Using the 3D Pantograph for Making a Proportional Copy of a Surface.

James Emery

8/18/2013

Contents

1 Introduction

2 A Proposition on Similarity of Triangles

Proposition. Given two triangles, if they have an equal angle, and if the sides forming the angle have sides such that the corresponding sides are proportional with the same proportionality constant then the triangles are similar.
Proof. Apply the law of cosines to show that the pair of third sides also have the same proportion.

3 The Planar Pantograph

Refer to figure 1. Let

$$A = (0, 0)$$

$$B = d_{AB}(\cos(\phi), \sin(\phi),$$

where d_{AB} is the distance from point A to B. Let D lie on a line from A to B so that

$$D = d_{AD}(\cos(\phi_1), \sin(\phi_1)$$

Define

$$F = d_{AF}(\cos(\theta), \sin(\theta)$$

Let C be chosen on the line through A and F, so that line segment BC is parallel to line segment DF. Then let E be a point on the line through D and F so that segments BD and CE are parallel. All of this simply means that $BCED$ is a parallelogram. We see that triangles ABC and ADF are similar. It follows that for any value of θ

$$\frac{d_{AF}}{d_{AC}} = \frac{d_{AD}}{d_{AB}}$$

Let α be the constant

$$\alpha = \frac{d_{AD}}{d_{AB}}.$$

If

$$C = (x_C, y_C)$$

and

$$F = (x_F, y_F)$$

we have

$$F = \alpha C$$

So if point C traces out a figure, then F traces out a similar figure enlarged by the scale factor α. Here is a Python program to define the points A, B, C, D, E, F and draw the figure with node definitions for program **cdiagram.ftn**.

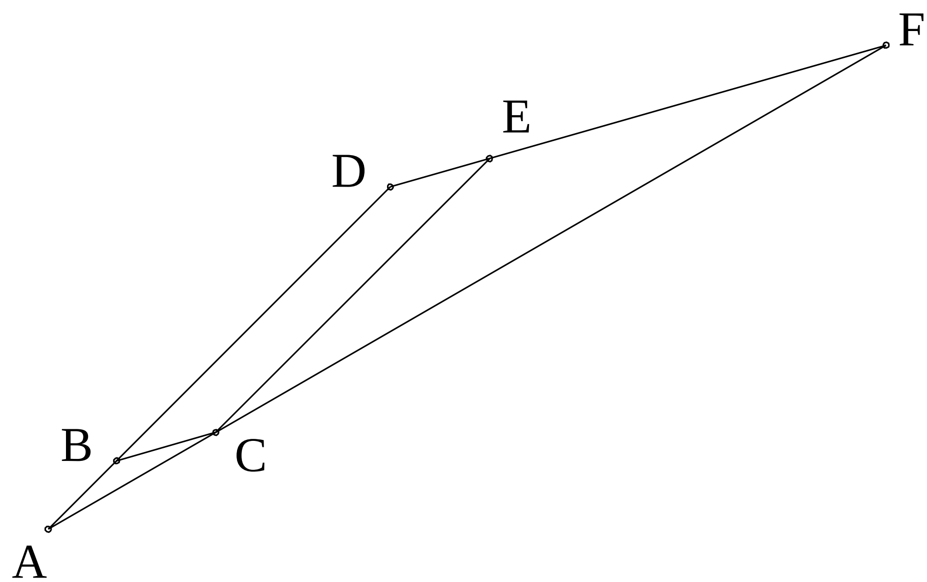

Figure 1: The pantograph scales a figure traced out at C to an enlarged figure, with scale factor α, traced out at F. A is a fixed point. $BCED$ is a parallelogram. The links pivot at the joints B, C, D and E. Scale factor α is equal to the ratio $AD/AB = AF/AC$. Points A, C, and F are collinear points, but lines AC and CF do not represent physical linkages, and their lengths vary. ABC and ADF are maintained as similar triangles by the parallel linkages..

```
# pantograph figure, program pantograph.py
import math
ax=0.
ay=0.
phi=math.pi/4.
dab=1.
bx=dab*math.cos(phi)
by=dab*math.sin(phi)
print "n ",1,ax,ay
print "n ",2,bx,by
dad=5.
dx=dad*math.cos(phi)
dy=dad*math.sin(phi)
print "n ",3,dx,dy
theta=math.pi/6.
daf=10.
fx=daf*math.cos(theta)
fy=daf*math.sin(theta)
print "n ",4,fx,fy
alpha = 5.
dac=daf/alpha
cx=dac*math.cos(theta)
cy=dac*math.sin(theta)
print "n ",5,cx,cy
ufx=fx-dx
ufy=fy-dy
ex=dx+ufx/alpha
ey=dy+ufy/alpha
print "n ",6,ex,ey
print "w 1 3"
print "w 1 4"
print "w 3 4"
print "w 2 5"
print "w 5 6"
print "a 1 -l   A"
print "a  2 -l   B"
print "a  3 -l D"
print "a  4 -l F"
print "a  5 -l C"
print "a  6 -l E"
print "con 1 "
print "con 2 "
print "con  3 "
print "con  4 "
print "con  5 "
print "con  6 "
```

4 Moving the Pantograph

So we align the pantograph so that the center, the probe position, and the copying position are colinear. When the probe point is moved we need to show that these three points are still colinear, so that the similarity of the triangles is maintained, and so the scaling parameter is constant. So the parallelogram maintains the included angle in the two triangles, that is the included angle between the sides whose lengths are fixed by the linkages. These distances are constant. By employing the law of cosines we see that the lengths of the variable sides are proportional with the same scale factor μ, thus the two triangles are still similar and thus the three points, namely center, probe, and copying point are colinear.

In more detail, refering to the pantograph figure we let the equal angles PTQ and PSR be called ϕ. Initially points P, Q and R are adjusted to be colinear. Then clearly triangles PTQ and PSR are similar triangles because their corresponding angles are equal. Let a_1 be the length of segment PT and a_2 the length of PS, b_1 the length of TQ b_2 the length of SR, and c_1 the length of PQ c_2 the length of PR. Then by similarity

$$a_1/a_2 = b_1/b_2 = c_1/c_2 = \alpha,$$

for some α.

Now supose we move point Q to a new location, then in general c_1, c_2 and angle ϕ will change. But the numbers a_1, a_2, b_1, b_2 do not change, so

$$a_1/a_2 = b_1/b_2 = \alpha$$

Using the law of cosines and the fact that the parallelogram linkage forces the angles PTQ and PSR to be equal to a single changed angle ϕ, we have by the law of cosines

$$c_1^2 = a_1^2 + b_1^2 - a_1 b_1 \cos(\phi)$$

while

$$\begin{aligned} c_2^2 &= a_2^2 + b_2^2 - a_2 b_2 \cos(\phi) \\ &= \alpha^2 (a_1^2 + b_1^2 - a_1 b_1 \cos(\phi)) \\ &= \alpha^2 c_1^2 \end{aligned}$$

So the ratio

$$c_1/c_2 = \alpha$$

is true for the new posiion, even though the distances have changed.

So the ratio has been maintained, and so the triangles are still similar, and in particular the points P, Q, R are colinear (because to be similar corresponding angles must be equal). So we have shown that the uniform scaling is maintained as point Q is moved.

5 A 3D Pantograph

A 3D pantograph can be used in creating molds for investment castings and for sculptures. If the pivot of the planar pantograph is a ball joint, then the pantograph mechanism produces a 3D scaling because certainly if the paralellograms of the linkage remain fixed and the whole thing is just pivoted on the ball joint, then the scale factor between the two pointers is preserved. It is also interesting to see that if we measure the angle between line AD and AF we can calculate the distance d_{AF}. Then if we measure the polar angle of AF and the azmuthal angle of AF, we have the spherical coordinates of point F. Thus we can use this device for mechanically scanning the surface of a 3d object. We can get to the back of the object if we are allowed to rotate the object knowing the angle of rotation.

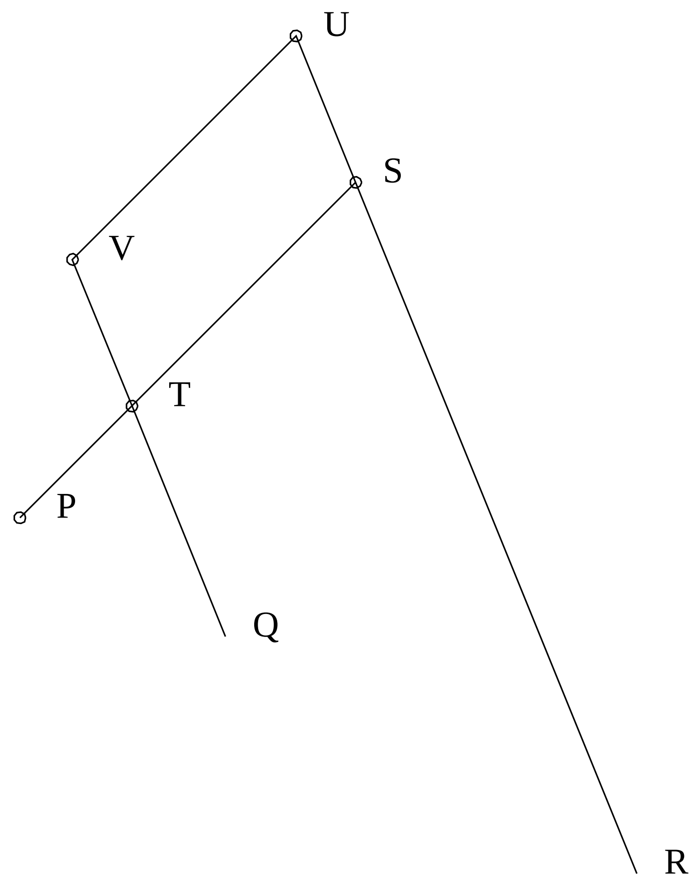

Figure 2: The pantograph scales a figure traced out at Q to an enlarged figure, with scale factor α, traced out at R. P is a fixed position, but the pantograph may rotate about it. The links can rotate about all pivot points. Scale factor α is equal to the ratio PT/PS. Points P, Q, and R are adjusted to be collinear points, and will remain so as the probe point Q and the copy point R move, as shown in the text. The lengths of the linkages are fixed, but the distances PQ and QR vary as point Q is moved. Angles PTQ and PSR are maintained equal by the parallelogram $TSUV$.