Quaternions

James Emery

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1 The Quaternions

The quaternions form a four dimensional vector space over the reals, with basis vectors $1, i, j, k$. Thus a general element is

$$x = x_0 + x_1 i + x_2 j + x_3 k,$$

where x_0, x_1, x_2, x_3 are real numbers. Multiplication is defined by

$$i^2 = -1, j^2 = -1, k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$$

If α, β are real numbers, and x, y are quaternions, then define

$$(\alpha x)(\beta y) = (\alpha\beta)(xy).$$

So if

$$x = x_0 + x_1 i + x_2 j + x_3 k,$$

and

$$y = y_0 + y_1 i + y_2 j + y_3 k,$$

then

$$\begin{aligned} xy &= x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3 \\ &+ (x_0 y_1 + x_1 y_0 + x_2 y_3 - x_3 y_2) i \\ &+ (x_0 y_2 + x_2 y_0 + x_3 y_1 - x_1 y_3) j \\ &+ (x_0 y_3 + x_3 y_0 + x_1 y_2 - x_2 y_1) k \end{aligned}$$

Define the norm of x as

$$\|x\| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$$

Define the conjugate of x as

$$x^* = x_0 - x_1 i - x_2 j - x_3 k.$$

Then

$$xx^* = x^* x = \|x\|^2.$$

So

$$\frac{x^*}{\|x\|^2}$$

is the inverse of x, if x is not zero. Thus every nonzero element has an inverse, that is every nonzero element in the ring is a unit, so the quaternions form a division ring. However it is not a field because multiplication is not commutative, that is in general $xy \neq yx$. Because it is also a vector space it is an algebra, and hence a division algebra.

Complex numbers form a two dimensional vector space over the reals and can be used as vectors in physics for two dimensional problems. Hamilton invented quaternions to extend complex numbers to three dimensions to be used for three dimensional vector analysis in physics. They were used widely for this purpose in the nineteenth century until J. Willard Gibbs introduced his simplified vector analysis. Notice that Gibbs took from quaternions the unit i, j, k vectors. The two dimensional subspace of the quaternions generated by bases vectors $1, i$ is the complex numbers.

A pure quaternion is one of the form

$$x = x_1 i + x_2 j + x_3 k.$$

If y also is a pure quaternion, then one finds that the product is

$$xy = x \times y - x \cdot y,$$

that is the cross product (outer product, vector product) minus the dot product (inner product). So we can begin to see the connection between Hamilton's quaternions and Gibb's vector analysis.

The quaternions are associative, but the vector product is not. For example

$$(i \times i) \times j = 0,$$

but

$$i \times (i \times j) = i \times k = -j.$$

2 The Quaternion Group

The elements 1,i,j,k,-1,-i,-j,-k, under quaternion multiplication, form a non-Abelian group of order eight.

3 The Use of Quaternions for Representing Rotations

Let x be a unit quaternion, meaning its norm equals 1. We claim that x represents a rotation. Let

$$x = x_0 + x_1 i + x_2 j + x_3 k$$

Let the vector part be

$$y = x_1 i + x_2 j + x_3 k$$

then

$$u = \frac{y}{\|y\|}$$

is a unit vector and

$$x = x_0 + \|y\| u$$

Then

$$1 = \|x\|^2 = x_0^2 + \|y\|^2 \|u\|^2 = x_0^2 + \|y\|^2$$

So for some angle θ

$$x_0 = \cos(\theta)$$

and

$$\|y\| = \sin(\theta).$$

So

$$x = \cos(\theta) + \sin(\theta)u$$

Then given a vector v, the vector

$$v' = xvx^{-1} = xvx^*,$$

where the multiplication is quaternion multiplication, is the vector obtained by rotating vector v about the u axis by angle 2θ (proof omitted).

Conversely, given a rotation axis specified by a unit vector u, and a rotation about this axis by angle 2θ, then the rotation is given by the quaternion

$$x = \cos(\theta) + \sin(\theta)u.$$

Rotations specified by matrices are 3 by 3 orthogonal matrices, that is the column vectors are unit vectors and the columns are orthogonal to each other. One can compute the rotation matrix from the quaternion, or the quaternion from the matrix. The quaternion is specified with just four numbers, and a nine number matrix need not be specified to do calculations.

4 Octonions

The octonions were discovered in 1843 by John T. Graves, a friend of William Hamilton, who called them octaves. They were discovered independently by Arthur Cayley (1845). They are sometimes referred to as Cayley numbers or the Cayley algebra.

The octonions are an eight dimensional vector space, with basis vectors $1, i, j, k, l, il, jl, kl$ with multiplication table:

1	i	j	k	l	il	jl	kl
i	-1	k	$-j$	il	$-l$	$-kl$	jl
j	$-k$	-1	i	jl	kl	$-l$	$-il$
k	j	$-i$	-1	kl	$-jl$	il	$-l$
l	$-il$	$-jl$	$-kl$	-1	i	j	k
il	l	$-kl$	jl	$-i$	-1	$-k$	j
jl	kl	l	$-il$	$-j$	k	-1	$-i$
kl	$-jl$	il	l	$-k$	$-j$	i	-1

The octonions are the largest of the four normed division algebras, which are, the Real Numbers, the Complex Numbers, the Quaternions, and the Octonions.

5 References

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