compatible with Maple 12. We read the following file into Maple:

```
% cat mlaplace
with(invtrans)
de:=diff(y(x),x,x)+2*diff(y(x),x)+y(x) = sin(2*x);
dsolve({de,y(0)=1,D(y)(0)=1},y(x));
laplace(de,x,s);
subs(laplace(y(x),x,s)=G,%);
solve(",G);
subs({D(y)(0)=1,y(0)=1},%);
invlaplace(%,s,x);
```

The above code was pasted into Maple 12. The laplace transform wouldnot work, until I blundered onto some information that the laplace transform and inverse laplace transform are in the inttrans package that must be loaded. Also the previous expression representation had to be changed to per cent sign from the double quote sign. Maple 12 gives equivalent though different forms for the results calculated by Maple 5, and which are listed here. The session is as follows:

```
> de:=diff(y(x),x,x)+2*diff(y(x),x)+y(x) = sin(2*x);
```

$$de := \left(\frac{\partial^2}{\partial x^2}\,\mathrm{y}(x)\right) + 2\left(\frac{\partial}{\partial x}\,\mathrm{y}(x)\right) + \mathrm{y}(x) = \sin(2\,x)$$

```
> dsolve({de,y(0)=1,D(y)(0)=1},y(x));
```

$$\mathrm{y}(x) = -\frac{4}{25}\cos(2\,x) - \frac{3}{25}\sin(2\,x) + \frac{29}{25}\,\mathrm{e}^{(-x)} + \frac{12}{5}\,\mathrm{e}^{(-x)}\,x$$

```
> laplace(de,x,s);
```

$$(\,\mathrm{laplace}(\,\mathrm{y}(x), x, s\,)\,s - \mathrm{y}(0)\,)\,s - \mathrm{D}(y)(0) + 2\,\mathrm{laplace}(\,\mathrm{y}(x), x, s\,)\,s \\ - 2\,\mathrm{y}(0) + \mathrm{laplace}(\,\mathrm{y}(x), x, s\,) = 2\,\frac{1}{s^2+4}$$

```
> subs(laplace(y(x),x,s)=G,%);
```

$$(\,G\,s - \mathrm{y}(0)\,)\,s - \mathrm{D}(y)(0) + 2\,G\,s - 2\,\mathrm{y}(0) + G = 2\,\frac{1}{s^2+4}$$

```
> solve(%,G);
```

$$-\frac{-s\,\mathrm{y}(0) - \mathrm{D}(y)(0) - 2\,\mathrm{y}(0) - 2\,\frac{1}{s^2+4}}{s^2+2\,s+1}$$

```
> subs({D(y)(0)=1,y(0)=1},%);
```

$$-\frac{-s - 3 - 2\,\frac{1}{s^2+4}}{s^2+2\,s+1}$$

```
> invlaplace(%,s,x);
```

$$-\frac{4}{25}\cos(2\,x) - \frac{3}{25}\sin(2\,x) + \frac{29}{25}\,\mathrm{e}^{(-x)} + \frac{12}{5}\,\mathrm{e}^{(-x)}\,x$$

The solution using dsolve, and the solution using the Laplace transform method are the same.

8 Solving Circuit Problems With the Laplace Transform

Resistor Capacitor Circuit Let a circuit consist of a constant voltage source V be in series with a Resistor R and a capacitor C. The voltage loop equation is

$$Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau + \frac{q_0}{C} = V,$$

where $i(t)$ is the current, and q_0 is the initial charge on the capacitor. We have

$$i(t) + \frac{1}{RC}\int_0^t i(\tau)d\tau + \frac{q_0}{RC} = \frac{V}{R}.$$

Taking the Laplace Transform

$$Li(s) + \frac{1}{RC}\frac{Li(s)}{s} + \frac{q_0}{RC}\frac{1}{s} = \frac{V}{R}\frac{1}{s}.$$

Then

$$Li(s)\left(1 + \frac{1}{RCs}\right) = \frac{CV - q_0}{RCs}$$

and so

$$Li(s) = \frac{CV - q_0}{RCs + 1} = \frac{CV/(RC) - q_0/(RC)}{s + 1/(RC)}$$

Then

$$Li(s) = (V/R - q_0/(RC))\frac{1}{s + 1/(RC)}.$$

So taking the inverse transform

$$i(t) = (V/R - q_0/(RC))e^{-t/(RC)}.$$

To find the charge we integrate

$$q(t) = (V/R - q_0/(RC))\int e^{-t/(RC)}$$

$$= (V/R - q_0/(RC))(-RC)e^{-t/(RC)} + K,$$

where K is a constant. So

$$q(t) = (q_0 - VC)e^{-t/(RC)} + K$$

At zero

$$q_0 = q(0) = (q_0 - VC) + K,$$

so $K = VC$. Finally

$$q(t) = q_0 e^{-t/(RC)} + VC(1 - e^{-t/(RC)}).$$

9 Bibliography