Relativity Discussion

4/19/2007

Jim Emery

Einstein and his assistants, Peter Bergmann, and Valentin Bargmann, on there daily walk to the Institute for advanced Study at Princeton.

Photograph by Lucien Aigner

Einstein, 1940, with his two co-workers, Valentin Bargmann (left) and the author (right), on their daily walk to the Institute for Advanced Study at Princeton.

Special Relativity

The **Lorentz** Transformation

Covariance, Four-Vectors

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$x' = \gamma(x - vt)$$

$$y' = y$$

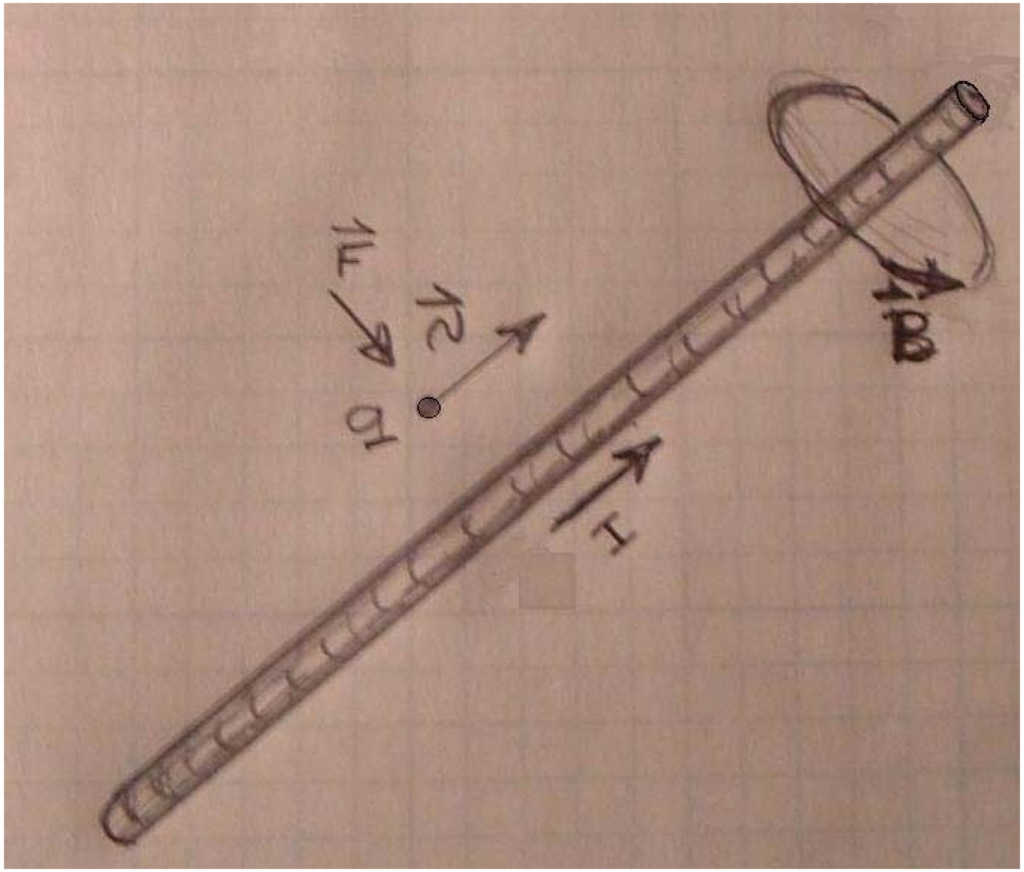
$$z' = z$$

$$t' = \gamma(t - \frac{xv}{c^2}).$$

The Metric
Distance Between Events

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

Magnetic Field Becomes an Electric Field

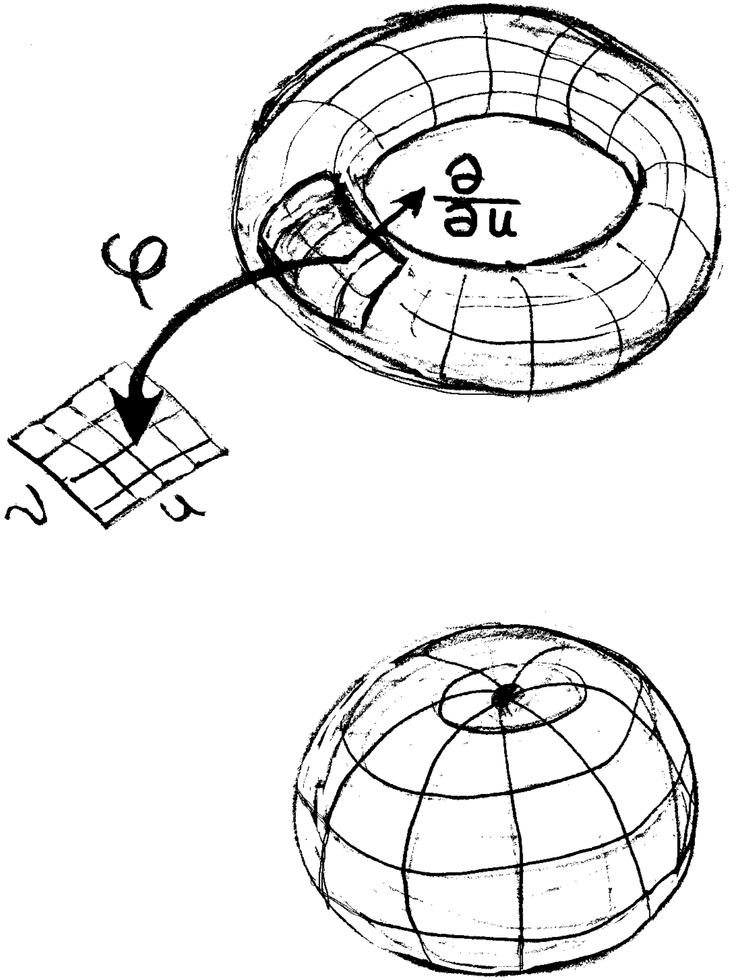


Feynmann Lectures on Physics

Differential Manifolds

- Coordinate Maps
- Tangent and Cotangent Spaces
- Covariant Derivative
- Geodesics
- Riemannian Space
- Metric Coefficients

A Manifold

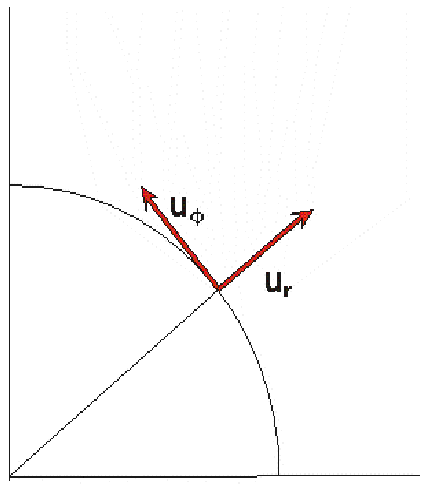


The Tangent Space

$$A_P[f] = \frac{d}{dt} f(P + tA)|_0$$

The directional derivative of a function f in the direction A, at P, is a linear functional on the space of functions, and is identified With the vector A, (or with a curve through P in the direction A). It has the properties of a derivation. Such derivations constitute The tangent space of the manifold at the point P.

Curvilinear Coordinates in a 2D Flat Space



Polar Coordinates

$$x = r\cos(\phi)$$

$$y = r\sin(\phi)$$

$$\mathbf{u}_\phi = \frac{\partial}{\partial\phi} \qquad \mathbf{u}_r = \frac{\partial}{\partial r}.$$

Polar Coordinate Example

$$x = r\cos(\phi)$$

$$y = r\sin(\phi)$$

The unit coordinate vectors are

$$\mathbf{u}_\phi = \frac{\partial}{\partial\phi}$$

and

$$\mathbf{u}_r = \frac{\partial}{\partial r}.$$

The metric is

$$ds^2 = g_{11}d\phi^2 + g_{12}d\phi dr + g_{21}dr d\phi + g_{22}dr^2$$
$$= r^2 d\phi^2 + dr^2.$$

As a quadratic form this may be written as

$$ds^2 = \begin{bmatrix} d\phi & dr \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} d\phi \\ dr \end{bmatrix},$$

where the matrix

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix},$$

is a positive definite symmetric matrix. A velocity $\mathbf{v}$ would be written as

$$\mathbf{v} = v_1\mathbf{u}_\phi + v_2\mathbf{u}_r.$$

To differentiate the velocity, we can not just differentiate the components v_1 and v_2, because the unit coordinate vectors $\mathbf{u}_\phi$, and $\mathbf{u}_r$, vary with position. This is why the Christoffel Symbols are needed for a covariant derivative, that is, a derivative independent of coordinate system.

Classical Tensors

$$dy_i = \sum_{j=1}^{3} \frac{\partial y_i}{\partial x_j} dx_j = \frac{\partial y_i}{\partial x_j} dx_j,$$

$$\frac{\partial f}{\partial y_i} = \frac{\partial x_j}{\partial y_i} \frac{\partial f}{\partial x_j}.$$

$$C^{nop}_{qr}(y) = \frac{\partial y_n}{\partial x_i} \frac{\partial y_o}{\partial x_j} \frac{\partial y_p}{\partial x_k} \frac{\partial x_l}{\partial y_q} \frac{\partial x_m}{\partial y_r} C^{ijk}_{lm}(x).$$

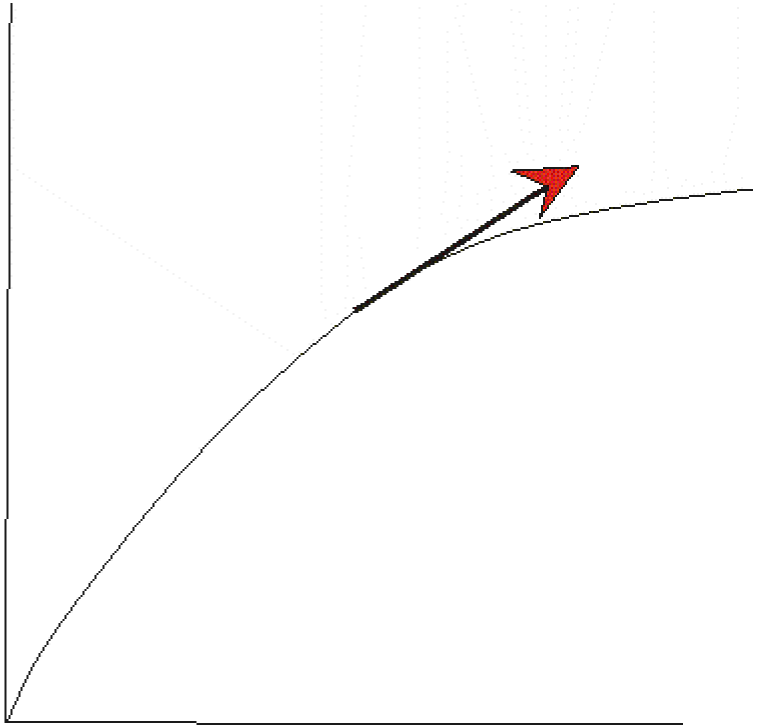
Basis Vectors for the Tangent and Cotangent Spaces

$$\mathbf{b}_i = \frac{\partial}{\partial q_i}.$$

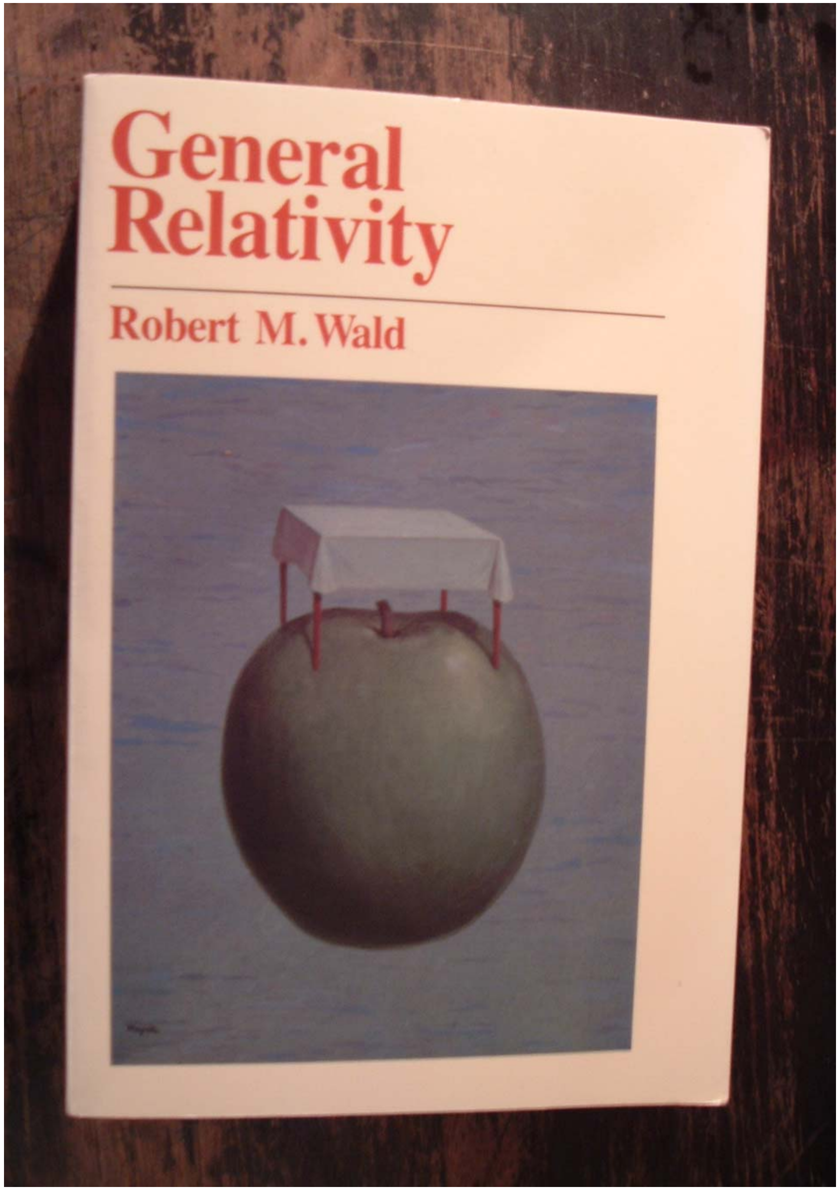
$$\mathbf{dq}^j(\frac{\partial}{\partial q_i}) = \delta_i^j.$$

The q are coordinates. The partial derivative operators are linear functionals, and so tangent vectors. They form a basis of the Tangent space at a point of the manifold. The differentials dq are duals, and so are a basis of the cotangent space. These are respectively contravariant and covariant vectors.

A Velocity Vector is in the Tangent Space of the Manifold

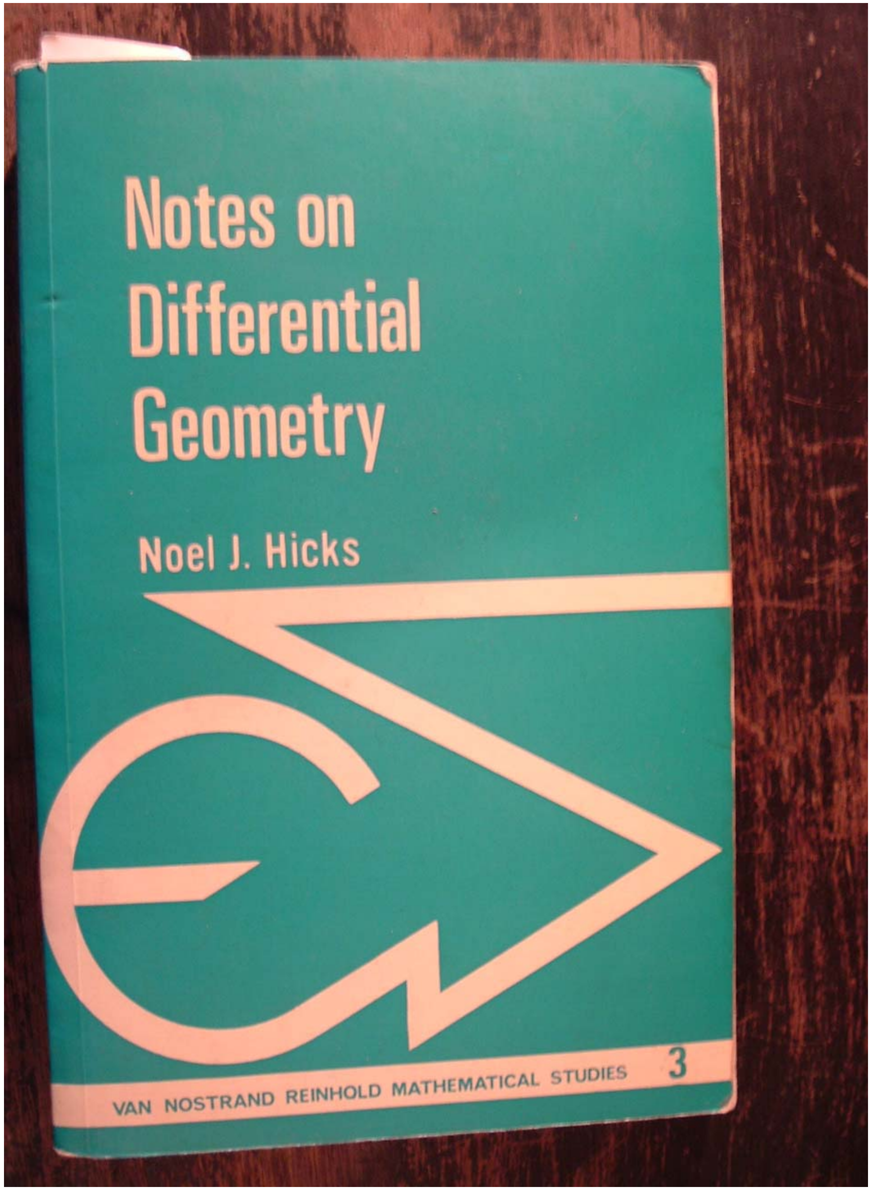


We differentiate to get the acceleration and the force on the particle. But we can’t just differentiate the vector components. We must have Covariance.



General Relativity

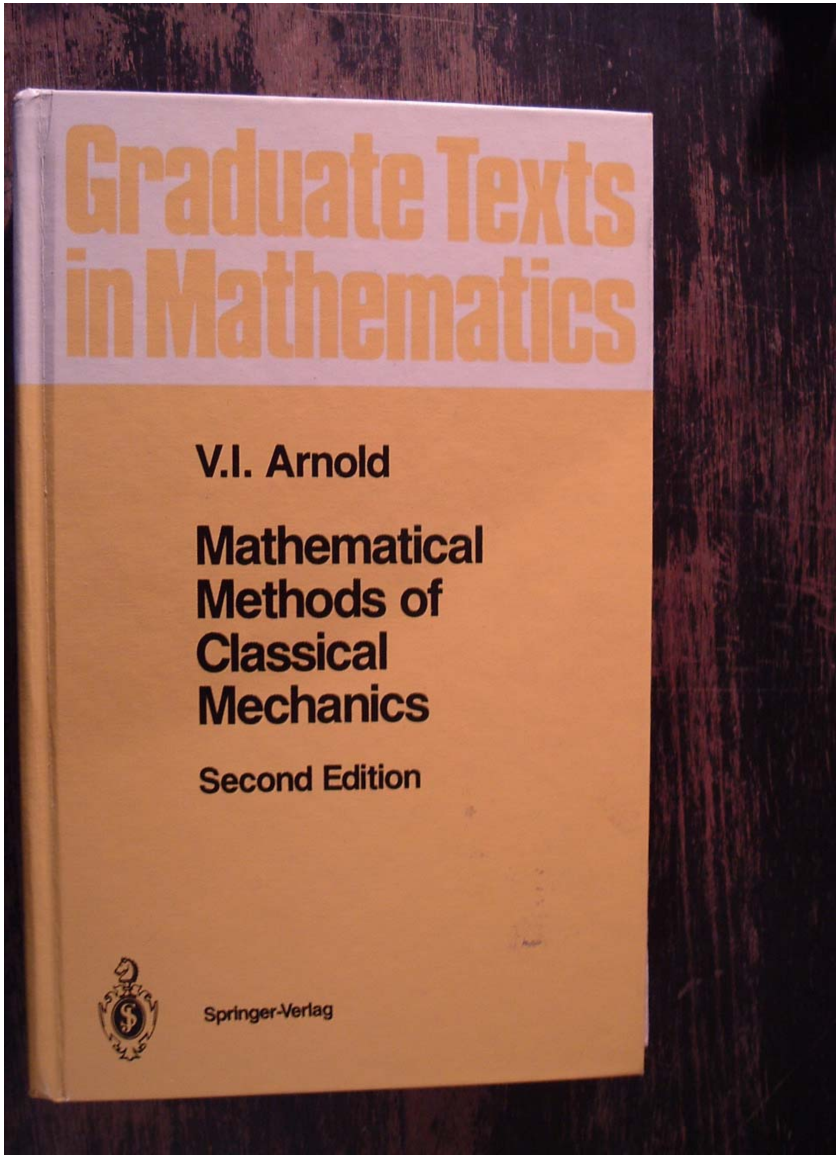
Robert M. Wald



Notes on Differential Geometry

Noel J. Hicks

VAN NOSTRAND REINHOLD MATHEMATICAL STUDIES 3



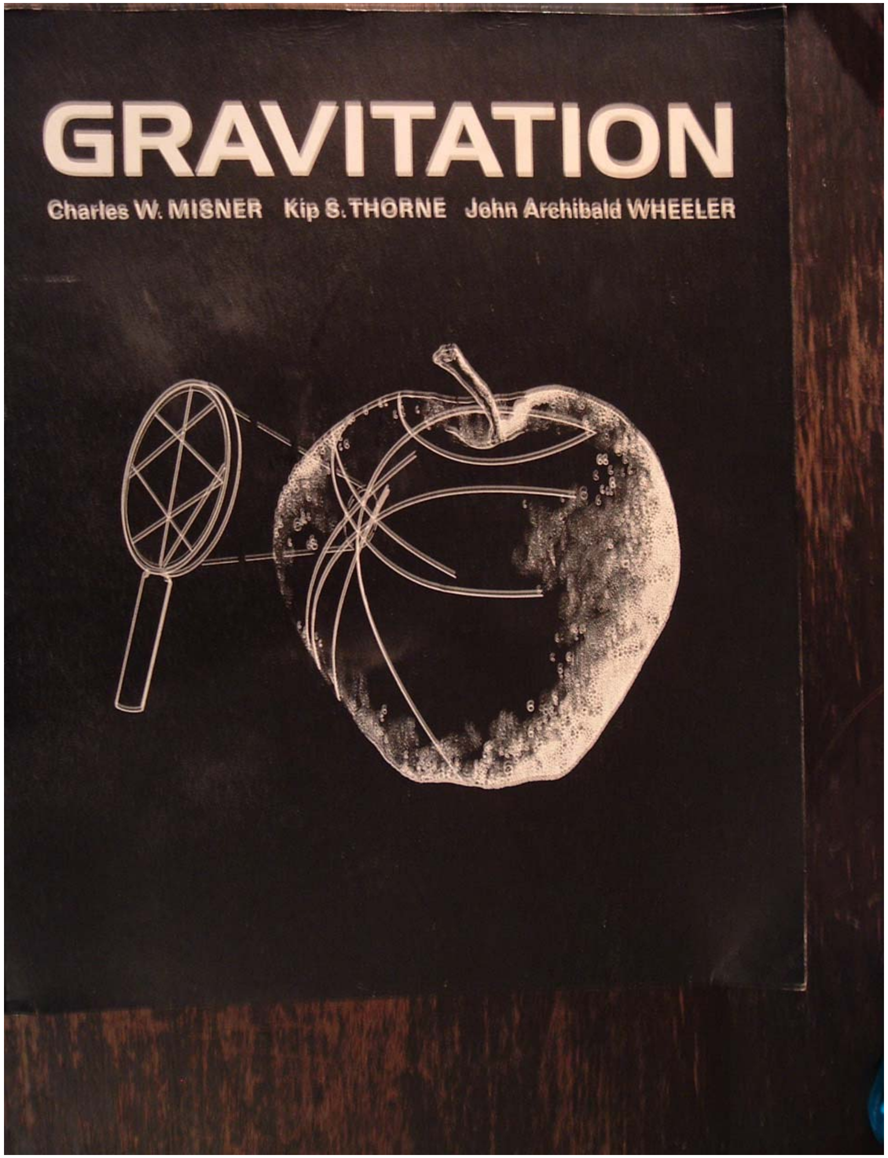
Graduate Texts in Mathematics

V.I. Arnold

Mathematical Methods of Classical Mechanics

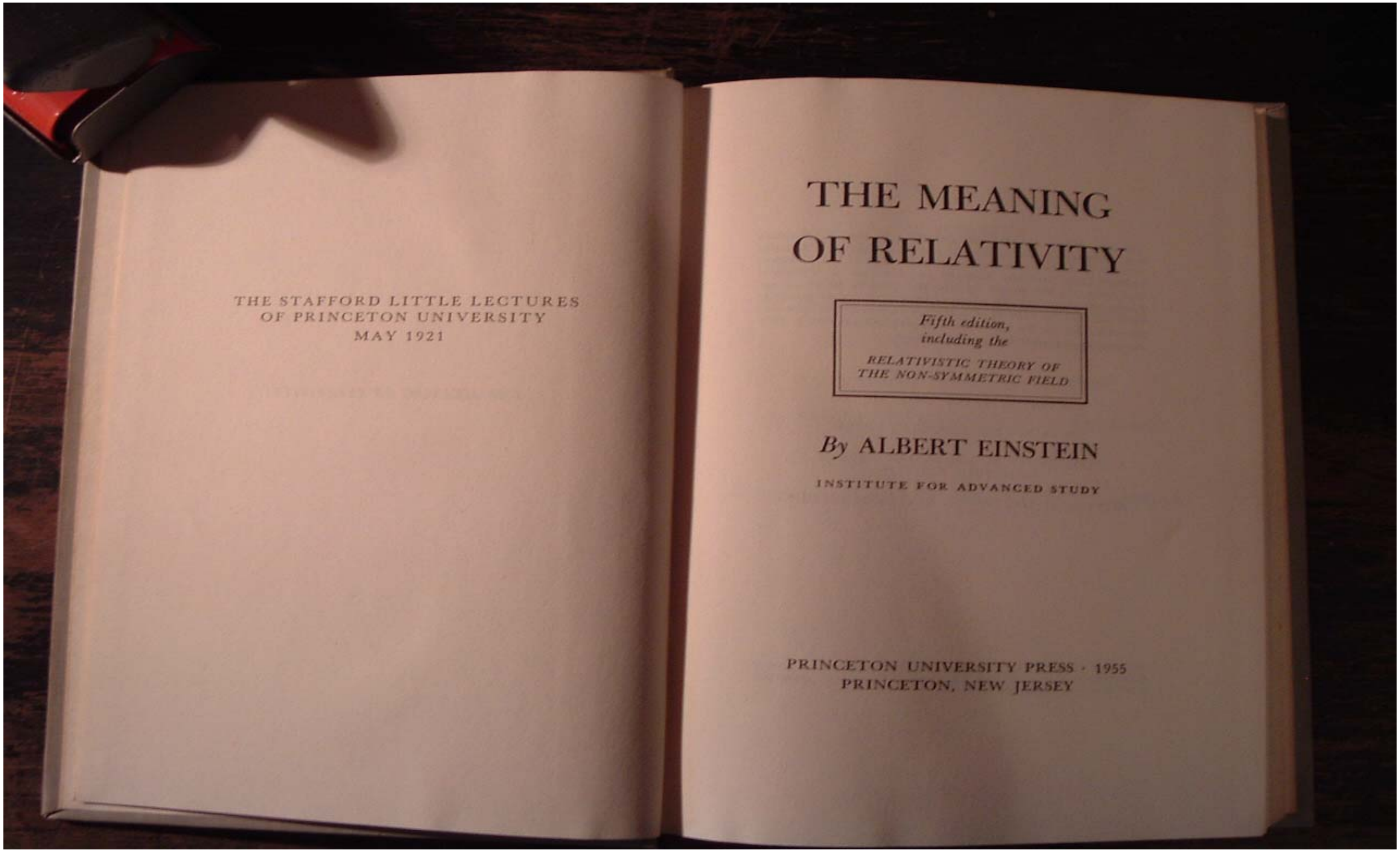
Second Edition

Springer-Verlag



GRAVITATION

Charles W. MISNER Kip S. THORNE John Archibald WHEELER



THE STAFFORD LITTLE LECTURES
OF PRINCETON UNIVERSITY
MAY 1921

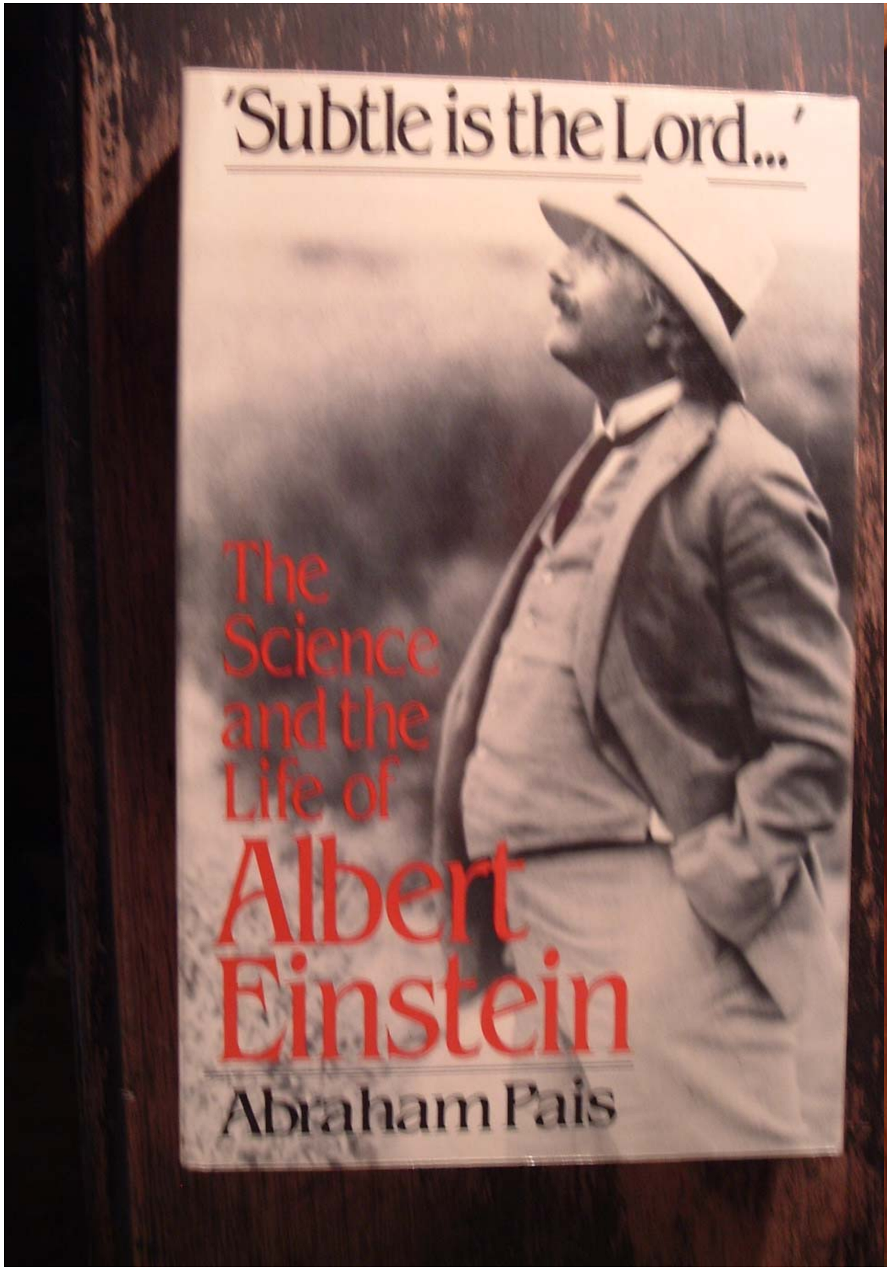
THE MEANING OF RELATIVITY

Fifth edition,
including the
RELATIVISTIC THEORY OF
THE NON-SYMMETRIC FIELD

By ALBERT EINSTEIN

INSTITUTE FOR ADVANCED STUDY

PRINCETON UNIVERSITY PRESS · 1955
PRINCETON, NEW JERSEY

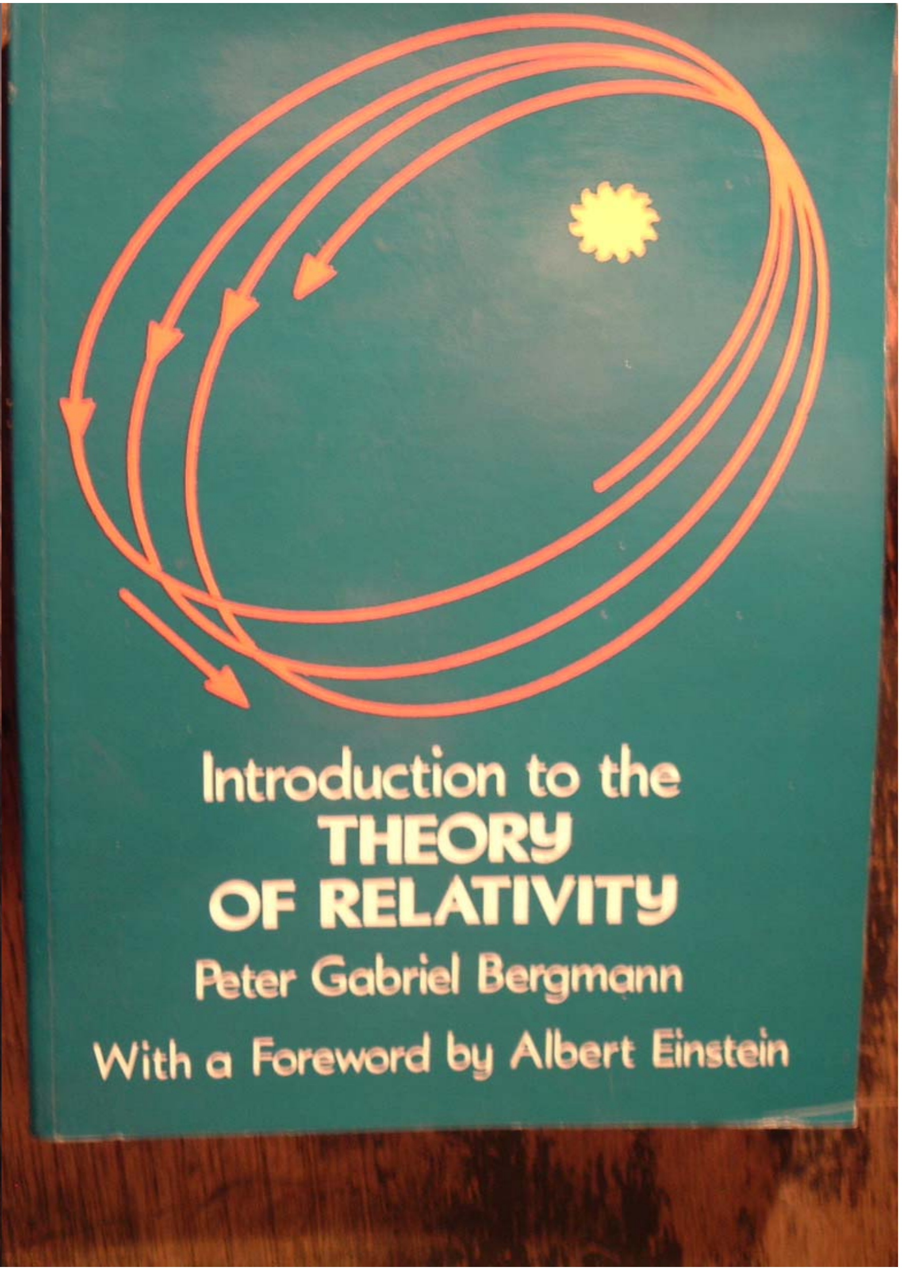


'Subtle is the Lord...'

The Science and the Life of

Albert Einstein

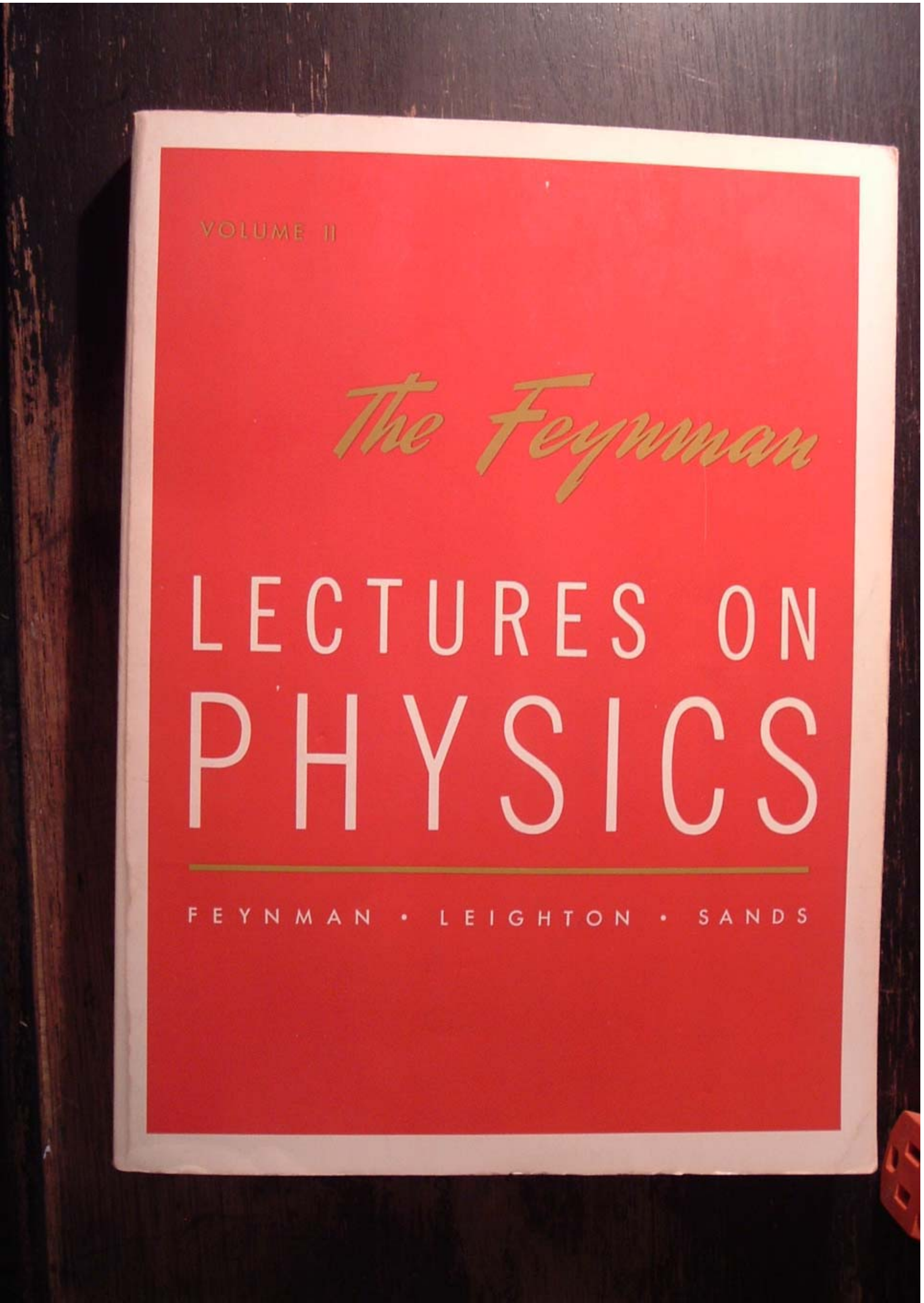
Abraham Pais



Introduction to the
THEORY
OF RELATIVITY

Peter Gabriel Bergmann

With a Foreword by Albert Einstein

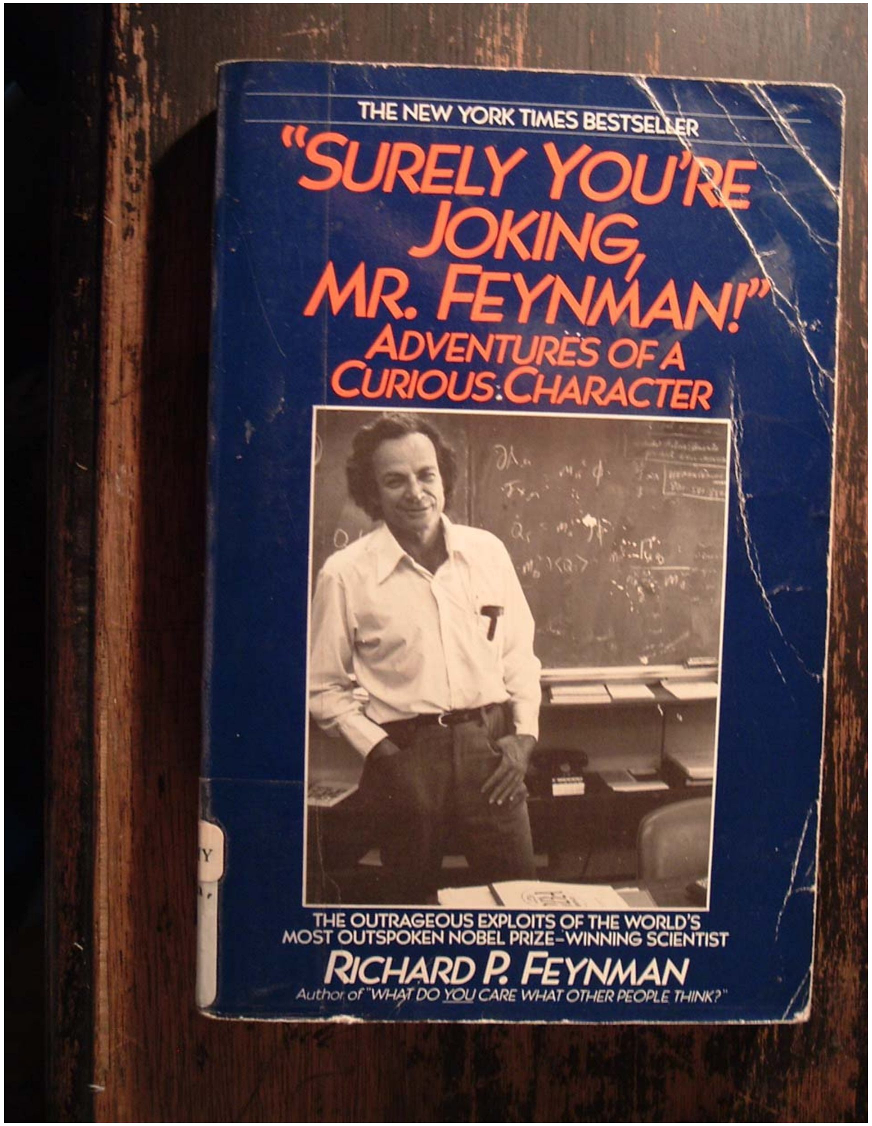


VOLUME II

The Feynman

LECTURES ON PHYSICS

FEYNMAN · LEIGHTON · SANDS



THE NEW YORK TIMES BESTSELLER

"SURELY YOU'RE JOKING, MR. FEYNMAN!"

ADVENTURES OF A CURIOUS CHARACTER

THE OUTRAGEOUS EXPLOITS OF THE WORLD'S MOST OUTSPOKEN NOBEL PRIZE–WINNING SCIENTIST

RICHARD P. FEYNMAN

Author of "WHAT DO YOU CARE WHAT OTHER PEOPLE THINK?"

The Covariant Derivative in Cartesian Coordinates is the directional derivative in the direction of a curve.

Given any curve α in R^n with $P = \alpha(0)$ and

$$\frac{d\alpha}{dt}(0) = X,$$

the covariant derivative of the vector field Y in the direction X at the point P is

$$\nabla_X Y(P) = \frac{d}{dt} Y(\alpha)(0)$$

Gauss' Intrinsic Geometry of Surfaces Theorem Egregium

$$ds^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} g_{ij} dx^i dx^j ,$$

Christoffel Symbols define the covariant derivative for the curvilinear coordinates in flat space.

$$g_{ij} = (\mathbf{b}_i, \mathbf{b}_j), \;\; g^{ij} = (\mathbf{b}^i, \mathbf{b}^j),$$

$$\frac{\partial \mathbf{b}_i}{\partial q^j} = \Gamma^k_{ij} \mathbf{b}_k, \qquad \mathbf{v}(t) = \dot{q}^i \mathbf{b}_i,$$

$$\frac{d\mathbf{v}}{dt} = (\ddot{q}^k + \dot{q}^i \dot{q}^j \Gamma^k_{ij}) \mathbf{b}_k.$$

In a Riemannian, or Semi-Riemannian Space, There is a unique Covariant Derivative defined via the Christoffel Symbols, which are in turn defined by the Metric coefficients.

$$\Gamma_{ij}^{k} = \frac{g^{kl}}{2}\left(\frac{\partial g_{jl}}{\partial u_i} + \frac{\partial g_{li}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_l}\right).$$

Given a curve with tangent T, and a vector field Y defined along the curve, if the covariant derivative of Y in the direction of T is zero, then Y is parallel translated along the curve. If the covariant derivative of T in the direction of the curve is zero, then the curve is a geodesic.

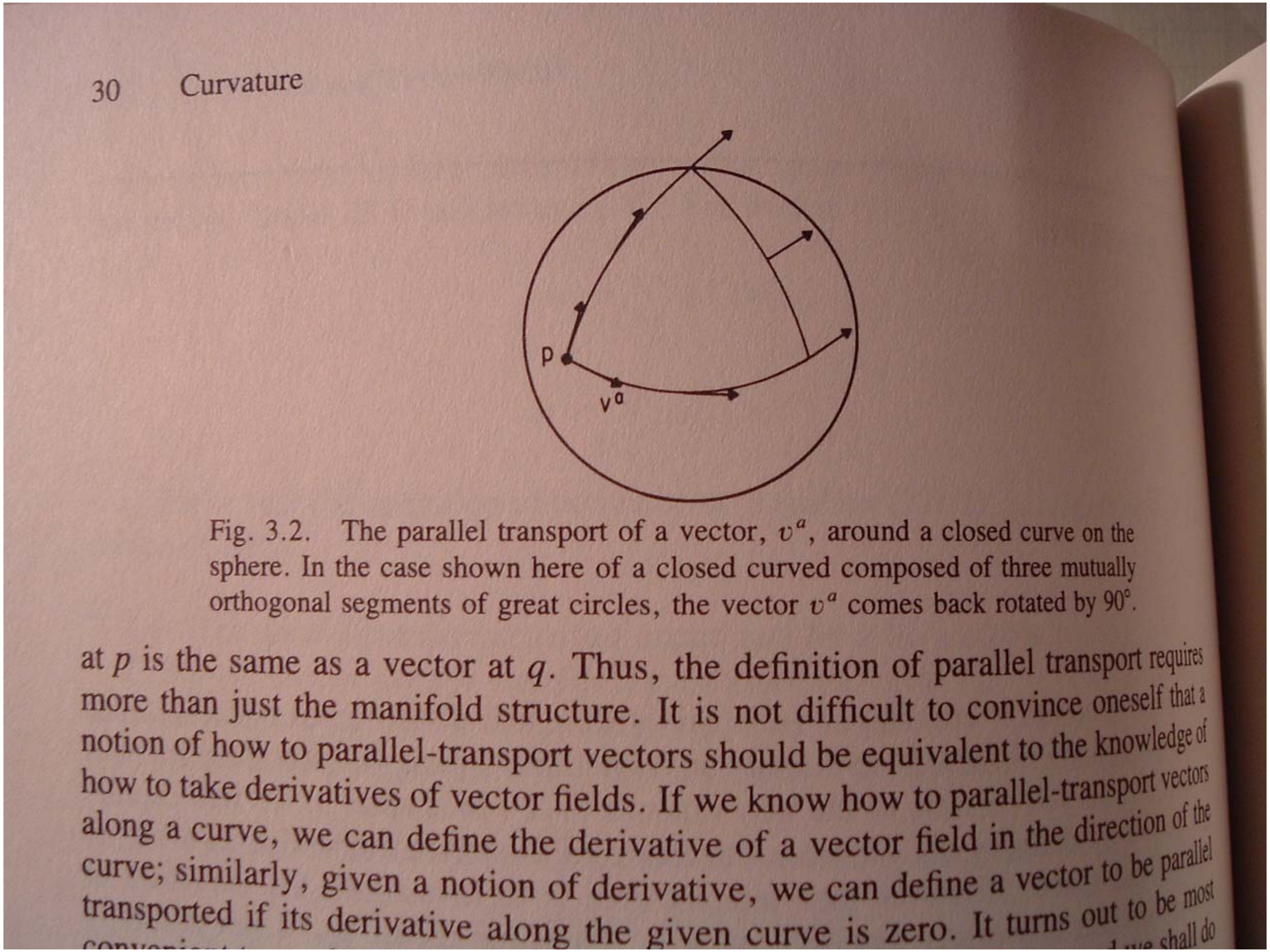
$$\nabla_T Y = 0,$$

$$\nabla_T T = 0,$$

In Space-Time a geodesic curve is the path of a particle moving in the curved space due to mass-energy and so is the analog of the straight line motion of an object not acted on by a force as given by Newton's first law.

So the task in General Relativity is to compute the metric coefficients g. These coefficients also define the Riemannian curvature of the space. So if the Riemannian curvature can be determined, then by inversion one can find the metric coefficients and thus solve the General Relativity problem. The equation to be solved that is determined by the curvature tensor is known as the Einstein equation.

Parallel translation on a 2-D surface defines the Riemann Curvature In higher dimensional Spaces we get the Riemann Curvature Tensor Again using parallel translation.



30 Curvature

Fig. 3.2. The parallel transport of a vector, v^a, around a closed curve on the sphere. In the case shown here of a closed curved composed of three mutually orthogonal segments of great circles, the vector v^a comes back rotated by 90°.

at p is the same as a vector at q. Thus, the definition of parallel transport requires more than just the manifold structure. It is not difficult to convince oneself that a notion of how to parallel-transport vectors should be equivalent to the knowledge of how to take derivatives of vector fields. If we know how to parallel-transport vectors along a curve, we can define the derivative of a vector field in the direction of the curve; similarly, given a notion of derivative, we can define a vector to be parallel transported if its derivative along the given curve is zero. It turns out to be most

Ricci tensor from contraction of the Riemann curvature tensor.
Poisson's Equation for classical gravitational potential.
The stress-energy tensor, a source of the field.

$$R_{ac} = R^b_{abc}$$

$$\nabla^2 \phi = 4\pi K \rho.$$

$$T_{ab} v^a v^b,$$

Einstein's equation.

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}.$$

Wald summarizes:

Spacetime is a manifold M
Lorentz metric g_{ab}
curvature of g_{ab} is related to matter by Einstein's equation.

The Schwarzschild Solution.

Verifications of the General Theory:

(1) The advance of the Perihelion of Mercury.

(2) The deviation of Light by the Field of the Sun, (Eclipse of 1919)

(3) Clocks slowed by a flight around the earth.