Quantum Mechanics

James Emery

7/13/2010

Contents

1 Introduction

Quantum Theory arose in order to explain serious problems with Classical Physics. One such problem occurred in the theory of black body radiation, and a second occurred in the observation of the discrete spectral lines of atoms. The original quantum theory was created by Bohr, and accurately explained the hydrogen spectra. Atoms were treated as little solar systems, with negative electrons orbiting around a positive nucleus. The electrons had to be confined to certain discrete orbits, and would have to undergo quantum jumps between orbits to alter the atomic energy levels. Later in 1925, a much more general theory of Quantum Mechanics was created by Heisenberg, Born, De Broglie, and Schrödinger. We outline some aspects of Quantum Mechanics below.

2 The Photoelectric Effect

If a metal plate such as zinc is placed on an electroscope and given an electric charge by electrostatic induction the leaves of the electroscope are repelled and so separate. Now if the plate is exposed to intense ultraviolet light, the electroscope begins to discharge. Photons transmit energy to the surface electrons and cause them to escape from the surface of the plate. By using a vacuum tube, and placing a retarding potential grid at a distance from the metal plate, the emission of photo electrons can be made to stop. This measures the energy of the electrons in electron volts. It is found that this stopping potential is independent of the intensity of the light. But is dependent on the frequency of the light. Einstein postulated that the energy of the photoelectron is proportional to the frequency where the constant of proportionality is equal to Planck's constant. That is

$$E = h\nu.$$

To make this experiment work, electromagnetic radiation of wavelength shorter than visable light is required. It is not clear whether the electroscope experiment occurred before the experiments of Lenard (Eisberg **Fundamentals of Modern Physics**. Lenard used aa evacuated glass tube with two plates, a perforated plate allowed UV to pass through to impinge on the secend plate. One would think that a source of ultraviolet radiation would not have been available earlier for the electroscope experiment. This photoelectric experiment is probably the most revolutionary experiment in the early 20th century and led to the quantum theory.

3 Operators and Hilbert Space

Mathematician John von Neumann introduced the concepts of Hilbert space and operator, theory into quantum mechanics. See his book **Mathematical**

Foundations of Quantum mechanics. Von Neumann pioneered much of the mathematics of operator theory and functional analysis.

4 Black Body Radiation, Antenna Temperature, and Brightness Temperature

We know from optics and physiology that the brightness of a visual image is independent of the distance of the image from the object. This is because the amount of radiation received from a point source in a unit time, by a receiver of a given size, such as a stopped lens, the eye, or a parabolic antenna, is inversely proportional to the square of the distance from the source to the receiver. For an extended source the radiation received is proportional to the area ΔA of the source and inversely proportional to the square of the distance. The farther away the receiver, the smaller the radiation received. But the area of the focused image of the source, say on the retina in the case of the eye, is also inversely proportional to the square of the distance. So brightness, being the total radiation received in a unit time divided by the area, is independent of the distance. Thus a headlight appears equally bright to the eye, whether it is a block away or a mile away. Of course it will appear much smaller and less spectacular if it is a mile away.

Note however, that when the object is at a very large distance, such as in the case of a telescope collecting the light from a star, the image is focused at a point, so the area of the image is independent of the distance. So a closer star is brighter. In such a case, for the apparent brightness, one must also take into account the spreading of the point image by diffraction, and the sensitivity of the receiver. In the case of the photographic image of stars, because of film chemical sensitivity, near stars will give larger images than will distant stars.

Brightness is also independent of distance for an antenna receiving black-body radiation.

Suppose a black body surface emits photons that are received by an antenna. The antenna forms a focused image in such a way that the sterance at the image equals the sterance at the source. So the brightness of the image does not depend on the distance between the source and the receiver, but only on the temperature of the blackbody source. Blackbody temperature can be measured with an optical pyrometer where one compares the brightness of the image against the brightness of a second image of known temperature. The second image can be a real heated body located near the first image. For example, it could be a heated wire, whose temperature is controlled by a rheostat. In that case the temperature of the wire is determined by the current passing through it, which can be calibrated to a dial setting. The heated body could even be a light bulb filament.

In the case of a parabolic antenna, rays parallel to the axis of the antenna are brought to a single focal point. This is the perfect geometrical case. In practice radiation received at the radiometer comes from some small solid direction angle, so again an image is brought to a focused image of nonzero area. And so again the brightness of the image equals the brightness of the object. If the object is not a black body, then the temperature measured by the antenna is the temperature of a blackbody that would produce the same amount of sensed radiation over the focal image. The equivalent blackbody temperature is called the brightness temperature of the object. For a perfect antenna, the antenna temperature would be the brightness temperature. But a modern radiometer usually incorporates some form of amplifier. Calibration constants must be applied to the antenna temperature to get the brightness temperature. The brightness temperature is not the actual temperature of the radiating body. For example, the brightness temperature of melting ice in the sea is not 273 degrees K, because the sea is not a black body. The difference between the brightness temperature and the true temperature may be thought of as being due to the emissivity of the surface not being equal to 1. The emissivity is not usually a constant. It varies with the wave length. If only a few frequencies are being measured, from the total of radiation frequencies, then the brightness temperature can be determined from values of emissivities for these few frequencies.

A formula for brightness temperature may be found in the article on Radiometry in **The American Institute of Physics Handbook**.

The Planck black body radiation formula is

$$W_\lambda(T,\lambda)d\lambda = \frac{c_1 d\lambda}{\lambda^5(\exp(c_2/\lambda T) - 1)}.$$

This is the power radiated by a black body surface, of unit area, and by the waves in the wave length interval λ to $\lambda + d\lambda$. It is the sum of the radiation in all directions outward from the radiating area. The constants are $c_1 = 2\pi hc^2$ and $c_2 = hc/k$. Using the values for Planck's constant

$$h = 6.6262 \times 10^{-34} JS,$$

the velocity of light

$$c = 3.03000 \times 10^8 MS{-}1,$$

and Boltzmann's constant

$$k = 1.3806 \times 10^{-23} JK^{-1},$$

we get

$$c_1 = 3.742 \times 10^{-16} WM^2$$

and

$$c_2 = 1.4388 \times 10^{-2} MK.$$

A Reference is **Optics**, Meyer-Arendt Page 467.

If wavelengths are measured in centimeters, the constant c_2 is 1.438 Cm K. This radiation obeys the Lambert cosine law, where the radiation in a given direction is proportional to the cosine of the angle between the radiating direction and the surface normal. One might think of this blackbody surface as being a small hole in a black body cavity, which contains isotropic radiation in equilibrium. Hence radiation that makes an angle of θ with the surface normal originates inside of the cavity, and such internal radiation in this direction sees only the projection of the hole inclined at angle θ. Hence the amount of radiation that exits the hole in the direction is proportional to $\cos(\theta)$. This explains Lambert's law. A body at temperature T in general emits less radiation than a black body. The power radiated is reduced by the emissivity multiplier.

We can show that the power radiated by a black body for all wave lengths is proportional to the fourth power of the absolute temperature. This is done by integrating over all frequencies,

Inside the black body cavity there is an equilibrium radiation density u_λ (energy per unit volume), which is dependent only on the temperature. The Planck law is usually derived for u. The relation between $u_\lambda(T, \lambda)$ and $W(T, \lambda)$ is

$$u_\lambda = \frac{4}{c} W,$$

where c is the velocity of light. Let us derive this relationship. The radiation passing through a small hole in the cavity of area ΔA, in unit time in a direction, with spherical coordinates ϕ, θ making an angle ϕ with the area normal, where the direction varies over the solid angle

$$d\Omega = d\phi \sin(\phi) d\theta,$$

is

$$c\Delta A \cos(\phi) du_s,$$

where c is the velocity of light, and where

du_s is the energy density of radiation that is in the solid angle direction. We have

$$du_s = u_\lambda \frac{d\phi \sin(\phi) d\theta}{4\pi}$$

Integrating over the direction hemisphere, we get the radiated power

$$\begin{aligned} P &= c\Delta A \int_0^{2\pi} \int_0^{\pi/2} \frac{u_\lambda \cos(\phi) \sin(\phi)}{4\pi} d\phi d\theta \\ &= \frac{u_\lambda c \Delta A}{2} \int_0^{\pi/2} \cos(\phi) \sin(\phi) d\phi \\ &= \frac{u_\lambda c \Delta A}{4} \int_0^{\pi/2} \sin(2\phi) d\phi \\ &= \frac{u_\lambda c \Delta A}{4}. \end{aligned}$$

So

$$W = \frac{P}{\Delta A} = \frac{u_\lambda c}{4}.$$

So in terms of energy density the Planck formula is

$$u_\lambda(T, \lambda) d\lambda = \frac{4c_1 d\lambda}{c\lambda^5(\exp(c_2/\lambda T) - 1)}.$$

The Planck formula is also written in terms of frequency ν rather than wave length. We have

$$\lambda = \frac{c}{\nu}$$

$$d\lambda = -\frac{c}{\nu^2} d\nu.$$

Introducing a new energy density function that is a function of temperature and frequency, and writing it as u_ν, we have

$$u_\nu(T,\nu)d\nu = -u_\lambda(T,\lambda)d\lambda = -\frac{4c_1 d\lambda}{c\lambda^5(\exp(c_2/\lambda T) - 1)} = \frac{4c_1\nu^3 d\nu}{c^5(\exp((c_2\nu)/(cT) - 1)}.$$

Recall that $c_1 = 2\pi hc^2$ and $c_2 = hc/k$.

So the formula becomes

$$u_\nu(T,\nu)d\nu = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{e^{h\nu/kT} - 1}.$$

This is the Planck formula as derived for example in **Quantum Mechanics**, by Powell and Crasemann.

Let us outline this derivation. Consider the general solution of the electromagnetic wave equation in a cubical cavity, with boundary condition of zero electric field at the walls of the cavity. Then any solution is a superposition of plane wave modes of the form

$$\exp(\omega t - r \cdot k),$$

where the wave number vector k has components k_x, k_y, k_z. There must be an integral number of wavelengths across each coordinate direction. Counting these we find that the number of modes in the interval k to $k + dk$ per unit volume is

$$\frac{8\pi k^2}{(2\pi)^3} dk$$

Using $\nu = kc/2\pi$, the number becomes

$$\frac{8\pi\nu^2}{c^3} d\nu.$$

In the classical case, each degree of freedom has an associated energy kT, so we get the Rayleigh-Jeans law

$$u_\nu = \frac{8\pi\nu^2}{c^3} kT.$$

In the quantum case we assume that each mode can have only the energies

$$0, \epsilon, 2\epsilon, 3\epsilon, ...,$$

with the probability of having energy $n\epsilon$ being

$$\exp(-n\epsilon/kT).$$

Then the expected value of the energy of a mode is

$$\frac{\epsilon}{\exp(\epsilon/kT) - 1}$$

Using this average in place of the classical average kT, we get for the energy density

$$u_\nu = \frac{8\pi\nu^2}{c^3}\frac{\epsilon}{\exp(\epsilon/kT) - 1}.$$

Wien showed, using thermodynamics, that the energy density must have the functional form $\nu^3 f(\nu/T)$. If we take the energy quantum to be

$$\epsilon = h\nu$$

for the constant h, then we get the quantum black body energy density formula

$$u_\nu(T,\nu)d\nu = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{e^{h\nu/kT} - 1},$$

and we see that the required functional relationship holds. We can determine the value of Planck's constant h from experimental data, using the Wein displacement law.

In order to measure the temperature of a black body source, clearly the temperature measured by the antenna must not depend on the distance between the source and the antenna. A larger antenna will receive more energy, and a closer antenna will measure more energy, so temperature can not be a simple measure of the radiation energy received.

The antenna focuses the radiation on a target radiometer. The amount of radiation could be measured by the equilibrium temperature of the target, or by some other measurement of the energy per unit area received at the focal image of the radiometer. Now in the case of radiation received from an image on the earth's surface, the radiation is not black body radiation. It is dependent on the properties of the earth material, on properties of the surface, and on the reflected radiation from the sun, from the sky, and from the clouds. A radiating surface has an emissivity constant that determines the amount of surface radiation as a fraction of blackbody radiation. The effect of the atoms of the radiating material is to absorb or scatter some of the black body radiation. In the general case radiation from a body at temperature T is not black body radiation because the equilibrium black body radiation is being partially absorbed and scattered by atoms of the material. That is why the amount of radiation from the surface of a real object is less than the amount radiated by a blackbody surface at that temperature.

5 Radio Astronomy of Planets

The antenna temperature may be determined by connecting it to a transmission line and measuring the transmitted power. Then one replaces the antenna with a resister. The temperature of the resistor is adjusted to give power equal to the power of the antenna. The Johnson noise generated by the resistor is proportional to the absolute temperature and has a black body distribution. Hence the temperature of the resistor can be used as the antenna temperature for the wavelength interval being transmitted down the transmission line. When measuring the temperature of a planet the reflected radiation from the sun must be ignored. But because this reflected light is in the visible spectrum, it will not be received by the radio antenna. For more information on this see Smith and Carr, **Radio Exploration of the Planetary System**.

6 The Wien Displacement Law

The Wien law says that the product of the wave length λ_{max}, at the peak of the energy density, with the temperature T, is a constant b.

$$\lambda_{max} T = b$$

There are two energy density functions, namely u_ν and u_λ, and we can show that the peaks of these two functions do not occur at the same frequency. Even so the functional relationship of the Wien law holds for both densities. We shall show this.

We write the frequency density function as

$$u_\nu = \frac{a\nu^3}{\exp(b\nu) - 1}.$$

The derivative is

$$\frac{du_\nu}{d\nu} = \frac{a\nu^2}{\exp(b\nu) - 1}\left[3 - \frac{b\nu\exp(b\nu)}{\exp(b\nu) - 1}\right]$$

Setting the derivative to zero, makes the quantity in brackets zero, which takes the form

$$x = 3(1 - \exp(-x),$$

if we set $x = b\nu$. The solution of this equation obtained numerically with Newton's method is

$$x = 2.8214393721221.$$

Since $b = h/kT$, the frequency where the peak occurs is

$$\nu = x/b = \frac{xkT}{h}$$

So we have Wien's displacement formula

$$\lambda T = \frac{ch}{kX},$$

where $\lambda = c/\nu$.

On the other hand we can get a similar result if we start with the wave length distribution function

$$u_\lambda = \frac{A}{\lambda^5(\exp(B/\lambda) - 1}.$$

The derivative is

$$\frac{du_\lambda}{d\lambda} = \frac{A}{\lambda^6}(\exp(B/\lambda) - 1)^2\left[\frac{B\exp(B/\lambda)}{\lambda}\right]$$

Setting this to zero forces the expression in brackets to zero. If we let $y = B/\lambda = \frac{hc}{kT\lambda}$, then we get the equation

$$5(1 - \exp(y)) + y\exp(y) = 0,$$

which has solution

$$y = 4.9651142317443$$

We get a form of Wien's equation

$$\lambda T = \frac{hc}{ky}$$

Notice that the constant on the left is different from the constant in the previous version of Wien's equation. So the frequency where the maximum energy per unit frequency interval occurs, is different from the frequency where the maximum energy per unit wavelength occurs.

By measuring the frequencies at the peaks for a set of temperatures one could determine Planck's constant experimentally using Wien's displacement law. We could do this provided we have Boltzmann's constant already. Wien's law and the Steven-Boltzmann's law allow the simultaneous determination of both h and k from experimental data.

7 The Stefan-Boltzmann Law

This law says that the total areance of a blackbody is proportional to the fourth power of the temperature.

$$W = \sigma T^4$$

We show this by integrating Planck's formula over all wavelengths.

Starting with the Planck formula

$$u_\nu(T,\nu)d\nu = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{e^{h\nu/kT} - 1}.$$

Let

$$x = \frac{h\nu}{kT}$$

then the total energy per unit volume is

$$u = \int_0^\infty u_\nu d\nu = \frac{8\pi k^4 T^4}{c^3 h^3} \int_0^\infty \frac{x^3}{\exp(x) - 1} dx.$$

We can evaluate the last integral as follows

$$\int_0^\infty \frac{x^3}{\exp(x) - 1} dx = \int_0^\infty \frac{x^3 \exp(-x)}{1 - \exp(-x)} dx$$

Now

$$\frac{1}{1 - \exp(-x)} = 1 + \exp(-x) + \exp(-2x) + ...$$

So

$$\int_0^\infty \frac{x^3 \exp(-x)}{1 - \exp(-x)} dx = \int_0^\infty (x^3 \exp(-x) + x^3 \exp(-2x) + ...) dx.$$

Using integration by parts we have

$$\int_0^\infty x^m \exp(-nx) dx = \frac{m}{n} \int_0^\infty x^{m-1} \exp(-nx) dx,$$

and

$$\int_0^\infty \exp(-nx) dx = \frac{1}{n}.$$

So

$$\int_0^\infty x^3 \exp(-nx) dx = \frac{6}{n^4}$$

Then

$$\int_0^\infty (x^3 \exp(-x) + x^3 \exp(-2x) + x^3 \exp(-3x) + ...) dx = 6(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ...) = \frac{\pi^4}{15}$$

Therefore

$$u = \frac{8\pi^5 k^4}{15 h^3 c^3} T^4 = \frac{4}{c} \sigma T^4$$

where u is in Joules per Meter squared. In terms of total areance this becomes

$$W = \frac{c}{4} u = \sigma T^4$$

Stefan's constant is

$$\sigma = \frac{2\pi^5 k^4}{15 h^3 c^2}$$

The total radiated power over all frequencies per unit area W has dimensions of Watts per unit area.

The constant b from Wien's law and Stefan's constant σ, when found experimentally from blackbody radiation data, give two equations involving k and h. So both h and k can be determined from such data.

8 A Program to Calculate and Plot the Planck Curves

Here is a program to make a plot of the Planck curves given as energy density per unit of wavelength, as a function of wavelength. The program makes a file called **Planck.eg**, which may be plotted or converted with one of the programs **wineg.cpp**, **pltvga.c eg2ps.c**, and **eg2raw.cpp**.

```
//planck.c blackbody energy density function
//energy density per unit of wavelength as a function of wavelength
//Writes plot file planck.eg
#include <stdio.h>
#include <stdlib.h>
#include <string.h>
#include <math.h>
#define pi 3.1415926535
main(int argc,char** argv){
 int i;
 int j;
 double lambdac;
 double u;
 double lambdamn;
 double lambdamx;
 double lambda1;
 double lambda2;
 double umn;
 double umx;
 int n=100;
 double h=6.6262e-34;
 double k=1.38e-23;
 double ts;
 double te;
 int nt;
 double t;
 double c=3.03e8;
 double c1;
 double c2;
 double lambda;
 double nu;
 FILE* fh;
```

```
if(argc<4){
 printf("planck.c Curves of the blackbody energy density function.\n");
 printf("Energy density per unit of wavelength as a function of wavelength.\n");
 printf("Writes plot file planck.eg\n");
 printf("Usage: planck Temperature1 Temperature2 Number_of_curves\n");
 printf("Example: planck 3000 6000 4\n");
 return(0);
}
ts=atof(argv[1]);
te=atof(argv[2]);
nt=atoi(argv[3]);
c1 = 2.*pi*h*c*c;
c2 = h*c/k;
fh=fopen("planck.eg","w");
lambda1 = (h*c)/(k*te*4.9651142317443);
lambda2 = (h*c)/(k*ts*4.9651142317443);
//dlambda=lambda2-lambda1;
lambdac=(lambda1+lambda2)/2.;
lambdamn=lambdac/4.;
lambdamx=4.*lambdac;
for(j=0;j<nt;j++){
 t=j*(te-ts)/(nt-1) + ts;
 for(i=0;i<n;i++){
  lambda=i*(lambdamx-lambdamn)/(n-1) + lambdamn;
  u=((4.*c1)/(c*pow(lambda,5))) / (exp(c2/(lambda*t))-1.);
  if((i==0)&&(j==0)){
   umn=u;
   umx=u;
  }
  else{
   if(u<umn)umn=u;
   if(u>umx)umx=u;
  }
 }
}
fprintf(fh,"v-1 1 -1 1\n");
fprintf(fh,"w %15.8g %15.8g %15.8g %15.8g \n",lambdamn,lambdamx,0.,1.05*umx);
printf("Spectrum:\n");
printf("Wave length (nanometers)   Color\n");
printf("       <  450              Violet\n");
printf("      450 500              Blue\n");
printf("      500 570              Green\n");
printf("      570 590              Yellow\n");
printf("      590 610              Orange\n");
printf("       >  610              Red\n");
printf("Melting point of Iron 1808 Kelvin.\n");
printf("T=Kelvin temperature\n");
printf("Lambda=wavelength of peak energy density per unit of wavelength\n");
printf("Nu=Corresponding frequency\n");
printf("Nu2=frequency of peak energy density per unit of frequency\n");
for(j=0;j<nt;j++){
 fprintf(fh,"c %d\n",j%4 + 1);
 t=j*(te-ts)/(nt-1) + ts;
 lambda = (h*c)/(k*t*4.9651142317443);
 nu = (2.8214393721221)*k*t/(h);
```

```
  u=((4.*c1)/(c*pow(lambda,5))) / (exp(c2/(lambda*t))-1.);
  fprintf(fh,"s %15.8g %15.8g 2 .02\n",lambda,u);
  printf(" T= %10.5g Lambda= %10.5g Nu=%10.5g Nu2=%10.5g\n",t,lambda,c/lambda,nu);
  for(i=0;i<n;i++){
   lambda=i*(lambdamx-lambdamn)/(n-1) + lambdamn;
   u=((4.*c1)/(c*pow(lambda,5))) / (exp(c2/(lambda*t))-1.);

   if(i==0){
    fprintf(fh,"m %15.8g %15.8g\n",lambda,u);
   }
   else{
    fprintf(fh,"d %15.8g %15.8g\n",lambda,u);
   }
  }
 }
 return(0);
}
```

9 Some Famous Experiments

The Franck-Hertz Experiment Using an electron tube filled with mercury vapor, mercury atoms are shown to absorb mechanical energy in quanta of 4.9 electron volts.
The Compton Effect Experiment The scattering of X-rays by electrons shows that photons are particles. The deflected photon has a single changed frequency, which depends on the angle of deflection.
The Davisson-Germer Experiment The scattering of electrons by a nickel crystal shows that electrons have wave like properties and thus produce interference effects. This validates the DeBroglie relations

$$E = h\nu = \frac{h\omega}{2\pi}$$

$$P = \frac{hk}{2\pi}$$

The Stern-Gerlach Experiment A beam of atoms passing through a nonuniform magnetic field experience a splitting of spectral lines, and show that the electron must have an intrinsic angular momentum or spin.

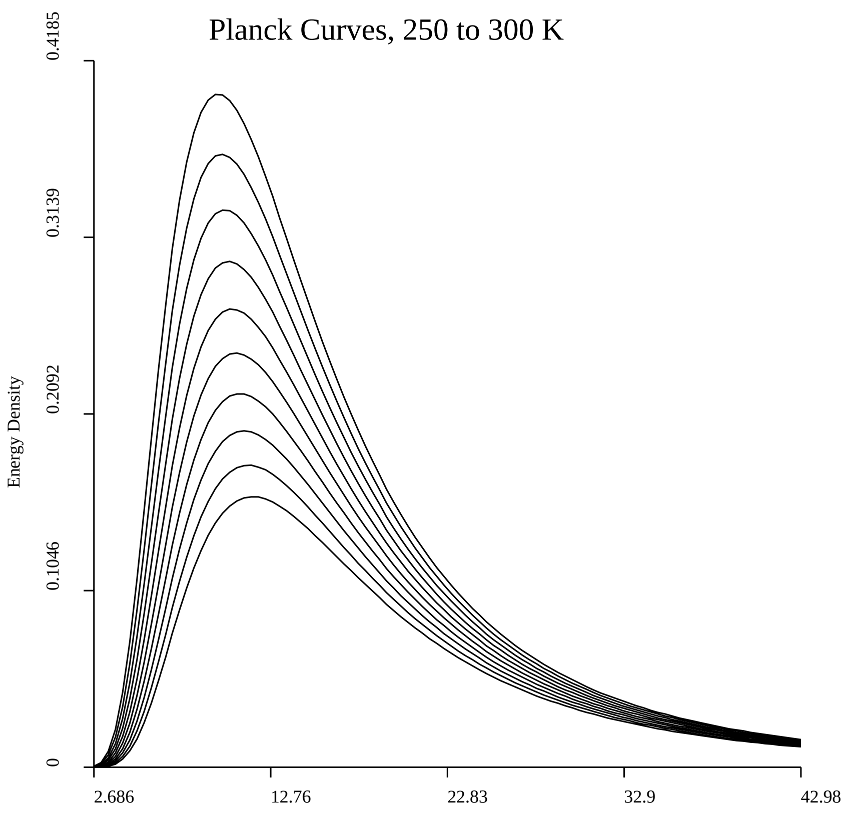

Figure 1: Blackbody radiation curves, energy density per unit wavelength.

10 Black Body Radiation

Plank's constant $h = 6.624(10^{34})$ joule·sec, arises in establishing Plank's law for black body radiation, which gives energy density as a function of frequency ν

$$u_\nu d\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{\exp(h\nu/kT - 1}kTd\nu.$$

Integrating this gives Stefen's law

$$u = \frac{4\sigma}{c}T^4,$$

where

$$\sigma = \frac{2\pi^5 k^4}{15h^3c^2}.$$

Together with Wien's law

$$\lambda_{max}T = b.$$

This gives h and k . Einstein gives an alternative derivation of Plank's law in terms of energy states of an atom bathed in radiation. This then relates black body radiation to the photoelectric effect and accounts for plank's constant in the photoelectric effect formula for the energy of a photon. (See Powell and Crasemann, **Quantum Mechanics**).

11 Quantum Radiation

The Einstein and the Weisskoph-Wigner theories of spontaneous emission are treated in the book Atomic Radiative Processes by Peter R Fontana 1982. This is the theory behind the laser.

12 The Schrödinger Wave Equation For a Free Particle

From the photoelectric effect, the energy of a photon is

$$E = h\nu,$$

where h is Planck's constant, and ν is the frequency of the photon. From relativity theory, the energy of a particle is given by the equation

$$E^2 = p^2 v^2 + m^2 c^4,$$

where v is the particle velocity and m is its mass. For a photon the velocity is the velocity of light c, and the mass is zero. Thus for a photon

$$E^2 = p^2 c^2.$$

This implies that the wavelength is

$$\lambda = c/\nu = \frac{E/p}{E/h} = h/p.$$

Let $\mathbf{k}$ be the wave number vector. By definition

$$k\lambda = 2\pi.$$

We have

$$\mathbf{p} = \frac{\mathbf{k}h}{k\lambda} = \hbar \mathbf{k},$$

where

$$\hbar = \frac{h}{2\pi}.$$

A plane wave takes the form

$$e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} = e^{(i/\hbar)(\mathbf{p}\cdot\mathbf{r} - Et)}$$

A wave packet is represented as a sum of such plane waves, written as the Fourier transform

$$\psi(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi h}} \int a(\mathbf{p}) e^{(i/\hbar)(\mathbf{p}\cdot\mathbf{r} - Et)} d\mathbf{p}.$$

This wave function satisfies the equation

$$(\hbar^2/2m)\nabla^2\psi - (\hbar/i)\partial\psi/\partial t = 0,$$

because $p^2/(2m) - E = 0$. It is called the Schrödinger wave equation. The group velocity of the wave packet is

$$v_g = \frac{d\omega}{dk} = \frac{dE}{dp}.$$

Using

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}},$$

and

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}}.$$

We find that

$$\frac{dE}{dp} = \frac{pc^2}{E} = v.$$

Thus the group velocity of the wave packet corresponds to the particle velocity. $|\psi^2|$ is proportional to the probability density function for the location of the particle. This is the Born postulate.

13 The Time Independent Schrödinger Equation

14 The Schrödinger Wave Equation For a Particle in a Potential Field

15 The Postulates of Quantum Mechanics

The dynamical variables of a mechanical system are interpreted as operators, operating on wave functions which are elements of a Hilbert space. The identification with classical mechanics is done somewhat in the manner of the mathematical theory of distributions. The expected values of a variable (i.e. the measured values) form a discrete set which is the set of eigenvalues of the operator.

16 The Bra and Ket Notation of Dirac

A ket $v>$ vector is an element of a Hilbert space (a complete complex inner product space). A bra vector $<u$ is an element of the dual space, that is, it is a linear operator. Thus the value of u on v is

$$u(v) =< u|v >,$$

which also may be interpreted as an inner product.

In another way of looking at it, a bra vector $< v$ is a row matrix of components with respect to some basis, of a vector v, that is if

$$v = \sum_{i=1}^{n} v_i b_i$$

then

$$v >= \left[\begin{array}{cccc} v_1 & v_2 & ... & v_n \end{array} \right]$$

and a ket vector $v>$ is the conjugate transpose, and thus a column vector of components.

From a mathematical point of view, this bra and ket business seems rather complicated, confused, and is not clearly necessary.

Chapter VII in Messiah outlines some linear algebra in this bra ket notation.

Section 2.3 of Shankar gives another explanation of this notation.

17 The Born Approximation

18 Perturbation Theory

19 The Relation Between Operator and Matrix Representation

Let Q be a Hermitian operator, and $\{\phi_i\}$ an orthogonal basis of the Hilbert space. Define a matrix with elements Q_{mn} by

$$Q_{mn} = \int \phi_m^* Q \phi_n d\tau.$$

Then this matrix has the same eigenvalues as the original operator Q and so there is a correspondence between the operator and the matrix. In the case of the spin operator, the matrix representation is easier to handle. See **Rae** chapter six.

20 Particle Scattering

21 Scattering Cross Section

Suppose rain falls on a flat roof. Suppose n drops of rain fall per second on each unit area. Suppose there is a hole of area A. Then the average number of drops falling through the hole in one second is nA. Given that a drop falls on a unit area containing the hole, the conditional probability that it falls through the hole in a second is

$$\frac{nA}{n} = A.$$

If there were m holes per unit area on the roof, then the probability of a given drop falling through any hole is mA. Then the probability that a single hole captures a drop in a second is

$$\frac{mA}{m} = A.$$

But this is paradoxical, because suppose our unit area is cm squared, and $A = 1/10$ square cm. Then the probability that a single hole captures a drop in a second is $1/10$. For if our unit area were meters squared, then the hole area would be

$$A = \frac{1}{10}\frac{1}{10000},$$

and so the probability that a single hole captures a drop in one second would be

$$\frac{1}{100000}?$$

If I is the average energy per unit time per unit area of a beam of electromagnetic radiation (X-Rays), and S is the energy scattered per unit time by an electron by classical Thompson scattering, then

$$S = \sigma_T I.$$

σ_T is called the Thomson cross section of the electron. It has units of area. To find S in a real experiment, we would measure the total scattered energy, and divide it by the number of electrons that lie within the finite beam path, where we assume that the material is thin enough that the intensity I stays essentially constant. The Thomson cross section of the electron is approximately (Eisberg p495)

$$\sigma_T = 6.66 \times 10^{-25}\text{cm}^2.$$

This classical treatment requires several assumptions such as: a relatively low beam energy, so that the Compton change of wavelength can be neglected, a weak atomic bonding of electrons so that the electrons may be considered free, and so on.

We may extend this to a beam of discrete particles, which are scattered by a particle. In this case S becomes scattered particles per unit time, and I becomes incident particles per unit area per unit time. So again we have

$$S = \sigma I$$

where σ is the scattering cross section of the particle, which again has units of area. Suppose the beam area is A, and the number of particles exposed to the beam is n, then the local area around an individual particle is

$$\Delta A = A/n.$$

The probability of an incident particle being scattered in unit time is

$$P = \frac{S}{IA} = \frac{\sigma}{A}.$$

Thus the probability of scattering in unit time, for a beam of area ΔA, directed at a target of a single particle is

$$P_s = \sigma/\Delta A.$$

Thus this probability P_s is the ratio of the cross section to the "area" of the target particle. This is why σ is called the cross section of the particle.

In nuclear reactions such as a fission reaction a particle collision with a nucleus splits the atom to produce two atoms, and the cross section takes on the character of a reaction rate constant.

In nuclear physics a cross section is usually given in barns. A barn (bn) is an area equal to 10^{-24}cm^2.

22 Mean Free Path

When a beam of particles meets an object there is scattering and absorption of energy. Suppose the incident beam has initial intensity $I(0)$ and intensity $I(x)$ at distance x within the object material. We can write

$$dI(x) = -\frac{1}{\ell} I(x) dx$$

where ℓ is a proportionality constant. Hence

$$\frac{dI}{dx} = -\frac{1}{\ell} I.$$

So

$$I(x) = I(0) e^{-x/\ell}.$$

For a particle in the beam, let X be a random variable, which is defined as the distance that the particle travels before being scattered or absorbed. Then

$$\begin{aligned} \text{Prob}\{X : X \le x\} &= 1 - \frac{I(x)}{I(0)} \\ &= 1 - e^{-x/\ell} = \int_0^x f(y) dy, \end{aligned}$$

where $f(x)$ is a probability density function. Differentiating, we get

$$f(x) = \frac{1}{\ell} e^{-x/\ell}.$$

Then the expected value of X is

$$E(X) = \int_0^\infty x f(x) dx = \ell.$$

ℓ is called the mean free path of the beam particle. The mean free path is related to scattering and absorption cross section.

23 Particle in a One Dimensional Box

24 Particle in a Two Dimensional Box

25 The Harmonic Oscillator

26 The Hydrogen Atom

27 Particle Physics

The Dirac theory of the electron, the positron.

Force as Particle exchange, Yukawa

Quantum Electrodynamics (QED), renormalization, Feynmann, Swinger, and Tomanga.

The Eightfold Way, Quarks, Murray Gell-Mann

Quarks: Up, Down, Strange, Charmed, Top, Bottom,

The Electroweak Theory, Weinberg, Glashow, Salam,

The fine structure constant is the ratio of the electron velocity in the first Bohr orbit to the velocity of light. In the cgs system

$$\frac{e^2}{\frac{h}{2\pi}c} = \frac{1}{137}$$

See physics5.xls.

28 Bibliography

[1]Powell John L, Crasemann Bernd, **Quantum Mechanics**, 1965, Addison-Wesley, My Quantum Mechanics course text.
[2]Rae Alastair I M, **Quantum Mechanics**, 1992, Institute of Physics, undergraduate text, operator approach, advanced applications.
[3]Eisberg Robert martin, **Fundamentals of Modern Physics**, John Wiley, 1961.
[4]von Neumann John, **Mathematical Foundations of Quantum mechanics**, Princeton University Press, 1955, This work introduced the concepts of Hilbert space and operator, theory into quantum mechanics..

[5]Dirac Paul A M, **The Principles of Quantum mechanics**, 4th edition, 1958, A bible of Quantum Mechanics.
[6]Pauling Linus, Wilson E Bright, **Introduction to Quantum mechanics**, 1935, McGraw-Hill, A book of practical quantum mechanics with, application to atomic structure by a famous chemist, and a well known applied mathematician..
[7]Shankar Ramamurti, **Principles of Quantum Mechanics**, Plenum Press, 1980, A postulatory approach, contains a discussion of Feynman's path integral approach., See page 13 for a good explanation of the Dirac bra and ket notation.
[8] Messiah Albert, **Quantum Mechanics**, Volumes I and II, 1958, (Translation from the French), North-Holland
[9] Williams, W.S.C. **Nuclear and Particle Physics,** 1991, Oxford. See page 176 for a simple explanation of exchange forces given by Yukawa.
[10] Kane, Gordon **Modern Elementary Particle Physics,** 1993, Addison-Wesley
[11] Glashow, Sheldon **Interactions,** 1993, Warner Books. Contains an appendix giving a chronology of major particle physics discoveries.
[12] Smith Alex G, Carr Thomas D **Radio Exploration of the Planetary System,** 1964, Van Nostrand.
[13] Cox P. A. **Introduction to Quantum Theory and Atomic Structure,** Oxfard University Press, 1996. (Oxford Chemistry Primers)