The Essence of Mechanics

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Contents

1 D'Alembert's Principle

A system of mass points may be thought of as a single point in a higher dimensional space. For example, consider a rigid body consisting of n mass points. It has $3n$ coordinates. If we select three noncolinear points of the body, $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, then the position of the kth point is uniquely located by the distances to these three points. There are three constraint equations

$$(x_k^1 - x_1^1)^2 + (x_k^2 - x_1^2)^2 + (x_k^3 - x_1^3)^2 = d_{k1}^2,$$

$$(x_k^1 - x_2^1)^2 + (x_k^2 - x_2^2)^2 + (x_k^3 - x_2^3)^2 = d_{k2}^2,$$

and

$$(x_k^1 - x_3^1)^2 + (x_k^2 - x_3^2)^2 + (x_k^3 - x_3^3)^2 = d_{k3}^2,$$

which fix the position of the kth point. Let E_{ij} be the equation expressing the distance constraint between the ith and the jth point. This equation E_{ij} is

$$(x_i^1 - x_j^1)^2 + (x_i^2 - x_j^2)^2 + (x_i^3 - x_j^3)^2 = d_{ij}^2,$$

The distances in the equations, the d_{ij}, are constant because the bodies are rigid. Then we may find six coordinates to specify the rigid body. For example, we may use the coordinates

$$(x_1^1, x_1^2, x_1^3, x_2^1, x_2^2, x_3^1).$$

From equations E_{12}, E_{13}, and E_{23}, we can solve for x_2^3, x_3^2, and x_3^3. So the three base points $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, are determined by the six coordinates. All other points can be located from the three base points. We have shown that the possible positions of the body represent a six dimensional manifold embedded in a space of dimension $3n$. And the six dimensional point, which represents the body, is constrained to lie on a surface defined by a large system of equations. Further constraints can be placed on the six coordinates, say that a point on the body must lie on some surface in three space. This would reduce the dimension of the manifold by one. The dimension of the constraint manifold for a body is called the number of degrees of freedom of the body. We may use any of a variety of local coordinate systems to represent the manifold. Such coordinates are called generalized coordinates.

Given n mass points of a rigid body, with the kth position vector $\mathbf{r}_k$, we have

$$\mathbf{F}_k^t - m_k \ddot{\mathbf{r}}_k = 0,$$

where $\mathbf{F}_k^t$ is the total force on the mass point. But part of this force may be a constraint force $\mathbf{F}_k^c$, orthogonal to the constraint surface. So if $\delta\mathbf{r}_k$ is a constrained displacement of the kth point, then

$$\mathbf{F}_k^c \cdot \delta\mathbf{r}_k = 0.$$

That is, the virtual work done by the constraint force is zero. Virtual work is defined to be the increment of work due to moving the system in an arbitrary tangent direction in the constraint manifold. Historically this work is called virtual, because real finite displacements in a tangent direction may be incompatable with the constaints. For example, suppose one end of a rod is constrained to lie on a circle, and the other end on a straight line. Then moving the first end along a line tangent to the circle by an arbitrary distance, and the other end along the straight line by an arbitrary distance, is in general incompatible with the length of the rod remaining constant. So such a displacement is virtual rather than real.

There may be internal forces between mass points which maintain the rigid structure of the body. By Newton's third law, such forces occur in pairs, so that even if these forces have components in the displacement directions, the net virtual work will be zero. This suggests that

$$\sum_{k=1}^{n} (\mathbf{F}_k - m_k \ddot{\mathbf{r}}_k) \cdot \delta\mathbf{r}_k = 0,$$

where $\mathbf{F}_k$ is the external force. $\mathbf{F}_k$ is the total force, minus the forces of constraint and the internal forces. This generalizes the usual conditions for static equilibrium to a "dynamic" equilibrium, and is called D'Alembert's principle. Using this principal we can solve problems without knowing the forces of constraint.

A simple form of a constraint is a mass point on a physical surface. A virtual displacement of a point P is a tangent vector in the tangent space at P. A simple case is the block sliding on an incline plane, with the incline plane as the constraint surface. This is shown in Figure 1.

We introduce coordinate vectors

$$\mathbf{u} = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}$$

and

$$\mathbf{v} = \sin(\theta)\mathbf{i} - \cos(\theta)\mathbf{j}.$$

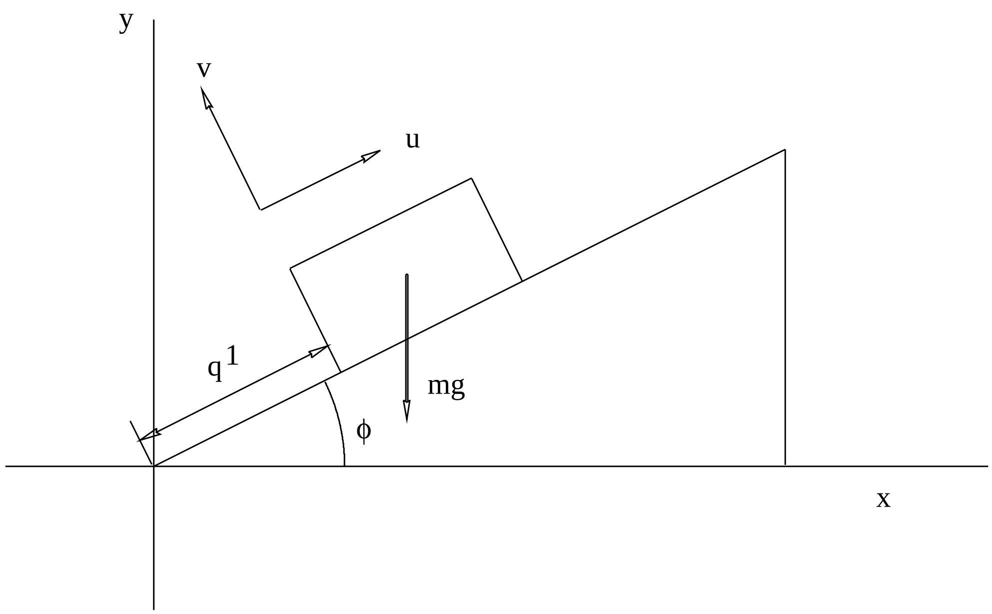

Figure 1: Incline plane as surface constraint.

The position vector of the block may be written as

$$\mathbf{r} = q^1\mathbf{u}.$$

The virtual displacement is

$$\delta\mathbf{r} = h\mathbf{u},$$

where h is an arbitrary length. Let $\mathbf{F}$ be a gravitational force

$$\mathbf{F} = -mg\mathbf{j} = -mg(\sin(\theta)\mathbf{u} + \cos(\theta)\mathbf{v}).$$

D'Alembert's equation is

$$(-mg(\sin(\theta)\mathbf{u} + \cos(\theta)\mathbf{v}) - m\ddot{\mathbf{r}}) \cdot h\mathbf{u} =$$

$$-m(g\sin(\theta) - \ddot{q}^1)h = 0.$$

So the equation of motion is

$$\ddot{q}^1 = -\sin(\theta)g.$$

It was not necessary to find the force of constraint. The force of constraint here, is the surface normal force supporting the block. Such force is easy to compute in this example, but forces of constraint are not necessarily easy to compute in general.

In summary, general contrained motion, with n degrees of freedom, is motion on a k-dimensional manifold, where k equals the degree of freedom. The manifold coordinates are called generalized coordinates.

2 Hamilton's Principle of Least Action

Hamilton's principle can be derived from D'Alembert's Principle. We integrate along a curve connecting two points. We let the curve vary between the points and call the variation of the kth position vector $\delta\mathbf{r}_k$. Then integrating D'Alembert's equation over time

$$\sum_{k=1}^{n}\int_{t_1}^{t_2}(\mathbf{F}_k - m_k\frac{d\mathbf{v}_k}{dt})\cdot\delta\mathbf{r}_k = 0$$

$$= \sum_{k=1}^{n}(-\delta\int_{t_1}^{t_2}V_k dt - \int_{t_1}^{t_2} m_k\frac{d\mathbf{v}_k}{dt}\cdot\delta\mathbf{r}_k dt)$$

$$= \sum_{k=1}^{n}(-\delta\int_{t_1}^{t_2}V_k dt - (m_k\mathbf{v}_k\cdot\delta\mathbf{r}_k)|_{t_1}^{t_2}$$

$$+ \int_{t_1}^{t_2} m_k\mathbf{v}_k\cdot\delta\mathbf{v}_k dt)$$

$$= \sum_{k=1}^{n}(-\delta\int_{t_1}^{t_2}V_k dt - 0 + \int_{t_1}^{t_2}\frac{m_k}{2}\delta(v_k)^2 dt)$$

$$= \sum_{k=1}^{n}(-\delta\int_{t_1}^{t_2}V_k dt + \delta\int_{t_1}^{t_2}T_k dt)$$

$$= \delta\int_{t_1}^{t_2}L dt.$$

We have used integration by parts. Forces are equal to the negative gradient of a potential function V. T is the kinetic energy, and $L = T - V$ is the Lagrangian. The integral of L over time is called the action. The unit of action is energy times time. Hamilton's principle states that motion takes place along a time path that minimizes action. We can write L as a function of $q_1, ..., q_k, \dot{q}_1, ..., \dot{q}_k$, and t, where k is the number of degrees of freedom of the system. From the calculus of variations, necessary conditions for an extrema of this integral are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, i = 1, .., k.$$

These are called Lagrange's equations of motion. They can also be deduced from tensor analysis.

3 Lagrange Equations Via Tensor Analysis

Let $x_1, ..., x_n$ be coordinates of a manifold and $\dot{x}_1, ...\dot{x}_n$ the components of a tangent vector. If $x_1, ..., x_n$ are functions of a time parameter t, then $\dot{x}_1, ...\dot{x}_n$ are curve velocities. Let F be a function of $x_1, ..., x_n, \dot{x}_1, ...\dot{x}_n$ and t. Let $y_1, ..., y_n$ be a second set of manifold coordinates. They are each a function of the first set of coordinates. Differentiating with respect to the parameter t we obtain the transformation rule for tangent vector components,

$$\dot{y}^i = \frac{dy^i}{dt} = \frac{\partial y^i}{\partial x^j}\frac{dx^j}{dt} = \frac{\partial y^i}{\partial x^j}\dot{x}^j.$$

So the y velocities are each a function of both the x values and the x velocities. But note that the first factor on the right is a function of only the x's. We can take partial derivatives with respect to an x^k and with respect to an $\dot{x}^k$. We have

$$\frac{\partial \dot{y}^i}{\partial x^k} = \frac{\partial^2 y^i}{\partial x^j \partial x^k}\dot{x}^j.$$

And we have

$$\frac{\partial \dot{y}^i}{\partial \dot{x}^k} = \frac{\partial y^i}{\partial x^k}.$$

Suppose we have a function G of $y_1, ..., y_n, \dot{y}_1, ...\dot{y}_n$ and t so that

$$F(x_1, ..., \dot{x}_n, t) = G(y_1, ..., \dot{y}_n, t),$$

that is F and G have the same physical value at a manifold point, but are in general different functions of their respective variables. We have

$$\frac{\partial F}{\partial x^k} = \frac{\partial G}{\partial x^k} = \frac{\partial G}{\partial y^p}\frac{\partial y^p}{\partial x^k} + \frac{\partial G}{\partial \dot{y}^p}\frac{\partial \dot{y}^p}{\partial x^k}$$

$$= \frac{\partial G}{\partial y^p}\frac{\partial y^p}{\partial x^k} + \frac{\partial G}{\partial \dot{y}^p}\frac{\partial^2 \dot{y}^p}{\partial x^j \partial x^k}\dot{x}^j.$$

And we have

$$\frac{\partial F}{\partial \dot{x}^k} = \frac{\partial G}{\partial \dot{x}^k} = \frac{\partial G}{\partial y^p}\frac{\partial y^p}{\partial \dot{x}^k} + \frac{\partial G}{\partial \dot{y}^p}\frac{\partial \dot{y}^p}{\partial \dot{x}^k} = 0 + \frac{\partial G}{\partial \dot{y}^p}\frac{\partial \dot{y}^p}{\partial \dot{x}^k} = \frac{\partial G}{\partial \dot{y}^p}\frac{\partial \dot{y}^p}{\partial x^k}.$$

Differentiating with respect to time

$$\frac{d}{dt}(\frac{\partial F}{\partial \dot{x}^k}) = \frac{d}{dt}\frac{\partial G}{\partial \dot{y}^p}\frac{\partial y^p}{\partial x^k} + \frac{\partial G}{\partial \dot{y}^p}\frac{d}{dt}\frac{\partial \dot{y}^p}{\partial x^k} =$$

$$= \frac{d}{dt}\frac{\partial G}{\partial \dot{y}^p}\frac{\partial y^p}{\partial x^k} + \frac{\partial G}{\partial \dot{y}^p}\frac{\partial^2 \dot{y}^p}{\partial x^j \partial x^k}\dot{x}^j.$$

Then we have

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{x}^i} - \frac{\partial F}{\partial x^i} =$$

$$(\frac{d}{dt}\frac{\partial G}{\partial \dot{y}^i} - \frac{\partial G}{\partial y^i})\frac{\partial y^i}{\partial x^k}.$$

Thus

$$\{\frac{d}{dt}\frac{\partial F}{\partial \dot{x}^i} - \frac{\partial F}{\partial x^i}\}$$

is a covariant tensor.

Suppose we have a system of m particles with $n = 3m$ Euclidean coordinates. Suppose that all forces are due to a potential function V. The force field could be a sum of external fields and of internal fields caused by the other particles. Let $L = T - V$. The force in the direction of the kth coordinate (which might correspond to the force on the jth particle in say the z direction) is

$$F_k = -\frac{\partial V}{\partial x^k}$$

The kinetic energy of the system is

$$T = \frac{1}{2}\sum_{i=1}^{n}(\dot{x}^i)^2 m_i$$

The Lagrangian is

$$L = T - V.$$

Hence

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^k} - \frac{\partial L}{\partial x^k} = \ddot{x}^k - F_k = 0.$$

Now L is a scalar invariant, because the kinetic energy T and the potential energy V are scalar invariants. So

$$\{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i}\}$$

is a covariant tensor. We have shown that in Euclidean coordinates

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0.$$

So for all coordinates q_i

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0.$$

Now if the system has $j < n$ degrees of freedom, then it is possible to find coordinates $q_1, ..., q_n$, so that all coordinates are determined by the first j. Then the equations of motion of the system are

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}^i}) - \frac{\partial L}{\partial q^i} = 0, i = 1, ..., j.$$

Of course T and V must be scalar invariants, so if for example we have a rigid body with six degrees of freedom then T and V must represent the whole kinetic energy and potential energy of the system, not just the inergies associated with say three particles, part of whose coordinates make up the coordinates of the rigid body. We started out by distributing the total energies among the individual particles.

4 Generalized Forces

The kinetic energy is a scalar invariant, so

$$Q_r = \frac{d}{dt}(\frac{\partial T}{\partial \dot{q}^i}) - \frac{\partial T}{\partial q^i}$$

is a tensor. But in Euclidean coordinates Q is a force, so Q_r is called the rth component of the generalized force.

The differential of work is also a scalar invariant

$$dW = Q_r dq^r.$$

5 Hamilton's Equations

Define

$$p_i = \frac{\partial L}{\partial \dot{q}^i}.$$

In Euclidean coordinates p is the momentum, so for generalized coordinates p is called the generalized momentum.

We can solve the following equations

$$p_j = \frac{\partial L}{\partial \dot{q}_j}(q, \dot{q}, t), j = 1, .., n$$

for $\dot{q}^1, ..., \dot{q}^n$. Then each $\dot{q}^j$ is a function of $q^1, ..q^n, p_1, ..., p_n$. Define the Hamiltonian

$$H(q, p, t) = p_j \dot{q}^j - L.$$

Differentiating with respect to q^i,

$$\frac{\partial H}{\partial q^i} = p_j \frac{\partial \dot{q}^j}{\partial q^i} - [\frac{\partial L}{\partial q^i} + \frac{\partial L}{\partial \dot{q}^j} \frac{\partial \dot{q}^j}{\partial q^i}]$$

$$= p_j \frac{\partial \dot{q}^j}{\partial q^i} - [\frac{\partial L}{\partial q^i} + p_j \frac{\partial \dot{q}^j}{\partial \dot{q}^i}]$$

$$= p_j \frac{\partial \dot{q}^j}{\partial q^i} - \frac{\partial L}{\partial q^i} - p_j \frac{\partial \dot{q}^j}{\partial q^i} = -\frac{\partial L}{\partial q^i}$$

Using Lagranges equation the last result becomes

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = -\frac{dp_i}{dt}.$$

So we have the first equation of Hamilton,

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}.$$

Differentiating with respect to p_i,

$$\frac{\partial H}{\partial p_i} = \dot{q}^i + p_j \frac{\partial \dot{q}^j}{\partial p_i} - (\frac{\partial L}{\partial \dot{q}^j} \frac{\partial \dot{q}^j}{\partial p_i} + 0) =$$

$$\dot{q}^i + p_j \frac{\partial \dot{q}^j}{\partial p_i} - p_j \frac{\partial \dot{q}^j}{\partial p_i} = \dot{q}^i.$$

So we have the second equation of Hamilton,

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p^i}.$$

For a system of particles in Euclidean coordinates we have

$$p_i \dot{q}^i = p_i v^i = m_i (v^i)^2 = 2T,$$

so

$$H = p_i \dot{q}^i - L = 2T - (T - V) = T + V,$$

which is the total energy of the system.

6 The Kinetic Energy of a Rigid Body

Let c be the center of mass of a moving rigid body. Let r be a vector to a point of the body. Let

$$p = r - c,$$

be the position of the point relative to the center of mass. The kinetic energy of the body is

$$\begin{aligned} k &= \frac{1}{2} \sum m_i \|\dot{r}_i\|^2 \\ &= \frac{1}{2} \sum m_i (\dot{p}_i + \dot{c}) \cdot (\dot{p}_i + \dot{c}) \\ &= \frac{1}{2} \sum m_i (\dot{p}_i \cdot \dot{p}_i + 2\dot{p}_i \cdot \dot{c} + \dot{c} \cdot \dot{c}) \\ &= \frac{1}{2} \sum m_i \|\dot{p}_i\|^2 + \dot{c} \sum m_i \dot{p}_i + \frac{M}{2} \|\dot{c}\|^2, \end{aligned}$$

where

$$M = \sum m_i,$$

is the total mass of the body. Now p_i is a vector from the center of mass of the body to the component particle of mass m_i, so that

$$\sum m_i \dot{p}_i$$

vanishes. The kinetic energy is

$$k = \frac{1}{2} \sum m_i \|\dot{p}_i\|^2 + \frac{M}{2} \|\dot{c}\|^2.$$

The kinetic energy of a rigid body with mass M is equal to the kinetic energy of a mass point with mass M, which is located at the center of mass of the body, plus the relative kinetic energy of the mass points with respect to a coordinate system located at the center of mass.

Proposition. The velocity of a point located on a fixed line segment in a rigid body is a linear combination of the velocities of the endpoints.

Proof. Let p be a point on a line with endpoints p_1 and p_2. Then

$$p = up_1 + (1-u)p_2,$$

for some parameter u. Differentiating, we have

$$\frac{dp}{dt} = u\frac{dp_1}{dt} + (1-u)\frac{dp_2}{dt}.$$

7 Mass Properties of an Assembly of Simple Shapes

There is a computer program called **mi.ftn** that computes the moment of inertia of an assembly of objects. It was used to give a dummy device specific mass properties by drilling holes in the potting material filler and adding lead weights. Also it was used to compute torque properties of a piezo electric motor. This required calculating the moment of inertia of a mechanism that included a ball bearing. See the document **massprop.tex**. Below we present relevant material. Also there is a program called **inertia.ftn** that computes the inertia tensor of a set of mass points. See the document

8 Changing the Center of Gravity of an Assembly

Let R_A be the center of gravity of a mass A.

(1) Given the center of gravity of a second mass element B the center of gravity of the combination is

$$R_C = \frac{M_a R_a + M_b R_b}{M_c}.$$

(2) Given a required new center of gravity R_c, the center of gravity of the second mass element B must be

$$R_b = \frac{M_c R_c - M_a R_a}{M_b}.$$

9 Properties of a Rectangular Block

The moment of inertia of a block, which is defined by $-a \leq x \leq a, -b \leq y \leq b, -c \leq z \leq c$, about an axis in the z direction is

$$I_{zz} = \int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2) \rho dx dy dz$$

$$= m \frac{a^2 + b^2}{3}.$$

Similarly;

$$I_{xx} = \int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (y^2 + z^2) \rho dx dy dz$$
$$= m \frac{b^2 + c^2}{3}.$$

$$I_{yy} = \int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + z^2) \rho dx dy dz$$

$$= m \frac{a^2 + c^2}{3}.$$

10 Properties of a Cylinder

Let a cylinder have density ρ, radius R, and height h. The moment of inertia about the cylinder axis, which we take to be the z direction is

$$I_{zz} = \int_{0}^{R} h \rho r^2 2 \pi r dr$$

$$= h \rho 2 \pi R^4 / 4$$

$$= h \rho \pi R^4 / 2$$

$$= \frac{MR^2}{2},$$

were M is the mass of the cylinder

$$M = h\rho\pi R^2.$$

The moment of inertia about the x axis, which is perpendicular to the cylinder axis is

$$I_{xx} = \int\int\int (y^2 + z^2) dm$$

$$= \int_{-h/2}^{h/2} \int_0^{2\pi} \int_0^R (r\sin(\theta)^2 + z^2)\rho r dr d\theta dz$$

$$= \rho \int_{-h/2}^{h/2} \int_0^{2\pi} \int_0^R (r^3 \sin^2(\theta) + rz^2) dr d\theta dz$$

$$= \rho \int_{-h/2}^{h/2} \int_0^{2\pi} \left[\frac{r^4}{4}\sin^2(\theta) + \frac{r^2}{2}z^2\right]_0^R d\theta dz$$

$$= \rho \int_{-h/2}^{h/2} \int_0^{2\pi} (\frac{R^4}{4}\sin^2(\theta) + \frac{R^2}{2}z^2) d\theta dz$$

$$= \rho \int_{-h/2}^{h/2} \int_0^{2\pi} (\frac{R^4}{4}\frac{1 - \cos(2\theta))}{2} + \frac{R^2}{2}z^2) d\theta dz$$

$$= M\left[\frac{R^2}{4} + \frac{h^2}{12}\right]$$

11 Properties of a Cylindrical Ring

The moment of inertia of a cylindrical ring is the moment of inertia of the difference between the outer cylinder of radius R_2 and the inner hole cylinder of radius R_1:

$$I_{ring} = \frac{\rho\pi}{2} h(R_2^4 - R_1^4).$$

The moment of inertia in the transverse direction is also a difference of cylindrical moments.

12 Properties of a Sphere

The moment in this case can be computed as an infinite sum of cylinders of height dx:

$$\begin{aligned} I_{sphere} &= \int_{-R}^{R} \frac{\rho\pi y^4}{2} dx \\ &= \frac{\rho\pi}{2} \int_{-R}^{R} (R^2 - x^2)^2 dx \\ &= \frac{\rho\pi}{2} \int_{-R}^{R} (R^4 - 2R^2x^2 + x^4) dx \\ &= \frac{\rho\pi}{2} \left[2R^5 - \frac{4}{3}R^5 + \frac{2}{5}R^5 \right] \\ &= \rho\pi R^5 \frac{8}{15} = \frac{2}{5}MR^2. \end{aligned}$$

13 Properties of a Torus

14 Properties of a Torus Truncated By a Cylinder

Consider the moment of inertia of a toroidal section. An example of a toroidal section is the volume removed from a cylindrical ring to form a bearing race. Consider a plane cross section of the torus with coordinates (x, y). If we take the volume to be a set of rings of depth dx, then the moment of inertia about the torus axis is

$$\frac{\rho\pi}{2} \int_{-x_0}^{x_0} (y^4 - y_0^4) dx,$$

where

$$y = r_2 + \sqrt{r_1^2 - x^2}.$$

The big torus radius is r_2, and the small torus radius is r_1. We assume that $x_0 < r_1$. The radius of the truncating cyclinder is y_0, which we assume is greater than r_2. We have

$$y_0 = r_2 + \sqrt{r_1^2 - x_0^2},$$

so that

$$(y_0 - r_2)^2 = r_1^2 - x_0^2.$$

Then

$$x_0 = \sqrt{r_1^2 - (y_0 - r_2)^2}.$$

Let us consider the integral

$$\int_{-x_0}^{x_0} y^4 dx.$$

We introduce the variable

$$u = \frac{x}{r_1}.$$

Then

$$du = \frac{dx}{r_1},$$

and

$$y(u) = r_1(\beta + \sqrt{1-u^2}),$$

where

$$\beta = \frac{r_2}{r_1}.$$

Then

$$\int_{-x_0}^{x_0} y^4 dx = r_1^3 \int_{-u_0}^{u_0} (\beta + \sqrt{1-u^2})^4 du.$$

The indefinite integral equals

$$(2\beta^3 + \frac{3}{2}\beta)\arcsin(u) + \frac{u^5}{5} - (\frac{2}{3} + 2\beta^2)u^3$$

$$+u(\beta^2(\beta^2+6)+1) + \beta u\sqrt{1-u^2}(2\beta^2 + \frac{5}{2} - u^2)$$

So definite integral equals twice J, where

$$J = (2\beta^3 + \frac{3}{2}\beta)\arcsin(u_0) + \frac{u_0^5}{5} - (\frac{2}{3} + 2\beta^2)u_0^3$$

$$+u_0(\beta^2(\beta^2+6)+1) + \beta u_0\sqrt{1-u_0^2}(2\beta^2 + \frac{5}{2} - u_0^2),$$

where

$$u_0 = \frac{x_0}{r_1}.$$

Then the moment of inertia of the toroidal slice is

$$\rho\pi(r_1^3 J - x_0 y_0^4).$$

15 Properties of a General Volume of Revolution

If a volume is generated by revolving an area in the upper half of the xy plane about the x axis, then the moment of inertia about the x axis is

$$\int \rho 2\pi y^3 dxdy.$$

This can be converted to a line integral using Green's Lemma:

$$\int udx + vdy = \int\int (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dxdy,$$

which is a special case of Stokes Theorem. One choice is

$$u = -\frac{\rho\pi}{2}y^4,$$

and

$$v = 0.$$

16 The Parallel Axis Theorem

The moment of inertia about an axis that does not pass through the center of gravity of a body is equal to the moment of inertia about a parallel axis that passes through the center of gravity plus d^2M where d is the distance between the two parallel axes, and M is the mass of the body. Without loss of generallity we shall assume that the parallel axes are parallel to the x axis and that the center of gravity of the body lies at $(0, y_0, z - 0)$, so that $\sqrt{y_0^2 + z_0^2} = d$. We introduce a center of mass coordinate system

$$x' = x, y' = y - y_0, z' = z - z_0$$

Let the moment of inertia of the body about its center of mass be

$$I_{xxcm} = \int\int (y'^2 + z'^2)dm$$

Then the moment of inertia in the unprimed system is

$$I_{xx} = \int\int (y^2 + z^2)dm$$

$$= \int \int ((y' + y_0)^2 + (z' + z_0)^2) dm$$

$$= I_{xxcm} + \int \int 2y' y_0 dm \int \int 2z' z_0 dm + \int \int (y_0^2 + z_0^2) dm$$

$$= I_{xxcm} + 0 + 0 + d^2 M = I_{xxcm} + d^2 M.$$

17 The Inertia Tensor, And Its Relation to the Moments of Inertia

The inertia tensor has components

$$I^{jk} = \int x^j x^k dm,$$

where dm is the element of mass. The inertia tensor is symmetric and represents a quadratic form. It can be diagonalized by rotating the coordinate system. In its diagonalized form, the terms on the diagonal are the principle inertia components. The directions of the eigenvectors are the principle directions. If the body is located with its center of mass at the origin and if the body has a plane of symmetry about one of the coordinate planes, say the plane 23, so that for every point in the body (x, y, z) there is a corresponding point $(-x, y, z)$, then clearly $I^{23} = 0$. That is, the corresponding so called products of inertia are zero. Now I can always be diagonalized. When I is diagonal, then of course the products of inertia are zero. But clearly the body need not have any symmetry.

There is a related "rotational moment of inertia" tensor J. Tensor J has components

$$J^{ik} = t(I)\delta_{ik} - I^{ik}.$$

In some books (e.g. Goldstein), J, rather than I, is called the inertia tensor.

The moment of inertia about an axis is the sum over all elements of mass times the square of the distance of each mass element to the axis.

The moment of inertia about any axis through the center of mass may be found from the inertia tensor. If f is the bilinear form corresponding to the symmetric matrix J, then the moment of inertia about a unit vector u is $f(u, u)$. See the section below called **Angular Momentum and the Inertia Tensor** for additional details.

18 The General Definition of Instantaneous Angular Velocity

Consider a rigid body, with a point at the origin fixed. Let P be a point in the body and let u_1, u_2, u_3 and u'_1, u'_2, u'_3 be two bases of a vector space. The unprimed basis is fixed in space. The primed basis is fixed in the body. Then

$$u'_i = R(u_i),$$

where R is an orthogonal transformation, which varies with time. Then the coordinates of P are related by

$$x'_i = a_{ij}x_j.$$

where (a_{ij}) is the inverse of the matrix $(\bar{a}_{ij})$ of R. By differentiating, and noting that the components of P are constant in the primed system, we find that (Lass page 46)

$$\frac{dx_i}{dt} = w_{ij}x_j,$$

for

$$w_{ij} = -\bar{a}_{ik}\frac{da_{kj}}{dt}.$$

By differentiating the norm of P, which is invariant, we find that (w_{ij}) is skew symmetric, that is

$$w_{ij} = -w_{ji}.$$

So (w_{ij}) can be represented as an axial vector. We have

$$c(w) = \begin{bmatrix} w_{32} \\ -w_{31} \\ w_{21} \end{bmatrix} = \begin{bmatrix} w_{32} \\ w_{13} \\ w_{21} \end{bmatrix}$$

and

$$v = \frac{dc(P)}{dt} = c(w) \times c(P).$$

w is the angular velocity vector. It transforms like a vector under an orthogonal transformation.

Example. Let rotation transformation R have matrix

$$(\bar{a}_{ij}) = \begin{bmatrix} c(t^2) & -s(t^2) & 0 \\ s(t^2) & c(t^2) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So

$$(a_{ij}) = (\bar{a}_{ij})^T = \begin{bmatrix} c(t^2) & s(t^2) & 0 \\ -s(t^2) & c(t^2) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$\frac{da_{ij}}{dt} = 2t \begin{bmatrix} -s(t^2) & c(t^2) & 0 \\ -c(t^2) & -s(t^2) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$(w_{ij}) = -2t \begin{bmatrix} c(t^2) & -s(t^2) & 0 \\ s(t^2) & c(t^2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s(t^2) & c(t^2) & 0 \\ -c(t^2) & -s(t^2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= -2t \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So

$$c(w) = \begin{bmatrix} 0 \\ 0 \\ 2t \end{bmatrix}.$$

Suppose $P = u_1'$. Let $c(P)$ be the coordinate vector of P in the unprimed system and $c'(P)$ the coordinate vector in the primed system. Then

$$c(P) = (\bar{a}_{ij})c'(P)$$

$$= (\bar{a}_{ij}) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c(t^2) \\ s(t^2) \\ 0 \end{bmatrix}.$$

Now

$$\frac{dc(P)}{dt} = 2t \begin{bmatrix} -s(t^2) \\ c(t^2) \\ 0 \end{bmatrix}.$$

This is the velocity computed with matrices. The velocity computed with the cross product is

$$c(w) \times c(P) = \begin{bmatrix} 0 \\ 0 \\ 2t \end{bmatrix} \times \begin{bmatrix} c(t^2) \\ s(t^2) \\ 0 \end{bmatrix}$$

$$= 2t \begin{bmatrix} -s(t^2) \\ c(t^2) \\ 0 \end{bmatrix}.$$

We get the same result with either method.

19 Angular Momentum and the Inertia Tensor

The angular momentum vector is defined as (Lass p.115)

$$L = \int r \times v dm,$$

where r is the position, v the velocity and m the mass. The kth component is

$$L_k = \int \epsilon_{ijk} x^i \dot{x}^j dm.$$

Then

$$\frac{dL_k}{dt} = \int \epsilon_{ijk} x^i \ddot{x}^j dm + \int \epsilon_{ijk} \dot{x}^i \dot{x}^j dm$$
$$= \int \epsilon_{ijk} x^i \ddot{x}^j dm.$$

The second term vanishes because the cross product of parallel vectors is zero. The torque vector T is

$$\int r \times df = \int r \times \frac{d^2 r}{dt^2} dm.$$

The kth component of the torque is

$$T_k = \int \epsilon_{ijk} x^i \ddot{x}^j dm.$$

Thus $dL/dt = T$. Because

$$\dot{x}^j = w^j_p x^p,$$

we have

$$L_k = \int \epsilon_{ijk} x^i w^j_p x^p dm = \epsilon_{ijk} \int x^i x^p dm$$
$$= \epsilon_{ijk} w^j_p I^{ip},$$

where I is the inertia tensor. The components of the inertia tensor are

$$I^{ip} = \int x^i x^p dm.$$

The moment of inertia about an arbitrary axis can be found from the inertia tensor. Given a straight line through the origin, let p be a function, whose value at a point is the distance from the point to the line. The moment of inertia about the line is then defined as (McConnel p.233)

$$\int p^2 dm.$$

Suppose u is a unit vector in the direction of the line. Let t be the trace. Then

$$p^2 = x^i x^i - (r \cdot u)^2$$
$$= \delta_{ij} x^i x^j - (\delta_{ij} x^i u^j)^2.$$

So the moment of inertia is

$$= \delta_{ij} I^{ij} - \delta_{ij} \delta_{kp} u^j u^p I^{ik}$$
$$= t(I) - I^{ik} u^i u^k$$
$$= t(I) \delta_{ik} u^i u^k - I^{ik} u^i u^k$$
$$= (t(I) \delta_{ik} - I^{ik}) u^i u^k$$
$$= J^{ik} u^i u^k.$$

The tensor J has components

$$J^{ik} = t(I) \delta_{ik} - I^{ik}.$$

In some books (e.g. Goldstein), J, rather than I, is called the inertia tensor. The explicit components of J are:

$$J^{11} = \int [(x^2)^2 + (x^3)^2] dm,$$

$$J^{22} = \int [(x^1)^2 + (x^3)^2] dm,$$

$$J^{33} = \int [(x^1)^2 + (x^2)^2] dm.$$

When $i \neq j$,

$$J^{ij} = -\int [(x^i)^2 + (x^j)^2] dm.$$

If f is the bilinear form corresponding to the symmetric matrix J, then the moment of inertia about a unit vector u is $f(u, u)$. Notice that the trace of J is twice the trace of I, hence we can also write I in terms of J. From

$$J^{ik} = t(I)\delta_{ik} - I^{ik}.$$

we have

$$J^{ik} = \frac{t(J)}{2}\delta_{ik} - I^{ik},$$

Or

$$I^{ik} = \frac{t(J)}{2}\delta_{ik} - J^{ik}.$$

20 Properties of a Symmetric Matrix, Principal Directions

The inertia tensor by definition is represented by a symmetric matrix. The eigenvalues of a symmetric matrix are real. Indeed, if

$$Av = \lambda v$$

and A is symmetric, then taking inner products,

$$\lambda(v, v) = (\lambda v, v) = (Av, v) = (v, Av) = (v, \lambda v) = \bar{\lambda}(v, v),$$

which means that the eigenvalue $\lambda = \bar{\lambda}$ and thus that λ is real.

Let λ_1, λ_2 be eigenvalues of A, with corresponding eigenvectors v_1, v_2, with λ_1 and λ_2 not equal. Then

$$\lambda_2(v_1, v_2) = (v_1, Av_2) = (Av_1, v_2) = \lambda_1(v_1, v_2)$$

It follows that $(v_1, v_2) = 0$. That is, any pair of eigenvectors corresponding to different eigenvalues are orthogonal. If an eigenvalue is repeated n times, then its corresponding eigenspace has dimension n. One can find an orthogonal basis for this subspace. Thus the eigenvectors define an orthogonal basis for the whole space.

If the eigenvalues of an inertia tensor are distinct, the orthogonal directions specified by the eigenvectors are called the principle directions.

21 The Moment of Inertia Integral as a Surface Integral

The divergence theorem is

$$\int_V \nabla \cdot F dv = \int_{\partial V} F \cdot n d\sigma.$$

Define a vector field by

$$F = \frac{x^3 i + y^3 j}{3}.$$

Then

$$\nabla \cdot F = x^2 + y^2.$$

So

$$I_{zz} = \int_V (x^2 + y^2)\rho dv = \int_V \nabla \cdot F \rho dv = \int_{\partial V} \frac{\rho}{3}(x^3 i + y^3 j) \cdot n d\sigma.$$

When the surface is triangulated, we may compute the surface integral by computing the integral over each triangle and taking the sum. Each surface integral may be reduced to a line integral on the triangle edges.

22 Moment of Inertia Using a Torsional Spring Balance

Let k be an elastic torsional spring constant.

$$k\theta = \tau = I\frac{d\omega}{dt}.$$

$$\frac{d^2\theta}{dt^2} = \frac{k}{I}\theta.$$

$$\theta = \sin\left(\sqrt{\frac{k}{I}}t\right).$$

The period is

$$T = 2\pi\sqrt{\frac{I}{k}}.$$

Let the subscript e refer to the empty table. The period of rotation of the empty table is T_e. Let I be the moment of inertia of an object placed on the table. And let T be the period of rotation of the object and the table. Then

$$\frac{T^2}{T_e^2} = \frac{I_e + I}{I_e}.$$

$$I_e = \frac{I}{(T/T_e)^2 - 1}.$$

We can determine Determine I_e using a body of known moment of inertia I, such as a cylinder. The moment of inertia of a cylinder of radius R and height h is given by

$$I = h\rho\int_0^R\int_0^{2\pi} r^2 r dr d\theta =$$

$$2\pi\rho h R^4/4.$$

Let M be the mass of the cylinder, then

$$M = \pi\rho R^2 h.$$

Then

$$I = \frac{1}{2}R^2 M.$$

Once we know I_e, we can determine an unknown I, by measuring the period.

23 Moment of Inertia by the Three Wire Method

See Crede, **Vibration and Shock Isolation** Let an equilateral triangular platform be suspended from three parallel strings or wires attached at the vertices. If the triangular platform is twisted about a vertical axis by an angle θ, then at each attachment point there will be a horizontal tangential force of magnitude

$$\frac{mgr\theta}{3L},$$

where m is the sum of the mass of the platform and the mass of the test object, which rests on the platform, r is the distance from the center of the platform to the triangle vertices, and L is the length of the strings. Since the angle of deflection of the strings from the vertical is small, we have replaced the sine of the angle by the angle. The torque is obtained by multiplying by r,

$$\tau = \frac{mgr^2}{L}\theta = k\theta.$$

Then the torsional spring constant is

$$k = \frac{mgr^2}{L}.$$

The period of rotational oscillation is

$$T = 2\pi\sqrt{I/k},$$

where I is the moment of inertia. Therefore we have

$$T^2 = 4\pi^2\frac{LI}{mgr^2}.$$

Let us define the radius of gyration R_g by

$$I = mR_g^2.$$

Then

$$T = \frac{2\pi}{\sqrt{g}}(R_g/r)^2\sqrt{L}.$$

Thus the period is proportional to the ratio of the radius of gyration to the distance from the center of the platform to the wire attachment points, and is also proportional to the square root of the length of the strings. For example if

$$R_g/r = .75,$$

and

$$L = 50cm,$$

then

$$T = 1.06 seconds.$$

The radius of gyration (and thus the moment of inertia) is obtained by measuring the period and computing

$$R_g = \frac{\sqrt{g}r}{2\pi\sqrt{L}}T.$$

24 Calculating the Inertia Tensor From Six Moments of inertia

Recall that the moment of inertia about an axis in the direction of unit vector u is

$$f(u,u) = J^{ij}u_i u_j.$$

By choosing u appropriately we can find the components of J from the moments of inertia, which can be measured.

Let

$$u = (1,0,0)^T,$$

then

$$f(u,u) = J^{11}.$$

Let

$$u = (0,1,0)^T,$$

then

$$f(u,u) = J^{22}.$$

Let

$$u = (0,0,1)^T,$$

then

$$f(u,u) = J^{33}.$$

Let

$$u = \frac{1}{\sqrt{(2)}}(1,0,1)^T,$$

then

$$g^{13} = f(u,u) = \frac{1}{2}J^{11} + \frac{1}{2}J^{13} + \frac{1}{2}J^{31} + \frac{1}{2}J^{33}$$

$$= \frac{1}{2}J^{11} + J^{13} + \frac{1}{2}J^{33}.$$

So that

$$J^{13} = g^{13} - \frac{J^{11} + J^{33}}{2}.$$

Let

$$u = \frac{1}{\sqrt{(2)}}(0, 1, 1)^T,$$

and

$$g^{23} = f(u, u) =$$

$$= \frac{1}{2}J^{22} + J^{23} + \frac{1}{2}J^{33}.$$

So that

$$J^{23} = g^{23} - \frac{J^{22} + J^{33}}{2}.$$

Let

$$u = \frac{1}{\sqrt{(2)}}(1, 1, 0)^T,$$

and

$$g^{12} = f(u, u) =$$

$$= \frac{1}{2}J^{11} + J^{12} + \frac{1}{2}J^{22}.$$

So that

$$J^{12} = g^{12} - \frac{J^{11} + J^{22}}{2}.$$

So J and hence I is determined by these six moments of inertia.

25 Dynamic Balancing

If a body is constrained to rotate about an axis or axel, it is in static balance if it is in static equilibrium for any static rotation position. It is in dynamic balance if while rotating there are no torques exerted on the axel. Static balance does not necesarily imply dynamic balance. There are three cases of dynamic balance or dynamic imbalance. The cases are: (1) the principle axis through the center of gravity coincides with the axis of revolution, (2) it intersects the axis of revolution, (3) it does not meet the axis of revolution.

26 The Gyroscope

We shall consider the gyroscope mathematically, but first let us give a qualitative discussion of the behavior of a gyroscope. Consider two particles rotating in a circle about an axis, where the axis itself is horizontal and is rotating about a vertical axis, where the circle is located some distance from the vertical axis (see the figure). Let gravity be acting downward. If we consider the curved path of one of the particles in 3-space, we note that the acceleration of such a particle is dependent upon the curvature of the path and on the particle speed. If the speed along the path is nearly constant, then the acceration is nearly normal to this curve. When one of the particles is at its highest vertical point, there will be a component of acceleration directed toward the vertical axis about which the axis of the particle circle rotates. This will also be true about the second particle at its lowest point, but because the circle is itself rotating about the verticle axis, the two particles have different speeds and a little different path curvatures. This difference means that there is a torque difference on this system of two particles. This difference is balanced by the gravitational torque. This rotation about the vertical axis is called precession. The rate of precession is such that the rotation about the vertical axis gives the torque difference necessary to balance the gravitational torque on the particles. For two particles like this which are not uniformly distributed around the circle, the balance can occur only while the points are in a certain orientation on the circle. So actually the rotating circle will have to bob up and down, or nutate. A rotating ring will not do this nutation. The mathematics governing the motion of the gyroscope is complex.

27 The Legendre Transformation

Method of changing roles of dual variables.

$$df = udx + vdy$$

$$g = f - vy$$

then

$$dg = udx - ydv$$

28 Emmy Noether's Theorem

Proven by Emmy Noether in 1915 and published in 1918. The essence of the theorem may be stated as: Physical quantities are conserved due to the symmetry of the Lagrangian. See V. I. Arnold, **Mathematical Methods of Classical Mechanics** P88, and an elementary treatment in, Louis N Hand Janet D Finch, **Analytical Mechanics** pp172-175.

29 A Mass Point in Circular Motion

Let a mass point point be specified in polar coordinates (θ, r). Let the point be constrained to lie on a circle of radius r. Let the polar coordinate unit vectors be

$$\mathbf{u}_r = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j},$$

$$\mathbf{u}_\theta = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}.$$

The first vector is perpendicular to the circle and the second is tangent to it. Let the position vector of the point be

$$\mathbf{p} = r\mathbf{u}_r.$$

The velocity is

$$\mathbf{v} = \frac{d\mathbf{p}}{dt} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\mathbf{u}_r}{dt} = r\frac{d\mathbf{u}_r}{dt},$$

because here r is constant. We have

$$\frac{d\mathbf{u}_r}{dt} = \frac{d\mathbf{u}_r}{d\theta}\frac{d\theta}{dt}$$

$$= \mathbf{u}_\theta\frac{d\theta}{dt}.$$

So

$$\mathbf{v} = r\frac{d\theta}{dt}\mathbf{u}_\theta = r\omega\mathbf{u}_\theta,$$

where ω is theangular velocity. The acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = r\frac{d\omega}{dt}\mathbf{u}_\theta + r\omega\frac{d\mathbf{u}_\theta}{dt}$$

$$= r\frac{d\omega}{dt}\mathbf{u}_\theta - r\omega^2\mathbf{u}_r$$

$$= r\frac{d\omega}{dt}\mathbf{u}_\theta - \frac{v^2}{r}\mathbf{u}_r,$$

where $d\omega/dt$ is the angular acceleration, and $v = r\omega$ is the tangential velocity. If the angular acceleration is zero then v^2/r is the magnitude of the centrepital acceleration directed toward the center of the circle.

30 The Kinetic Energy of a Mass Point

Let a mass point of mass m be accelerated from an initial velocity 0 to a velocity v by a force f through a distance s. The work done and thus the kinetic energy of the mass point is

$$\int f ds = \int m\frac{dv}{dt}ds$$

$$= \int m\frac{ds}{dt}dv$$

$$= \int mvdv$$

$$= m\frac{v^2}{2}.$$

The kinetic energy of a rigid body of mass M moving with linear velocity v is the sum of the kinetic energies of its mass points,

$$M\frac{v^2}{2}.$$

If a mass point is revolving uniformly with angular velocity ω in a circle of radius r, then its kinetic energy is

$$m\frac{v^2}{2} = m\frac{r^2\omega^2}{2}.$$

The moment of inertia of such a mass point is

$$I = mr^2,$$

hence its kinetic energy is

$$I\frac{\omega^2}{2}.$$

It follows that a body rotating about a fixed point with moment of inertia I has a kinetic energy of rotation

$$I\frac{\omega^2}{2}.$$

31 The Simple Rotation of a Rigid Body

Although the general motion of a rigid body is rather complicated, the case of rotation about a fixed axis is simple, so we shall derive the needed formulas here.

Suppose a mass point, with mass m_i, is constrained to move on a circle with radius r_i. Suppose a force F_i is applied to this point. The forces of constraint may be ignored, since they do no work. Then the work done is

$$dW = r_i d\theta F_{ti} = d\theta \tau_i,$$

where F_{ti} is the tangential component of the force and τ_i is the torque. Then the rate of doing work is

$$\frac{dW}{dt} = \omega\tau_i.$$

This is equal to the change in kinetic energy of the particle

$$\omega\tau_i = \frac{d((1/2)r_i^2\omega^2 m_i)}{dt} = r_i^2 m_i \frac{d\omega}{dt}\omega.$$

Thus

$$\tau_i = r_i^2 m_i \frac{d\omega}{dt}.$$

Summing over all particles in the rotating rigid body, we get

$$\tau = \sum \tau_i = (\sum r_i^2 m_i)\frac{d\omega}{dt} = I\frac{d\omega}{dt},$$

where I is the rotational moment of inertia.

32 Uniform Acceleration

Suppose a mass point undergoes uniform acceleration a. Then

$$a = \frac{d^2 s}{dt^2},$$

where s is the location coordinate of the point. Integrating, the velocity becomes

$$v = \int a dt = v_0 + at$$

where v_0 is the initial velocity. Integrating again, the position is

$$s = \int v dt = s_0 + v_0 t + a\frac{t^2}{2},$$

where s_0 is the initial position. The change in kinetic energy is

$$m(\frac{v^2}{2} - \frac{v_0^2}{2}) = f(s - s_0) = ma(s - s_0),$$

where f is a constant force of acceleration. Dividing by $m/2$ we get

$$v^2 - v_0^2 = 2a(s - s_0).$$

The velocity increases linearly with the time, so the average velocity multiplied by the time is the distance traveled

$$\frac{v + v_0}{2}t = \frac{v_0 + at + v_0}{2}t$$

$$= (v_0 + \frac{at}{2})t$$

$$= v_0 t + \frac{at^2}{2}$$

$$= (s - s_0).$$

The four equations

$$v = v_0 + at,$$

$$s = s_0 + v_0 t + a\frac{t^2}{2},$$

$$v^2 - v_0^2 = 2a(s - s_0),$$

and

$$\frac{v + v_0}{2} t = (s - s_0),$$

usually suffice to solve problems of this type.

Example 1. Suppose a ball is rolled off of the edge of a table top. Suppose the table is at height height h and the ball lands on the floor at a distance s from a point on the floor below the table edge. What was the horizontal velocity of the ball as it left the table edge?

Solution The ball is accelerated by the acceleration of gravity g downward. The initial vertical velocity is zero, and the initial vertical position is zero. So using the second formula

$$h = g\frac{t^2}{2},$$

from which we find the time at which the ball hits the floor to be

$$t = \sqrt{2h/g}.$$

Then again from the first equation with the horizontal acceleration $a = 0$, we have

$$v_0 t = s.$$

So we find the initial horizontal velocity to be

$$v_0 = \frac{s}{t} = s\sqrt{\frac{g}{2h}}.$$

In a real experiment let a be the distance up a ramp, so that the ball starting at a gains velocity v, in rolling down the ramp. Let $h = 0.9$ meters. We measure the following values and compute v.

a	s	v
$10cm$	$.48m$	$1.11m/s$
20	.68	1.59
30	.79	1.84
40	.89	2.08
50	1.08	2.52
60	1.13	2.63
70	1.18	2.77
80	1.29	3.01

Example 2. Suppose a stone is dropped from a one hundred meter tall bridge. What is the velocity of the stone as it hits the water, neglecting air resistance?
Solution The acceleration of gravity is $g = 9.81$ meters per second squared. Then from the third equation

$$v^2 = 2gs,$$

$$v = \sqrt{2gs} = \sqrt{2(100)(9.81)} = 44.29$$

meters per second, or 159.5 killometers per hour.

33 The Acceleration of a Rolling Body

Let a sphere roll down a ramp that is inclined to the horizontal by angle α. Let s be the distance from the bottom of the ramp to the contact position of the ball. Let the ball have radius r and let θ be the angle of rotation of the ball. Then let

$$s = r\theta.$$

The kinetic energy of the ball is

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}I(v/r)^2$$

$$= \frac{1}{2}(M + I/r^2)v^2$$

The potential energy is

$$V = hgM,$$

where

$$h = s\sin(\theta)$$

is the height of the ball. The Lagrangian is

$$L = T - V = \frac{1}{2}(M + I/r^2)v^2 - s\sin(\theta)gM.$$

$$\frac{\partial L}{\partial v} = (M + I/r^2)v$$

$$\frac{d}{dt}\frac{\partial L}{\partial v} = (M + I/r^2)\frac{dv}{dt}$$
$$\frac{\partial L}{\partial s} = -\sin(\theta)gM.$$

The equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial s} = 0$$

So

$$(M + I/r^2)\frac{dv}{dt} + \sin(\theta)gM = 0$$

or

$$\frac{dv}{dt} = -g\sin(\theta)\frac{M}{M + I/r^2}$$

So the effective acceleration has been reduced by the factor

$$\frac{M}{M + I/r^2}$$

In the case of a sphere

$$I = \frac{2}{5}Mr^2,$$

so the factor becomes

$$\frac{M}{M + (2/5)M} = \frac{5}{7}.$$

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