STEM Society Meeting, October 14, 2014

James Emery

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Contents

1 About the STEM Society and the STEM Society Website

STEM is an abbreviation for Science, Technology, Engineering and Mathematics. The acronym STEM is commonly associated with K-12 education, but our use of the term is only slightly bound to this meaning. There are over one hundred people on the mailing list, although a much smaller group attends any one meeting. We meet on the second Tuesday of each month at the Trailside Center at 99th and Holmes in Kansas City, Missouri. The meetings are open to all. The start time is 6PM. We make presentations, have discussions, and have demonstration experiments. These relate to Science, the History of Science, Mathematics, Engineering, Philosophy and Technology at all levels. The topics have ranged from a technical discussion of the mathematics of General Relativity to scientific experiments for young students.

These meeting notes contain links to many other documents, which may be viewed or downloaded by clicking the link. A partial list of documents can be reached by clicking the heading **Documents**. The meeting notes may also be viewed in an archive file (archive.pdf), which is in the list of documents. Many of the documents are PDF files. They may be viewed or downloaded to the computer by clicking, provided Adobe Reader, or another program capable of reading PDF files, is present. There are many more documents available at the site than are listed under **Documents** because the documents.htm file is not at all up to date. The last time I checked, about March 2014, there were about 350 document files on the site. We are in the process of creating better techniques for finding documents and authors. The first meeting of the STEM Society was in November of 2006. For several years we used the content management program called Joomla. It had a fancy looking interface, but was hard to use. It overran the space somehow at our internet provider Bluehost. So we now have a very simple HTML site. It is not so slick looking as Joomla, but very easy to maintain and modify.

The web site is:

`http://www.stem2.org/`

Direct to the documents list:

`http://www.stem2.org/je/documents.htm`

Direct to the archive file:

http://www.stem2.org/je/archive.pdf

2 The October Meeting Announcement

The October meeting of the STEM Society will take place on the second Tuesday of the month, October 14, 2014, at the Trailside Center at 99th and Holmes in Kansas City, Missouri. The starting time is 6PM.
Here are some possible topics:

(1) For myself, I have thought about presenting various topics, but I have not really fixed upon definite ones at this time.

I would like to review some of the topics we have treated over the years, and give some tools for locating such topics on our website.

Under consideration are the following: Information about scripting in, Windows, Macintosh, Linux. A review of Taylor's Formula and power series. The electret microphone, Opamps, and an amplifier for the electret piezoelectric device. The Theory of Waves, and their representation, Topics in Differential Equations, The relation between entropy in thermodynamics, and entropy in information theory. Physics Experiments, The adventures of Paul Erdös, the documentary film "N is a Number," Digital Signal Processing, Shannon's Sampling Theorem. Algebraic Topology.

(2) Someone could perhaps lead a discussion of the Ebola virus, and its appearance in neighboring states.

(3) Next month (November) we shall have a demonstration by a member of the Johnson County Crime Lab. This demonstration will feature video equipment to document a crime scene. This will include actual crime scene pictures.

(4) Books.

(5) Surprise projects and discussions.

The STEM Society Website:

http://www.stem2.org/

3 Crazy Quotes, and Useless Babies

Niels Bohr to Wolfgang Pauli ”Your theory is crazy, but not crazy enough,”

”What is the use of a newborn baby?”, Michael Faraday or James Clerk Maxwell?

4 Scripting

Writing batch files for Windows (.bat), and shell scripting files (.sh) for Unix and Mac OS. See the document called **Batching and Scripting**

http://www.stem2.org/je/batch.pdf

5 Calculus Review, Taylor's Formula and Power Series

The following material about Taylor's formula and Taylor's Series in the following sections comes from the document called **Quick Calculus**:

http://www.stem2.org/je/calcq.pdf

6 Taylor's Formula

Functions can be approximated by polynomials. Consider the polynomial

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + c_3(x - x_0)^3 + ... + c_n(x - x_0)^n$$

The kth derivative at x_0 is

$$p^{(k)}(x_0) = k!c_k.$$

So

$$c_k = \frac{p^{(k)}(x_0)}{k!}.$$

So the polynomial can be written as

$$p(x) = \sum_{k=0}^{n} \frac{p^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Now consider an arbitrary function with n derivatives. The polynomial

$$p(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,$$

has derivatives that agree with the derivatives of function f at x_0, that is

$$f^k(x_0) = p^k(x_0)$$

for $k = 0, 1, 2, 3, ..., n$.

So in a neighborhood of x_0, $f(x)$ is approximated by the polynomial $p(x)$. $p(x)$ is called the nth degree Taylor polynomial for function f. This is an approximation at x in general so there is an error term. The folowing theorem gives an expression for the error.

Theorem *Taylor's Formula With Remainder*. Let f be a function that has n derivatives at each point in the interval (a, b). Then given x_0 and x in (a, b) there is a number c, between x_0 and x, so that

$$f(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + f^{(2)}(x_0)\frac{(x - x_0)^2}{2!} + f^{(3)}(x_0)\frac{(x - x_0)^3}{3!} + ... +$$

$$f^{(n-1)}(x_0)\frac{(x - x_0)^{n-1}}{(n-1)!} + f^{(n)}(c)\frac{(x - x_0)^n}{n!}$$

.

This c depends on x, x_0 and n

Proof. Let p be the Taylor polynomial of degree $n - 1$

$$p(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + f^{(2)}(x_0)\frac{(x - x_0)^2}{2!} + ... + f^{(n-1)}(x_0)\frac{(x - x_0)^{n-1}}{(n-1)!}.$$

The strategy is to take the difference between $f(x)$ and the right side of the Taylor formula, where an unknown constant M replaces the derivative value of $f^{(n)}(c)$ in the error term. The difference is set equal to a function $g(x)$. Function $g(x)$ is differentiated n times, applying Rolles Theorem repeatedly. This will allow us to find the constant c, and determine the constant M so that

$$M = f^{(n)}(c).$$

Without loss of generality we shall assume $x_0 < x$.

Let M be defined by

$$f(x) = p(x) + \frac{M(x - x_0)^n}{n!}.$$

We define a function g by

$$g(t) = f(t) - [p(t) + \frac{M(t - x_0)^n}{n!}].$$

This will allow us to apply Rolles theorem to g on the interval $[x_0, x]$.

So by our definitions we have

$$g(x_0) = f(x_0) - f(x_0) + 0 = 0,$$

and

$$g(x) = 0.$$

By Rolle's' Theorem there is a number x_1, $x_0 < x_1 < x$, so that $g'(x_1) = 0$. Because $g'(x_0) = 0$, we may apply Rolle's Theorem again to $g' = g^{(1)}$ and obtain a number $x_2, x_0 < x_2 < x_1$, so that $g^{(2)}(x_2) = 0$.

Continuing in this way, after n steps, we find that there is an x_n so that $x_0 < x_n < x$, and letting $c = x_n$, so that

$$f^{(n)}(c) - M = g^{(n)}(c) = 0$$

(Notice that we have annihilated the polynomial p by n differentiations). Therefore $M = f^{(n)}(c)$ and so we have

$$f(x) = p(x) + f^{(n)}(c)\frac{(x - a)^n}{n!}.$$

This is Taylor's Formula.

If we let n go to infinity, we get the formal power series

$$\sum_{k=0}^{\infty} f^{(k)}(a)\frac{(x - a)^k}{k!}$$

known as the Taylor series for the function $f(x)$. Such a series may or may not converge and may or may not represent the function $f(x)$.

Theorem If the error term in Taylor's Formula goes to zero as n goes to infinity then the Taylor Series for $f(x)$ converges and represents the function $f(x)$.

Proof.

See the section on Taylor Series.

The converse of this theorem is false. That is there exists a function $f(x)$ which has derivatives of all orders (called a C^∞ function), that has a Taylor Series that converges, but the series does not represent the function. See the section on Taylor Series.

Examples of Taylor's Formula and Taylor's series.

Let the nth degree Taylor polynomial for the function $f(x)$ be

$$p_n(x) = \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!}$$

Then by Taylor's formula, the Taylor series converges pointwise at x provided

$$f(x) - p_n(x) = f^{(n)}(c(x,n))\frac{(x-x_0)^{n+1}}{(n+1)!} \to 0.$$

We write $c(x,n)$ because the constant c in Taylors formula depends upon x and n.

The Taylor Series for the exponential function $\exp(x)$**.**

Each derivative of the exponential function $\exp(x)$ equals the function itself. So the Taylor series for the exponential function developed about 0 is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Given $a > 0$, for all $x \in [-a, a]$

$$|\exp(x) - p_n(x)| \le \exp(c(x,n))\frac{a^{n+1}}{(n+1)!} \le \exp(a)\frac{a^{n+1}}{(n+1)!}.$$

The right side goes to zero as n goes to ∞. Thus the series represent the function for all real $x \in [-a, a]$. And the convergence is uniform on this interval (See a later section on uniform convergence of a sequence of functions).

Since $a > 0$ is arbitrary, the series converges and represents the function $\exp(x)$ for all real x. However, the convergence of the polynomial functions p_n is not uniform on the whole real line. It will take more terms for the series to converge to $\exp(x)$ for large x.

Approximating the Sine Function With a Taylor Polynomial.

The derivative of $\sin(x)$ is $\cos(x)$, of $\cos(x)$ is $-\sin(x)$. It follows that the Taylor series about 0 for $\sin(x)$ is

$$\begin{aligned}\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + ... \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)!}.\end{aligned}$$

Keeping the 19 nonzero terms in Taylors Formula up to degree $n = 37$ we find that the Taylor approximation is accurate to a full 14 decimal places over the interval $[0., 2\pi]$, as is verified in the following table produced by computer program **sine.ftn**.

x	$\sin(x)$	Taylor Approximation
.00000000000000	.00000000000000	.00000000000000
.26179938779915	.25881904510252	.25881904510252
.52359877559830	.50000000000000	.50000000000000
.78539816339745	.70710678118655	.70710678118655
1.0471975511966	.86602540378444	.86602540378444
1.3089969389957	.96592582628907	.96592582628907
1.5707963267949	1.0000000000000	1.0000000000000
1.8325957145940	.96592582628907	.96592582628907
2.0943951023932	.86602540378444	.86602540378444
2.3561944901923	.70710678118655	.70710678118655
2.6179938779915	.50000000000000	.50000000000000
2.8797932657906	.25881904510252	.25881904510252
3.1415926535898	0.0	0.0
3.4033920413889	-.25881904510252	-.25881904510252
3.6651914291881	-.50000000000000	-.50000000000000
3.9269908169872	-.70710678118655	-.70710678118655
4.1887902047864	-.86602540378444	-.86602540378444
4.4505895925855	-.96592582628907	-.96592582628907
4.7123889803847	-1.0000000000000	-1.0000000000000
4.9741883681838	-.96592582628907	-.96592582628907
5.2359877559830	-.86602540378444	-.86602540378444
5.4977871437821	-.70710678118655	-.70710678118655
5.7595865315813	-.50000000000000	-.50000000000000
6.0213859193804	-.25881904510252	-.25881904510252
6.2831853071796	0.0	0.0

The figure called **Taylor Approximation** shows the approximation for the two polynomials of degree 5 and 11 on the interval $[0, 2\pi]$. For x near zero the approximation is quit good with a low degree polynomial. Because of the nature of the periodic function $\sin(x)$, any value is determined by values on the interval $[0, \pi/2]$.

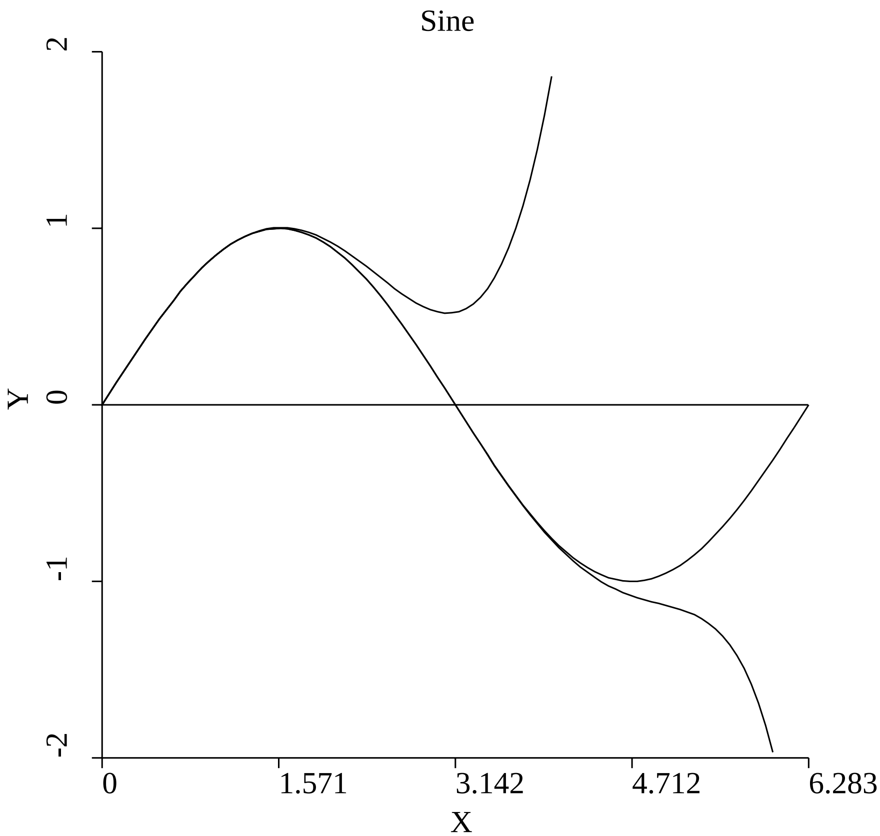

Figure 1: **Taylor Approximation**. The approximation of the function $y = \sin(x)$ on the interval $[0, 2\pi]$ by the 5th degree Taylor polynomial (upper curve), and by the 11th degree polynomial (lower curve).

7 A Non-analytic Smooth Function, A C^∞ Function Without a Taylor Series

Cauchy discovered the function

$$\exp(-1/x^2)$$

which has continuous derivatives of all orders, and whose nth derivative values at 0 are all 0, thus which has a zero Taylor series about zero, so does not represent the function, except at 0. We have

$$\lim_{x\to 0} \exp(-1/x^2) = 0$$

and all derivatives, using an induction argument, are equal to a finite sum of terms like

$$\exp(-1/x^2)\left[\frac{c}{x^k}\right],$$

where $k \geq 1$, and so which all go to zero as $x \to 0$.

$$f(x) = \exp(-1/x^2)$$

$$Df(x) = \exp(-1/x^2)[2/x^3]$$

$$D^2 f(x) = \exp(-1/x^2)[4/x^6 - 6/x^4]$$

$$D^3 f(x) = \exp(-1/x^2)[24/x^5 - 36/x^7 + 8/x^9]$$

$$D^4 f(x) = \exp(-1/x^2)[300/x^8 - 120/x^6 - 144/x^{10} + 16/x^{12}]$$

Now

$$D\exp(-1/x^2)/x^k) = 2\exp(-1/x^2)/x^{k+3} - k\exp(-1/x^2)/x^{k+1}$$

So all derivatives will be a sum of terms, each of which is a product of some constant c_k, $\exp(-1/x^2)$, and $1/x^k$, for some positive integer k. Now

$$\lim_{x\to 0}(\exp(-1/x^2)/x^k) = 0,$$

because writing $y = 1/x$ this is equivalent to

$$\lim_{y\to\infty}(y^k/\exp(y^2)) = 0.$$

Another function of this sort is one that equals $\exp(-1/x)$ for $x > 0$ and zero elsewhere.

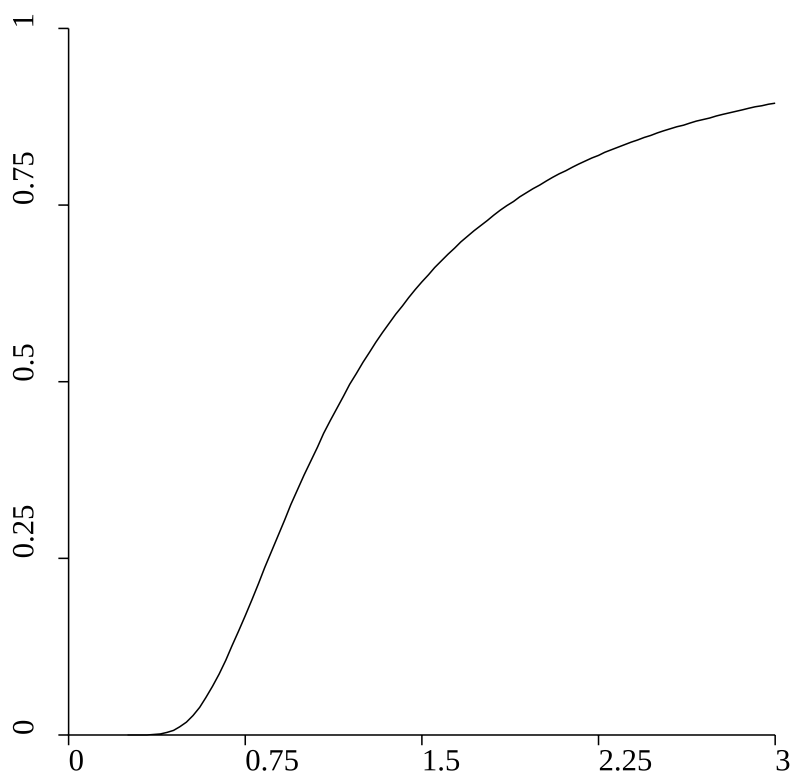

Figure 2: **A Function not represented by its Taylor Series**. $f(x) = \exp(-1/x^2)$. All derivatives at zero are zero, so the Taylor Series is the zero power series.

8 The Binomial Theorem

We have results such as

$$(a+b)^2 = a^2 + 2ab + b^2,$$

and

$$(a+b)^3 = a^3 + 3a^2b + 3a^1b^2 + b^3.$$

The general result is called the binomial theorem.
Binomial Theorem For each positive integer n we have

$$(a+b)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{n-k}b^k,$$

where

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)...(n-k+1)}{k!},$$

is the number of ways of choosing k things from n things. We read

$$\left(\begin{array}{c} n \\ k \end{array} \right),$$

as n choose k. So

$$\left(\begin{array}{c} n \\ 0 \end{array} \right) = 1$$

and

$$\left(\begin{array}{c} n \\ n \end{array} \right) = 1.$$

Proof. Suppose the theorem holds for n, then we have

$$(a+b)^{n+1} = a(a+b)^n + (a+b)^n b$$

$$= \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{(n+1)-k}b^k + \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{n-k}b^{k+1}$$

$$= \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{(n+1)-k}b^k + \sum_{k=1}^{n+1} \left(\begin{array}{c} n \\ k-1 \end{array} \right) a^{(n+1)-k}b^k$$

$$= \left(\begin{array}{c} n \\ 0 \end{array} \right) a^{n+1}b^0 + \sum_{k=1}^{n} \left[\left(\begin{array}{c} n \\ k \end{array} \right) + \left(\begin{array}{c} n \\ k-1 \end{array} \right) \right] a^{(n+1)-k}b^k + \left(\begin{array}{c} n \\ n \end{array} \right) a^0 b^{n+1}$$

$$= a^{n+1}b^0 + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] a^{(n+1)-k} b^k + a^0 b^{n+1}.$$

The expression

$$\left[\binom{n}{k} + \binom{n}{k-1} \right]$$

is

$$\frac{n!}{k!(n-k)!)} + \frac{n!}{(k-1)!(n-(k-1))!)}$$

$$= \frac{n!}{k!(n-k)!)} + \frac{n!}{(k-1)!((n+1)-k))!)}$$

$$= \frac{n!((n+1)-k))}{k!((n+1)-k))!)} + \frac{kn!}{k!((n+1)-k))!)}$$

$$= \frac{(n+1)!}{k!((n+1)-k))!)}$$

$$= \binom{n+1}{k}.$$

So the sum above becomes

$$(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{(n+1)-k} b^k.$$

Therefore by induction, the binomial theorem holds for all integers n.

9 The Binomial Series

A binomial series is an infinite series of the form

$$(1+x)^r = \sum_{k=0}^{\infty} \binom{a}{k} x^k,$$

where r is any real number, and

$$\binom{r}{k} = \frac{r(r-1)(r-2)...(r-k+1)}{k!}.$$

So for example

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1024}x^6 + ...$$

It is clear that if $0 < x < 1$ that this series converges because it is an alternating series, and the terms are decreasing in magnitude.

Consider

$$(1+x)^{-5/3} = 1 - \frac{5}{3}x + \frac{20}{9}x^2 - \frac{220}{81}x^3 + \frac{770}{243}x^4 - \frac{2618}{729}x^5 + \frac{26180}{6561}x^6 - ...$$

For x outside the interval $(-1, 1)$ this series diverges because the terms do not go to zero.

And it is not quite so obvious that this series converges for $-1 < x < 1$.

Theorem.

$$(1+x)^r = \sum_{k=0}^{\infty} \left(\begin{array}{c} a \\ k \end{array} \right) x^k,$$

for $-1 < x < 1$.

Proof. $(1+x)^r$ is defined to be $\exp(r \ln(1+x))$. Let $f(x) = (1+x)^r$, then the derivatives of f are

$$f^{(1)}(x) = r(1+x)^{r-1},$$

$$f^{(2)}(x) = r(r-1)(1+x)^{r-2},$$

$$f^{(3)}(x) = r(r-1)(r-2)(1+x)^{r-3},$$

and so on. So for any positive integer k

$$f^{(k)}(x) = [r(r-1)(r-2)...(r-k+1)](1+x)^{r-k},$$

and

$$f^{(k)}(0) = r(r-1)(r-2)...(r-k+1).$$

So the Taylor series for $(1+x)^r$ is

$$(1+x)^r = \sum_{k=0}^{\infty} \frac{r(r-1)(r-2)...(r-k+1)}{k!} x^k.$$

The convergence of the binomial series for $-1 < x < 1$ and r any real number can be proven in various ways, so for example, as a consequence of Bernstein's convergence theorem (Sergei Natanovich Bernstein 1880-1968). See p244 of **Mathematical Analysis**, 2nd edition, 1975, by Tom M. Apostol.

10 The Electret Microphone

Charles Platt, **The Eclectic Electret Microphone**, Make Magazine Volume 39, June/July/2014 p62.

`http://makezine.com/magazine/make-39/electronics-fun-fundamentals-the-eclectic-electret-microphone/`

An electret microphone is a version of a condenser microphone. A capacitor used to be called a condenser. Consider two parallel conducting plates, one plate having a positive charge Q and the other plate a negative charge also of magnitude Q. By using Gauss's law we can find the electric field E between these plates. Gauss's law is that the total electric field flux crossing a surface is equal to the the total charge contained inside the surface. This is expressed as one of Maxwell's equations

$$\nabla \cdot \mathbf{D} = \rho,$$

where ρ is the charge density, the charge per unit volume. Now $\mathbf{D} = \epsilon \mathbf{E}$ As a consequence in the region between the plates we have

$$E = \frac{\sigma}{\epsilon},$$

where σ is the charge density and ϵ is the permitivity of the dielectric between the plates. This determines E the electric field. This is the force on a unit charge. So the change in potential energy transporting a unit charge across the capacitor plates of distance d is Ed. This is the electric potential of voltage V,

$$V = dE = d\frac{\sigma}{\epsilon}$$

$$\sigma = \frac{Q}{A},$$

where A is the area of the capacitor plates. Thus

$$V = \frac{Q}{C},$$

where

$$C = \frac{\epsilon A}{d},$$

is the capacitance.

In a condenser microphone the sound waves vibrate the capacitor, changing the distance d between the capacitor plates. The charge Q is maintained relatively constant, so that the voltage V changes as d changes, thus generating a voltage signal, which is fed to an amplifier.

In an electret microphone a polarized electric material, called an electret, is placed between the capacitor plates. The name comes from electric, and magnetic, because it has a permanent polarization with electric poles, like a permanent magnetmagnetic poles. So there are permanent plus charges on one side, and negative on the other. So this electret serves as the "charge" in the capacitor. So no external power is required to supply a charge. So this microphone works something like the condensor microphone, although the analysis of its behavior will be a bit different, it having a permanent electric field.

In this circuit example, an operational amplifier is used to amplify the tiny voltage measured in millivolts, to volts. This signal is fed to a power amplifier IC.

See the behavior of the inverting amplifier in my document **Operational Amplifiers**:

`http://stem2.org/je/opamp.pdf`

11 Famous Physics Experiments

From Rich Kaufman:

Your item about physics experiments interested me. Two experiments I know of which were monumental were the Davison-Germer, and the Michelson-Morley. Also pretty significant was Meitner-Hahn.

We shall discuss these later.