# Vector Analysis

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# 1 Introduction

Vector Analysis is a classical subject dealing with those aspects of vectors which have application in Applied Mathematics and Physics. The Physicist J. Willard Gibbs is considered the founder of Vector Analysis. It has some historical connection with Hamilton's theory of Quaternions. Linear Algebra, which is the algebraic study of finite dimensional vector spaces, also bears some relationship to Vector Analysis. But it does not involve calculus. Recall that Calculus in its advanced treatment is called Analysis. It is called analysis because it involves the tiny infinitesimal details of mathematics, sort of mathematics with a microscope. Vector Analysis could easily have been called Vector Calculus. It is not just vector algebra.

Vector Analysis tends to be a subject in Applied Mathematics, and is used extensively in Physics and Engineering. Vector Analysis is usually confined to two or three dimensional Euclidean space. Related, but more advanced subjects include: Differential Geometry, Differential Forms, Tensor Analysis, and the Theory of Differential Manifolds. These subjects extend some of the ideas of vector analysis to higher dimensional and abstract spaces. As is often the case with abstraction in mathematics, ideas often become conceptually simpler, more general, and in many cases proofs become easier, provided one has a flair for the abstract. Here however we shall confine ourselves just to Vector Analysis.

## 2 The Inner Product

We shall prove the law of cosines. Suppose we have three points

$$p_0 = (0,0), p_1 = (b,0), p_2 = (x,y) = (a\cos(\theta), a\sin(\theta)).$$

These points form a triangle with sides $p_0p_2, p_0p_1, p_2p_1$. These sides have lengths $a, b, c$. The angle between side $p_0p_1$ and side $p_0p_2$ is $\theta$. We have

$$c^2 = (x-b)^2 + y^2$$

$$= (x-b)^2 + a^2 - x^2$$
$$= x^2 - 2xb + b^2 + a^2 - x^2$$
$$= a^2 + b^2 - 2xb = a^2 + b^2 - 2ab\cos(\theta).$$

Thus we have the law of cosines, namely the square of the side opposite an angle of a triangle, is equal to the sum of the squares of the adjacent sides, minus two times the product of the sides and the cosine of the angle. That is,

$$c^2 = a^2 + b^2 - 2ab\cos(\theta).$$

The inner product (dot product) of two vectors, $A$ and $B$, is defined as

$$A \cdot B = a_1b_1 + a_2b_2 + a_3b_3.$$

Then the dot product of a vector with itself is the square of its length. That is,

$$A \cdot A = a_1a_1 + a_2a_2 + a_3a_3 = \|A\|^2.$$

Let

$$C = B - A.$$

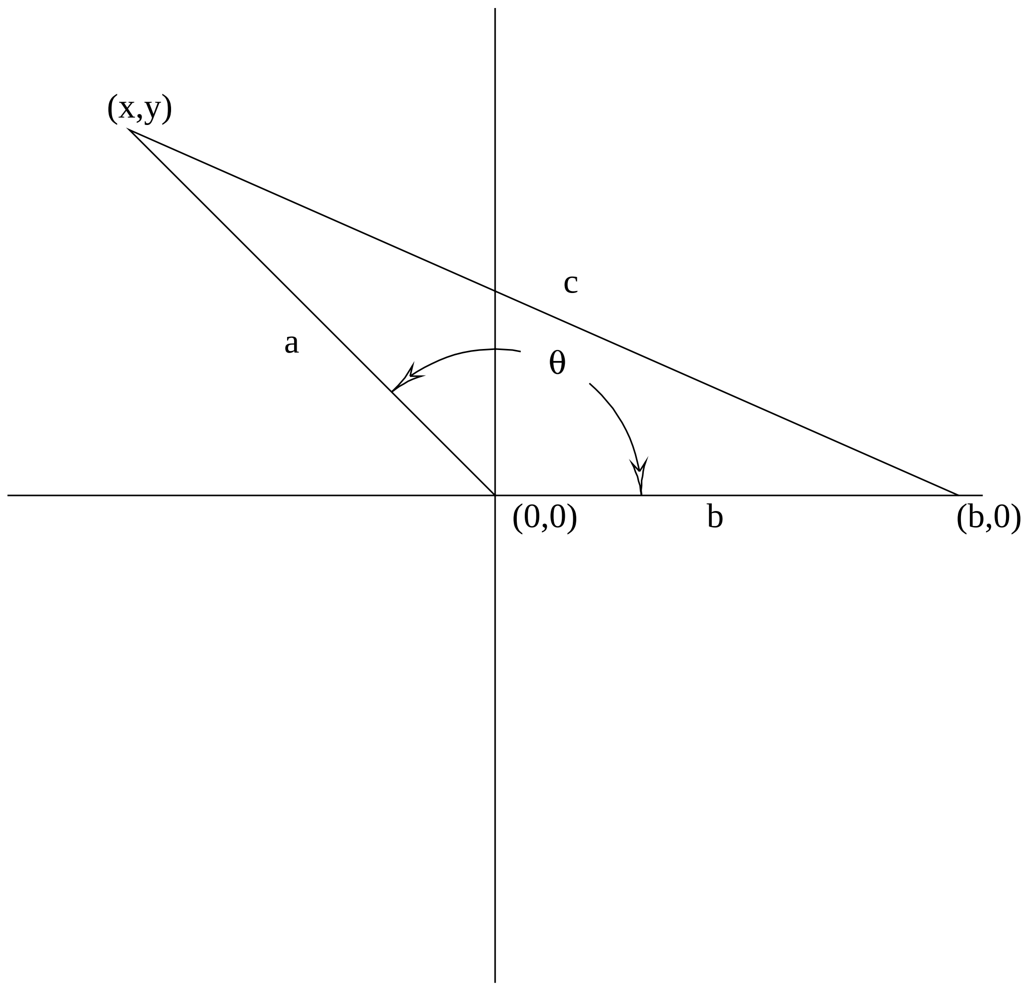


Figure 1: Derivation of the law of cosines. $x = \cos(\theta)$, $y = \sin(\theta)$. Computing $c^2$, we find that $c^2 = a^2 + b^2 - 2ab\cos(\theta)$.

Then

$$\begin{aligned} \|C\|^2 &= (B - A) \cdot (B - A) \\ &= B \cdot B - B \cdot A - A \cdot B + A \cdot A \\ &= \|B\|^2 - 2A \cdot B + \|A\|^2. \end{aligned}$$

From which it follows that

$$2A \cdot B = \|A\| + \|B\|^2 - \|C\|^2.$$

But the right hand side is, by the law of cosines,

$$2\|A\|\|B\| \cos(\theta),$$

where $\theta$ is the angle between vectors $A$ and $B$. Hence

$$A \cdot B = \|A\|\|B\| \cos(\theta).$$

Thus if the the dot product is zero, then the cosine is zero, and so the angle between the vectors is plus or minus $\pi/2$, and the vectors are perpendicular.

# 3 The Vector Product

The vector product of two vectors $A$ and $B$, (the cross product), is defined to be

$$A \times B = (a_2 b_3 - a_3 b_2)i + (a_3 b_1 - a_1 b_3)j + (a_1 b_2 - a_2 b_1)k,$$

where $i, j, k$ are the unit coordinate vectors. This may be written as a determinant with $i, j, k$ in the first row, the components of $A$ in the second, and the components of $B$ in the third row.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

When the rows of a determinant are interchanged, the sign of the determinant changes, hence

$$A \times B = -B \times A.$$

Then

$$A \times A = -A \times A.$$

But this can be true only if

$$A \times A = 0.$$

We have shown that the vector product of any two parallel vectors is zero.

Given three vectors $A, B, C$, we see that

$$A \cdot (B \times C),$$

is given as the determinant that has rows $A, B$, and $C$. By interchanging these rows twice, we see that

$$A \cdot (B \times C) = (A \times B) \cdot C.$$

That is, in the scalar triple product, the dot and the cross may be interchanged. Now using this result, we see that

$$(A \times B) \cdot B = A \cdot (B \times B) = A \cdot 0 = 0.$$

Then $A \times B$ is perpendicular to $B$. Similarly it is perpendicular to $A$. Therefore we have shown that the vector product of two vectors is perpendicular to each of them. This establishes the direction of the vector product, except possibly for sign. One may further establish the right hand rule. The direction of $A \times B$ is given by the right hand rule: Curl the fingers of your right hand from $A$ to $B$, then $A \times B$ is in the direction of your thumb. One may verify directly that if $V$ is a vector in the upper $xy$ half plane that

$$i \times V$$

points in the positive $z$ direction. This verifies the right hand rule in this case. One may also show the invariance of the cross product to a rigid motion, which establishes the right hand rule in general.

By direct computation one may verify that the vector triple product satisfies

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B).$$

This is the "Back Minus Cab Rule". We have established the direction of the cross product, now we shall find its magnitude. Let

$$C = A \times B.$$

Then

$$\begin{aligned}
\|C\|^2 &= C \cdot C \\
&= (A \times B) \cdot C \\
&= A \cdot (B \times C) \\
&= A \cdot (B \times (A \times B)) \\
&= A \cdot (A(B \cdot B) - B(B \cdot A) \\
&= (A \cdot A)(B \cdot B) - (A \cdot B)^2 \\
&= \|A\|^2 \|B\|^2 (1 - \cos^2(\theta)) = \|A\|^2 \|B\|^2 \sin^2(\theta).
\end{aligned}$$

The magnitude of the cross product is the product of the lengths of the vectors, times the sine of the angle between them,

$$\|A \times B\| = \|A\| \|B\| \sin(\theta).$$

**Example** The equation of a plane. Let the plane have a unit normal vector $N$. Let $P = (x, y, z)$ be a point on the plane. Let $d$ be the distance from the origin to the plane. Then $d$ is equal to the length of $P$ times the cosine of the angle between $P$ and the normal $N$. Hence

$$d = P \cdot N.$$

Therefore the equation of the plane is

$$P \cdot N - d = xn_1 + yn_2 + zn_3 - d = 0.$$

Suppose we are given three points $P_1, P_2, P_3$ and we wish to find the equation of the plane passing through these points. The normal to the plane is perpendicular to each of $P_2 - P_1$ and $P_3 - P_1$. Therefore

$$N = \frac{(P_2 - P_1) \times (P_3 - P_1)}{\|(P_2 - P_1) \times (P_3 - P_1)\|}$$

Also $d$ is equal to the inner product of $N$ with any one of the three points. For example

$$d = P_1 \cdot N.$$

Then the equation of the plane is

$$P \cdot N - P_1 \cdot N = xn_1 + yn_2 + zn_3 - d = 0.$$

# 4 Some Elementary Geometric Theorems and Formulas Derived With Vector Analysis

## 4.1 Heron's Formula for the Area of a Triangle

Let $T$ be the area of a triangle with sides given by vectors $A$, $B$, and $C$, and corresponding side lengths $a, b$ and $c$. Let $s$ be one half of the perimeter of the triangle

$$s = \frac{a+b+c}{2}.$$

Heron's formula for the area is

$$T = \sqrt{(s-a)(s-b)(s-c)s}.$$

We can derive this formula using the side vectors. The area is one half of the magnitude of the cross product of the vectors $A$ and $B$. That is,

$$2T = \|A \times B\|.$$

So

$$4T^2 = a^2b^2\sin^2(\theta) = a^2b^2(1-\cos^2(\theta)) = a^2b^2 - \|A \cdot B\|^2.$$

Also

$$c^2 = \|C\|^2 = \|A-B\|^2 = (A-B)\cdot(A-B) = a^2 - 2A\cdot B + b^2.$$

Then

$$\|A \cdot B\|^2 = \frac{(c^2-(a^2+b^2))^2}{4}.$$

Substituting this into the equation that we found above, namely

$$4T^2 = a^2b^2 - \|A \cdot B\|,$$

we get

$$\begin{aligned} 16T^2 &= 4a^2b^2 - (a^2+b^2-c^2)^2 \\ &= [2ab-(a^2+b^2-c^2)][2ab+(a^2+b^2-c^2)] \\ &= [c^2-(a-b)^2][(a+b)^2-c^2] \\ &= [c-(a-b)][c+(a+b)][a+b-c][a+b+c] \end{aligned}$$

$$= [c+b-a][c+a-b][a+b-c][a+b+c]$$
$$= [a+b+c-2a][a+b+c+-2b][a+b+c-2c][a+b+c].$$

Dividing each product on the right by 2, we get

$$T^2 = (s-a)(s-b)(s-c)s,$$

where

$$s = \frac{a+b+c}{2},$$

is the half perimeter of the triangle. Taking the square root, we get Heron's formula,

$$T = \sqrt{(s-a)(s-b)(s-c)s}.$$

This derivation is suggested in a problem in Apostol's Calculus.

# 5 Curl, Divergence, Gradient, and Laplacian

The curl of a vector field $\mathbf{A}$ in cartesian coordinates is

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$
$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i}$$
$$- \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\mathbf{j}$$
$$\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k}.$$

The divergence of a vector field $\mathbf{A}$ in cartesian coordinates is

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

The gradient of a function $f$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

The divergence of a gradient is the Laplacian

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

# 6 Stokes's Theorem, The Divergence Theorem

If a surface $S$ has bounding curve $\partial S$, Stokes theorem is

$$\int_S \nabla \times \mathbf{A} \cdot \mathbf{n} dS = \int_{\partial S} \mathbf{A} \cdot d\mathbf{r},$$

which allows a surface integral to be evaluated as a line integral around the boundary of the surface. The surface normal is $\mathbf{n}$.

The divergence theorem allows a volume integral to be evaluated as a surface integral. Let $V$ be a volume and $\partial V$ be it enclosing surface. Then

$$\int_V \nabla \cdot \mathbf{A} dv = \int_{\partial V} \mathbf{A} \cdot \mathbf{n} ds.$$

# 7 The Physical Meaning of Curl, Divergence, Gradient, and Laplacian

By Stokes' Theorem

$$\int_S \nabla \times \mathbf{A} \cdot \mathbf{n} dS = \int_{\partial S} \mathbf{A} \cdot d\mathbf{r}.$$

Consider the limit

$$\frac{1}{\mu(S)} \int_S \nabla \times \mathbf{A} \cdot \mathbf{n} ds,$$

as the area $\mu(S)$ of the surface element shrinks to zero around a point $P$. Assuming that $A$ is a continuous function, it is intuitive that, the expression would converge to the curl at the point $P$,

$$\nabla \times \mathbf{A}(P).$$

(Warning, there is an "ass" in assumption.) That is, the curl evaluated at a point $P$ is the limit of

$$\frac{1}{\mu(S)} \int_{\partial S} \mathbf{A} \cdot d\mathbf{r},$$

as the curve $\partial S$ surrounding a point $P$ shrinks to zero. And clearly this is a measure of how the vector field $\mathbf{A}$ "curls" around the point $P$. Clearly if there was no change of direction of $\mathbf{A}$ the curls is zero .

Similarly the divergence of a vector field at a point may be defined as the limit of a surface integral divided by the surface area, as the surface surrounding a point goes to zero. This says that the divergence measures how a source of **A** "diverges" from a point.

The gradient is clearly the analogue of a one dimensional rate of change, namely a derivative of **A** in a given direction.

Similarly the Laplacian is the analogue of a one dimensional second derivative in one dimension, extended to space.

# 8 Green's Theorem in the Plane.

Green's Theorem in the plane is a special case of Stokes Theorem, and conversely can be used in an intuitive proof of Stokes' Theorem. If $S$ is an area in the $x, y$ plane, and **A** is a vector function of only $x, y$ but not $z$, then we have

$$\begin{aligned}\mathbf{A} &= A_1(x,y)\mathbf{i} + A_2(x,y)\mathbf{j} + A_3(x,y)\mathbf{k} \\ &= A_1(x,y)\mathbf{i} + A_2(x,y)\mathbf{j},\end{aligned}$$

because $A_3(x,y) = 0$. Also notice that derivatives of the components of **A** with respect to $z$ are zero. So

$$\nabla \times \mathbf{A} = (\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})\mathbf{k}$$

Also the vector differential surface element is

$$d\mathbf{S} = dxdy\mathbf{k}.$$

Thus Stokes theorem is

$$\int_S \left( \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) ds$$

$$\int\int \left( \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) dxdy$$

$$= \int_{\partial S} \mathbf{A} \cdot d\mathbf{r}$$

$$= \int \mathbf{A} \cdot d\mathbf{r}$$

$$= \int_{\partial S} (A_x dx + A_y dy),$$

where $\mathbf{r}$ is the boundary curve bounding this area $A$.

$$\int\int \left( \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) dxdy$$

$$= \int_{\partial S} (A_1 dx + A_2 dy),$$

is called Green's Theorem in the Plane.

# 9 Applications of Green's Theorem in the Plane

**A Formula For the Area Enclosed by a Curve**. As an application of Green's Theorem we can find the area enclosed by a curve by evaluating a line integral around the curve. So let $A_x = -y/2$ and $A_y = x/2$, then

$$\int_S ds =$$

$$\int_S \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) ds$$

$$= \int_{\partial S} (A_x dx + A_y dy).$$

$$= (1/2) \int_{\partial S} (-ydx + xdy).$$

**An Example of Calculating the Area**. Let the region $S$ be a circle bounded by the curve

$$\mathbf{r} = r\cos(t)\mathbf{i} + r\sin(t)\mathbf{j},$$

for

$$0 \leq t \leq 2\pi.$$

Then

$$dx = -r\sin(t)dt$$

$$dy = r\cos(t)dt.$$

Then the area $\alpha$ is

$$\alpha = (1/2) \int_{\partial S} (-ydx + xdy).$$

$$= \frac{r^2}{2}\int_0^{2\pi}(\sin^2(t)+\cos^2(t))dt = \frac{r^2}{2}2\pi = r^2\pi.$$

**A Center of Mass Formula**. Now let us find the $x$ coordinate of the center of mass of a region $S$ bounded by the curve $\mathbf{r}(t)$. Let

$$A_x = 0,$$

and

$$A_y = \frac{x^2}{2}.$$

Then

$$\int_S xds$$

$$= \int_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) ds$$

$$= \int_{\partial S} (A_x dx + A_y dy)$$

$$= \int_{\partial S} \frac{x^2}{2} dy.$$

So the center of mass $x$ coordinate is

$$x_{cm} = \frac{1}{\alpha}\int_{\partial S} \frac{x^2}{2}dy,$$

where $\alpha$ is the area of region $S$. Similarly using

$$A_x = -\frac{y^2}{2},$$

and

$$A_y = 0,$$

we find

$$y_{cm} = -\frac{1}{\alpha}\int_{\partial S} \frac{y^2}{2}dx.$$

**An Example of Calculating the Center of Mass** . Let the area be the right half circle of radius $r$. Let the area be bounded by the curve

$$\mathbf{r} = r\cos(t)\mathbf{i} + r\sin(t)\mathbf{j},$$

for

$$-\pi/2 \leq t \leq 2\pi,$$

and by the straight line from $(0, r)$ to $(0, -r)$.

Now

$$dy = r\cos(t)dt,$$

So

$$\int_{\partial S} \frac{x^2}{2} dy$$

$$= \int_{-\pi/2}^{\pi/2} \frac{r^3}{2} \cos^3(t)dt + \int_{-r}^{r} 0dy = \frac{2}{3}r^3.$$

Hence the $x$ coordinate of the center of mass is

$$x_{cg} = \frac{(2r^3/3)}{\pi r^2/2} = \frac{4r}{3\pi}$$

**An Area Moment of Inertia Formula**. Letting

$$A_x = 0$$

and

$$A_y = x^3/3$$

We get for the moment of inertia about the $y$ axis

$$I_y = \int_S x^2 ds$$

$$= \int_S \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} ds$$

$$= \int_{\partial S} A_x dx + A_y dy$$

$$= \int_{\partial S} (x^3/3)dy$$

**An Example of Calculating the Area Moment of Inertia** . Let the area be a circle of radius $r$. Let the area be bounded by the curve

$$\mathbf{r} = r\cos(t)\mathbf{i} + r\sin(t)\mathbf{j},$$

for

$$0 \leq t \leq 2\pi.$$

Then

$$x = (r\cos(t))^3$$

and

$$dy = rcos(t)dt$$

So

$$I_y = r^4 \int_0^{2\pi} \cos^4(t)/3dt$$
$$= \frac{\pi r^4}{4}.$$

By the parallel axis theorem, the moment of inertia about an axis through the center of mass may be obtained from the moment of inertia about a parallel axis at a distance $d$ from the center of mass axis.

These formulas allow us to compute areas, centers of mass, and moments of inertia for areas bounded by piecewise defined curves.

# 10 A Proof of Green's Theorem in the Plane

Suppose a vector $\mathbf{A}$ is in the $xy$ plane and its components are functions of only $x$ and $y$. Let

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j}$$

Suppose we have a region in the plane with a bounding curve $C$ which has the property that each horizontal line meets it in at most two points. Then curve $C$ consists of a left hand portion $C^L$ and a right hand portion $C^R$. Similarly suppose a vertical line meets $C$ in at most two points so that $C$ consists of a bottom curve $C^B$ and a top curve $C^T$. Integrating over the area enclosed by the curve we have

$$\int\int \frac{\partial A_2}{\partial x} dxdy = \int (A_2^R - A_2^L)dy = \int_C A_2 dy,$$

where $A_2^R$ is the value of component function $A_2$ on the right side of curve $C$, and $A_2^L$ is the value of component function $A_2$ on the left side of curve $C$. Similarly

$$\int\int \frac{\partial A_1}{\partial y} dydx = \int (A_1^T - A_1^B)dx = -\int_C A_1 dx.$$

So we obtain Green's Theorem for this simple case. To generalize the proof, we do a bit of hand waving, and state that in the case of a more complex curve

$C$, we can decompose the area into simple regions. The line integral on the boundary between such simple regions will vanish because we integrate twice on such boundary in opposite directions. Thus we have Greens Theorem in the plane.

$$\int\int \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) dxdy$$

$$= \int_{\partial S} (A_1 dx + A_2 dy).$$

For a mathematically rigorous proof of Green's Theorem and Stokes Theorem, we must introduce some machinery that allows us to give a rigorous definition of a surface and its boundary curve. This is done in more advance books with differential forms. For example see **Calculus on Manifolds** by Spivak.

# 11 A Proof of Stokes' Theorem: A Special Case

So Green's theorem in the plane is Stokes' theorem in the plane. Hence it should be possible to prove Stokes Theorem by projecting into a plane and applying Green's theorem. In fact this is possible.

**Stokes' Theorem** Suppose in this special case the surface $S$ and its boundary curve $\partial S$ have the property that the projections into each of the coordinate planes is one to one, and that these projections are simply connected. Then we have

$$\int_S \nabla \times \mathbf{A} \cdot \mathbf{n} dS = \int_{\partial S} \mathbf{A} \cdot d\mathbf{r}.$$

**Proof**.

If $g(x_1, ..., x_n)$ is a function of n variables, we write $D_k g$ for the $k$th partial derivative of $g$. Suppose

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}.$$

Let us suppose first that only $A_1$ is not zero. Then

$$\nabla \times \mathbf{A} \cdot \mathbf{n} dS = (\mathbf{j}\frac{\partial A_1}{\partial z} - \mathbf{k}\frac{\partial A_1}{\partial y}) \cdot \mathbf{n} dS.$$

$$= \frac{\partial A_1}{\partial z}\mathbf{j} \cdot \mathbf{n} dS - \frac{\partial A_1}{\partial y}\mathbf{k} \cdot \mathbf{n} dS.$$

Notice that

$$\mathbf{k} \cdot \mathbf{n} dS = \cos(\theta) dS,$$

where $\theta$ is the angle between $\mathbf{n}$ and $\mathbf{k}$. This is the projection of area element $dS$ onto an area element $dxdy$ in the $xy$ plane. We can transform

$$\mathbf{j} \cdot \mathbf{n} dS$$

to also have this form.

Let the surface be defined by

$$z = f(x, y).$$

Let

$$F(x, y) = A_1(x, y, f(x, y)).$$

If the partial derivative of this function times the projection of the surface area element $dS$ were

$$-\frac{\partial F}{\partial y} dxdy$$

we could apply Green's Theorem in the plane to get a line integral of

$$F dx$$

around the projection of the boundary curve.

So let us compute the partial derivative with respect to $y$

$$\frac{\partial F}{\partial y} = D_2 F(x, y) = D_2 A_1(x, y, f(x, y)) + D_3 A_1(x, y, f(x, y)) D_2 f(x, y).$$

So let us convert

$$\mathbf{j} \cdot \mathbf{n}$$

to an expression involving

$$\mathbf{k} \cdot \mathbf{n}$$

in the equation

$$\nabla \times \mathbf{A} \cdot \mathbf{n} dS = \frac{\partial A_1}{\partial z}\mathbf{j} \cdot \mathbf{n} dS - \frac{\partial A_1}{\partial y}\mathbf{k} \cdot \mathbf{n} dS.$$

It will turn out that after the conversion we have

$$\int_S \nabla \times \mathbf{A} \cdot \mathbf{n} dS = \int_{S'} -\frac{\partial F}{\partial y} dx dy,$$

where $S'$ is the projection of $S$ to the $xy$ plane.

To motivate this conversion, suppose the normal vector $\mathbf{n}$ is parallel to the $y = 0$ plane. Let $\theta$ be the angle between $\mathbf{n}$ and $\mathbf{k}$. Then

$$\cos(\theta) = \mathbf{n} \cdot \mathbf{k},$$

$$-\sin(\theta) = \cos(\pi/2 + \theta) = \mathbf{n} \cdot \mathbf{j}$$

and

$$\tan(\theta) = \frac{dz}{dy}.$$

Hence

$$\mathbf{n} \cdot \mathbf{j} = -\sin(\theta) = -\cos(\theta)\tan(\theta) = -\mathbf{n} \cdot \mathbf{k}\frac{\partial f}{\partial y}.$$

We can prove this is true in the general case, even when the surface normal is not parallel to the $y = 0$ coordinate plane. So suppose we consider our surface

$$\mathbf{R}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}.$$

If we differentiate this partially with respect to $y$, we get a vector tangent to the surface, which is a $y$ coordinate tangent vector.

$$\frac{\partial \mathbf{R}}{\partial y} = \mathbf{j} + D_2 f(x, y)\mathbf{k}.$$

Now this surface tangent vector is perpendicular to the surface normal, so taking the dot product with $\mathbf{n}$ we have

$$0 = \frac{\partial \mathbf{R}}{\partial y} \cdot \mathbf{n} = \mathbf{j} \cdot \mathbf{n} + D_2 f(x, y)\mathbf{k} \cdot \mathbf{n}.$$

So

$$\mathbf{j} \cdot \mathbf{n} = -D_2 f(x, y)\mathbf{k} \cdot \mathbf{n}.$$

Substituting in

$$\nabla \times \mathbf{A} \cdot \mathbf{n} dS = (\mathbf{j}\frac{\partial A_1}{\partial z} - \mathbf{k}\frac{\partial A_1}{\partial y}) \cdot \mathbf{n} dS$$

$$= -(D_3A_1D_2f + D_2A_1)\mathbf{k}\cdot\mathbf{n}dS$$

Above we showed that

$$\frac{\partial F}{\partial y} = D_2F(x,y) = D_2A_1(x,y,f(x,y)) + D_3A_1(x,y,f(x,y))D_2f(x,y).$$

Hence

$$\int_S \nabla\times\mathbf{A}\cdot\mathbf{n}dS = \int_{S'} -\frac{\partial F}{\partial y}dxdy,$$

which by Green's theorem in the plane equals the line integral

$$\int_{\partial S'} Fdx = \int_{\partial S} A_1dx.$$

This last integral equals the line integral of $A_1$ around the boundary of $S$ with respect to $dx$.

We can do a similar calculation when only $A_2$ is not zero, or when only $A_3$ is not zero. When we add up these three cases, we have proved Stokes' Theorem for the case, namely

$$\int_S \nabla\times\mathbf{A}\cdot\mathbf{n}dS = \int_{\partial S}\mathbf{A}\cdot d\mathbf{r}.$$

# 12 Stokes's Theorem in the General Case

To prove Stokes Theorem in the general case, we break up the surface into a set of small surface patches where the conditons of the previous section hold. Then each internal boundary line between patches is integrated twice in opposite directions, and thus cancells out. Intuitively this proves Stokes Theorem in the general case. However, to be rigorous we must introduce machinery for decomposing such surfaces, and characterizing those surfaces where this can be done. Surfaces in general can be quite strange, for example they can be noncompact, unbounded, nondifferentiable, self-intesecting, and nonorientable. The proper area of mathematics for this is the theory of differential manifolds. For more information consult a book on Differential Forms and manifolds such as Spivak's "Calculus on Manifolds." It turns out that this theorem can be extended to higher dimensions. It involves the concept of the exterior derivative.

# 13 An Application of Stokes Theorem: Faraday's Law of Induction and the Corresponding Maxwell Equation

Faraday's Law Of Induction says that the EMF (ElectoMotive Force) around a circuit path is equal to the negative rate of change of magnetic flux passing through the interior of the path.

$$EMF = -\frac{d\phi}{dt}.$$

Faraday would say the EMF is proportional to the number of lines of flux cutting the circuit. The lines of flux of a magnetic field were made visible to Faraday by looking at the aligned iron fillings. The EMF is an electric potential which drives current flow around a circuit and is given as the line integral of the electric field around the circuit.

$$EMF = \int \mathbf{E} \cdot d\mathbf{r}.$$

The rate of change of flux is

$$\frac{d\phi}{dt} = \int_S \frac{d\mathbf{B}}{dt}.$$

By Stokes Theorem

$$EMF = \int \mathbf{E} \cdot d\mathbf{r} = \int_S \nabla \times \mathbf{E} \cdot d\mathbf{S},$$

where $S$ is the region surrounded by the circuit. Thus Faraday's law is

$$-\int_S \frac{d\mathbf{B}}{dt} \cdot d\mathbf{S} = \int_S \nabla \times \mathbf{E} \cdot d\mathbf{S},$$

where

$$d\mathbf{S} = \mathbf{n} dS,$$

is the product of the surface unit normal $\mathbf{n}$ and the differential surface area element $dS$. Assuming continuity, and taking the limit as the area shrinks to zero, we arrive at

$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt},$$

which is Maxwell's version of Faraday's law, and one of Maxwell's famous equations describing electromagnetism.

## 14 The Parallel Axis Theorem

The moment of inertia about an axis that does not pass through the center of mass of a body is equal to the moment of inertia about a parallel axis that passes through the center of gravity plus $d^2M$ where $d$ is the distance between the two parallel axes, and $M$ is the mass of the body. Without loss of generality we shall assume that the parallel axes are parallel to the $x$ axis and that the center of gravity of the body lies at $(0, y_0, z_0)$, so that $\sqrt{y_0^2 + z_0^2} = d$. We introduce a center of mass coordinate system

$$x' = x, y' = y - y_0, z' = z - z_0$$

Let the moment of inertia of the body about its center of mass be

$$I_{xxcm} = \int\int (y'^2 + z'^2)dm$$

Then the moment of inertia in the unprimed system is

$$I_{xx} = \int\int (y^2 + z^2)dm$$

$$= \int\int ((y' + y_0)^2 + (z' + z_0)^2)dm$$

$$= I_{xxcm} + \int\int 2y'y_0 dm + \int\int 2z'z_0 dm + \int\int (y_0^2 + z_0^2)dm$$

$$= I_{xxcm} + 0 + 0 + d^2M = I_{xxcm} + d^2M$$

## 15 An Intuitive Classical Treatment of Orthogonal Curvilinear Coordinates

Let

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

be the vector of a point.

Let $u_1, u_2, u_3$ be a set of coordinates so that the cartesian coordinates $x, y, z$ are functions of these coordinates.

$$x = x(u_1, u_2, u_3),$$

$$y = y(u_1, u_2, u_3),$$
$$z = z(u_1, u_2, u_3).$$

These coordinates could be the cylindrical coordinates or the spherical coordinates. When we let only one of the three coordinates vary, we get coordinate curves $\mathbf{c}_i(u_i) = \mathbf{r}(u_1, u_2, u_3)$, where $u_j$ is fixed if $j$ is not equal to $i$. We assume that these coordinate curves are orthogonal. So at any point where these curves intersect the curve tangent vectors are perpendicular to one another. A tangent vector to coordinate curve $\mathbf{c}_i$ is

$$\frac{d\mathbf{c}_i}{du_i} = \frac{\partial \mathbf{r}}{\partial u_i}.$$

Let $s_1$ be the arclength along this coordinate curve. Then

$$\frac{d\mathbf{c}_i}{du_i} = \frac{\partial \mathbf{r}}{\partial u_i} = \frac{\partial \mathbf{r}}{\partial s_i}\frac{ds_i}{du_i}.$$

Define

$$\mathbf{u}_i = \frac{\partial \mathbf{r}}{\partial s_i},$$

and

$$h_i = \frac{ds_i}{du_i}.$$

Each $\mathbf{u}_i$ is a unit vector, and these three unit vectors are orthogonal. It is assumed that the three coordinates are listed in an order so that these three vectors form a right handed system, so that

$$\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3.$$

Now let $\mathbf{c}(s)$ be any curve paramatized by arc length.

$$\mathbf{c} = \mathbf{c}(u_1(s), u_2(s), u_3(s)).$$

$$\frac{d\mathbf{c}}{ds} = \frac{d\mathbf{r}}{ds}.$$

$$\frac{d\mathbf{c}}{ds} = \sum_{i=1}^{3} \frac{\partial \mathbf{r}}{\partial u_i}\frac{du_i}{ds}$$

$$= \sum_{i=1}^{3} h_i \mathbf{u}_i \frac{du_i}{ds}$$

Now

$$\frac{d\mathbf{c}}{ds}$$

is a unit tangent vector, so the square of the magnitude is

$$1 = \sum_{i=1}^{3} h_i \mathbf{u}_i \frac{du_i}{ds} \cdot \sum_{i=1}^{3} h_i \mathbf{u}_i \frac{du_i}{ds}.$$

Because the $\mathbf{u}_i$ are a system of unit orthogonal vectors we get

$$1 = \sum_{i=1}^{3} h_i^2 \left(\frac{du_i}{ds}\right)^2.$$

So the differential distance $ds$ along the curve is given by

$$ds^2 = \sum_{i=1}^{3} h_i^2 du_i^2.$$

## 15.1 The Gradient in Orthogonal Curvilinear Coordinates

Given a function $f(x, y, z)$, the rate of change of $f$ on a curve in a direction given by a unit vector $\mathbf{n}$ per unit distance is the directional derivative

$$\frac{df}{ds} = \nabla f \cdot \mathbf{n}.$$

and

$$df = \nabla f \cdot \mathbf{n} ds.$$

If $\mathbf{c}(t)$ is a curve with parameter $t$, we have

$$\frac{df(\mathbf{c}(t))}{dt} = \nabla f \cdot \frac{d\mathbf{c}}{dt}.$$

If the curve is paramatized in arc length $s$ then

$$\frac{d\mathbf{c}}{ds}$$

is a unit vector. We can write

$$df = \nabla f \cdot d\mathbf{s} = \frac{\partial f}{du_1} h_1 du_1 \frac{\partial f}{du_2} h_2 du_2 + \frac{\partial f}{du_3} h_3 du_3.$$

We have in $x, y, z$ coordinates the defining expression for the gradient

$$df = \nabla f \cdot d\mathbf{r},$$

where

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

We want to find an expression for the gradient in the $u_1, u_2, u_3$ coordinate system involving the basis vectors of that system.

Let us write this expression for the gradient as $\nabla_u f$. Then we are looking for the defining equation for the gradient written as

$$df = \nabla_u f \cdot d\mathbf{r},$$

when $df$ and $d\mathbf{r}$ have been written in $u_1, u_2 u_3$ coordinates. Let us write

$$\nabla_u f = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \lambda_3 \mathbf{u}_1,$$

where $\lambda_1, \lambda_2, \lambda_3$ are to be determined.

We can express $d\mathbf{r}$ in $u_1, u_2, u_3$ coordinates. We have

$$d\mathbf{r} = \sum_{i=1}^{3} \frac{\partial \mathbf{r}}{\partial u_i} du_i$$

$$= \sum_{i=1}^{3} h_i \mathbf{u}_i du_i.$$

Considering $f$ a function of $u_1, u_2, u_3$, we have

$$df = \frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2 + \frac{\partial f}{\partial u_3} du_3$$

Let us substitute the expressions for $d\mathbf{r}$, $df$ and $\nabla_u f$ into the defining equation

$$df = \nabla_u f \cdot d\mathbf{r}.$$

We are greatly aided in this substitution by the fact that the unit vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, are orthogonal. We get

$$\sum_{i=1}^{3} \frac{\partial f}{\partial u_i} du_i = \sum_{i=1}^{3} h_i \lambda_i du_i.$$

We conclude that

$$\lambda_i = \frac{1}{h_i}\frac{\partial f}{\partial u_i}.$$

So the expression for the gradient in curvilinear coordinates is

$$\nabla_u f = \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial f}{\partial u_i}\mathbf{u}_i.$$

And the expression for the operator $\nabla_u$ is

$$\nabla_u = \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial}{\partial u_i}\mathbf{u}_i.$$

## 15.2 The Divergence in Orthogonal Curvilinear Coordinates

We use the vector identity

$$\nabla \cdot f\mathbf{v} = f\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla f.$$

Let

$$\mathbf{v} = v_1\mathbf{u}_1 + v_2\mathbf{u}_2 + v_3\mathbf{u}_3.$$

The divergence of the first term is

$$\nabla \cdot v_1\mathbf{u}_1 = v_1\nabla \cdot \mathbf{u}_1 + \mathbf{u}_1 \cdot \nabla v_1.$$

Now

$$\mathbf{u}_1 = \mathbf{u}_2 \times \mathbf{u}_3$$

so

$$\frac{\mathbf{u}_1}{h_2 h_3} = \frac{\mathbf{u}_2}{h_2} \times \frac{\mathbf{u}_3}{h_3}$$

$$= \nabla_u u_2 \times \nabla_u u_3.$$

Because the divergence of the cross product of two gradients vanishes, we have

$$\nabla \cdot \left[\frac{\mathbf{u}_1}{h_2 h_3}\right] = 0$$

So let us write the divergence of the first term differently as

$$\nabla \cdot \left[ h_2 h_3 v_1 \frac{\mathbf{u}_1}{h_2 h_3} \right] = h_2 h_3 v_1 \nabla \cdot \frac{\mathbf{u}_1}{h_2 h_3} + \frac{\mathbf{u}_1}{h_2 h_3} \cdot \nabla h_2 h_3 v_1.$$

So the first term on the left vanishes giving

$$\nabla \cdot v_1 \mathbf{u}_1 = \nabla \cdot \left[ h_2 h_3 v_1 \frac{\mathbf{u}_1}{h_2 h_3} \right] = \frac{\mathbf{u}_1}{h_2 h_3} \cdot \nabla h_2 h_3 v_1 = \frac{1}{h_1 h_2 h_3} \frac{\partial h_2 h_3 v_1}{\partial u_1}.$$

Getting similar results for the other two terms of $\mathbf{v}$ we have the divergence in these coordinates is

$$\nabla \cdot \mathbf{v} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial (h_2 h_3 v_1)}{\partial u_1} + \frac{\partial (h_3 h_1 v_2)}{\partial u_2} + \frac{\partial (h_1 h_2 v_3)}{\partial u_3} \right].$$

## 15.3 The Laplacian in Orthogonal Curvilinear Coordinates

The laplacian is the divergence of the gradient

So the Laplacian operator is

$$\nabla^2 = \nabla \cdot \nabla = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial u_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left( \frac{h_3 h_1}{h_2} \frac{\partial}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial}{\partial u_3} \right) \right].$$

## 15.4 The Curl in Orthogonal Curvilinear Coordinates

## 15.5 Cylindrical Coordinates

## 15.6 Spherical Coordinates

Let the spherical coordinates of a point in Euclidean space be $(u_1, u_2, u_3)$. The first coordinate $u_1$ is the distance from the origin to the point written $r$. $u_3$ is the angle from the $x$ axis to the line through the projection of the point to the $xy$ plane, which we shall call $\theta$. $u_2$ is the angle measured from the $z$ axis through the line through the point. We shall call this angle $\phi$. So we write

$$(u_1, u_2, u_3) = (r, \phi, \theta).$$

This is the way that spherical coordinates are represented in most mathematics books. However, in some books, especially physics books, the labels

$\theta$ and $\phi$ are interchanged, with $\theta$ being the angle measured from the $z$ axis. However this labelling is done, the spherical coordinates $(u_1, u_2, u_3)$ must form a right handed system, so that if $u_1 = r$, then the second coordinate $u_2$ must be the angle from the $z$ axes, so that a right handed system is obtained. So we use the mathematical choice $(r, \phi, \theta)$ with

$$x = r\sin(\phi)\cos(\theta),$$

$$y = r\sin(\phi)\sin(\theta),$$

$$z = r\cos(\phi),$$

where

$$0 \le \phi \le \pi$$

$$0 \le \theta < 2\pi$$

So the vector from the origin to a point is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$= r\sin(\phi)\cos(\theta)\mathbf{i} + r\sin(\phi)\sin(\theta)\mathbf{j} + r\cos(\phi)\mathbf{k}.$$

Then

$$h_1\mathbf{u}_r = \frac{\partial \mathbf{r}}{\partial r} = \sin(\phi)\cos(\theta)\mathbf{i} + \sin(\phi)\sin(\theta)\mathbf{j} + \cos(\phi)\mathbf{k}.$$

And

$$h_2\mathbf{u}_\phi = \frac{\partial \mathbf{r}}{\partial \phi} = r\cos(\phi)\cos(\theta)\mathbf{i} + r\cos(\phi)\sin(\theta)\mathbf{j} - r\sin(\phi)\mathbf{k},$$

$$h_3\mathbf{u}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = -r\sin(\phi)\sin(\theta)\mathbf{i} + r\sin(\phi)\cos(\theta)\mathbf{j}.$$

So

$$h_1 = 1, h_2 = r, h_3 = r\sin(\phi)$$

$$\mathbf{u}_r = \sin(\phi)\cos(\theta)\mathbf{i} + \sin(\phi)\sin(\theta)\mathbf{j} + \cos(\phi)\mathbf{k}.$$

$$\mathbf{u}_\phi = \cos(\phi)\cos(\theta)\mathbf{i} + \cos(\phi)\sin(\theta)\mathbf{j} - \sin(\phi)\mathbf{k},$$

$$\mathbf{u}_\theta = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}.$$

Notice that in curvilinear coordinates, because $h_i u_i$ must measure distance, the scale factor $h_i$ can usually be determined without calculation. So the Laplacian in spherical coordinates is

$$\nabla^2 = \nabla\cdot\nabla = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right].$$

$$= \frac{1}{r^2\sin(\phi)}\left[\frac{\partial}{\partial r}\left(r^2\sin(\phi)\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial}{\partial \phi}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{\sin(\phi)}\frac{\partial}{\partial \theta}\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial}{\partial \phi}\right) + \frac{1}{r^2\sin^2(\phi)}\frac{\partial^2}{\partial \theta^2}.$$

# 16 A Slightly More Modern Treatment of Curvilinear Coordinates

Let $u_1, u_2, u_3$ be a system of curvilinear orthogonal coordinates in Euclidean three space. Let $t$ be a tangent vector (such as a velocity). Then it may be written as a linear combination of basis vectors, which we shall define. We have

$$t = c^1\partial/\partial u_1 + c^2\partial/\partial u_2 + c^3\partial/\partial u_3,$$

where the differential operators play the role of basis vectors. Let $<,>$ be the Euclidean inner product. We may consider $\partial/\partial u_i$ to be the tangent vector to the $i$th coordinate curve. Thus if $C_i(u_i)$ is the $i$th coordinate curve, then we identify the differential operator

$$\partial/\partial u_i$$

with the curve tangent vector

$$dC_i/du_i.$$

(Actually this is a special case of the natural isomorphism between curve tangent vectors and the linear functionals defined by the curves, which are called derivations. Thus if $\alpha(u)$ is a curve and $f$ a function, then

$$\frac{df(\alpha)}{dt}(t_0),$$

maps $f$ to a real number. And this gives the same value for any other curve that has the same tangent.)

Example: Let $x_1, x_2, x_3$ be the usual Euclidean coordinates. Let the first coordinate curve be

$$C_1(x_1) = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k},$$

where $x_2$ and $x_3$ are held fixed. Then

$$\partial/\partial x_1 = \partial C_1/\partial x_1 = \mathbf{i}$$

Example: Spherical coordinates $r, \theta, \phi$. We have

$$x = r\sin(\theta)\cos(\phi)$$

$$y = r\sin(\theta)\sin(\phi)$$

$$z = r\cos(\theta)$$

Define coordinate curve $C(r)$ by holding $\theta$ and $\phi$ fixed. We find that

$$\partial/\partial r = \sin(\theta)\cos(\phi)\mathbf{i} + \sin(\theta)\sin(\phi)\mathbf{j} + \cos(\theta)\mathbf{k}$$

This is a unit vector. In general coordinate tangent vectors are not unit vectors. We wish to work with unit vectors, so we define unit vectors $\mathbf{a}_i$ in the direction of $\partial/\partial u_i$. Then the unit vectors in spherical coordinates are

$$\mathbf{a}_r = \sin(\theta)\cos(\phi)\mathbf{i} + \sin(\theta)\sin(\phi)\mathbf{j} + \cos(\theta)\mathbf{k}$$

$$\mathbf{a}_\theta = \cos(\theta)\cos(\phi)\mathbf{i} + \cos(\theta)\sin(\phi)\mathbf{j} - \sin(\theta)\mathbf{k}$$

$$\mathbf{a}_\phi = -\sin(\phi)\mathbf{i} + \cos(\phi)\mathbf{j}$$

The length of the tangent vector $t$ is

$$ds^2 =< t, t >= \sum_{i=1}^{3}\sum_{j=1}^{3} c^i c^j < \partial/\partial u_i, \partial/\partial u_j >$$

Assuming an orthogonal system

$$ds^2 =< t, t >= \sum_{i=1}^{3} (c^i)^2 < \partial/\partial u_i, \partial/\partial u_i > .$$

Define dual vectors $d\mathbf{u}_i$ by

$$d\mathbf{u}_i(\partial/\partial u_j) = \delta^i_j$$

($\delta^i_j$ equals 1 if $i = j$ and zero otherwise). Then $d\mathbf{u}_i(t) = c^i$. In the old days, they did not carefully distinguish a function from its value, and wrote $du_i = d\mathbf{u}_i(t)$. Then $c_i = du_i$, so

$$t = du_1\partial/\partial u_1 + du_2\partial/\partial u_2 + du_3\partial/\partial u_3.$$

They then succumbed to the temptation to substitute $dt$ for $t$. And if the length of $t$ is written as $ds$, then one gets

$$ds^2 =< t, t >= \sum_{i=1}^{3} du_i^2 < \partial/\partial u_i, \partial/\partial u_i > .$$

This is old notation, which was devised because of imperfect understanding. It is still widely used and is somewhat intuitive, but it is found to be confusing when examined closely. Define

$$g_{ij} =< \partial/\partial u_i, \partial/\partial u_j > .$$

and

$$h_i^2 = g_{ii},$$

which is the square of the length of the coordinate tangent vectors.

Then for orthogonal coordinates the length element becomes

$$ds^2 =< t, t >= \sum_{i=1}^{3} h_i^2 du_i^2.$$

For spherical coordinates $r, \theta$ and $\phi$

$$h_1 = 1, h_2 = r, h_3 = r\sin(\theta)$$

The spherical volume element is $dv = h_1h_2h_3drd\theta d\phi = r^2\sin(\theta)drd\theta d\phi$. For cylindrical coordinates $r, \theta$ and $z$ ,

$$h_1 = 1, h_2 = r, h_3 = 1.$$

The volume element is $dv = rdrd\theta dz$.

Example: divergence in spherical coordinates. We use the fundamental definition:

$$div\mathbf{F} = lim_{V\to 0}\frac{1}{V}\int_S \mathbf{F}\cdot\mathbf{n}da$$

where the volume element is a small nearly cubical element with edges along coordinate curves and of lengths $h_1 dr$, $h_2 d\theta$, and $h_3 d\phi$. We find

$$div\mathbf{F} = \frac{1}{r^2}\frac{\partial(F_r r^2)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\theta \sin(\theta))}{\partial\theta} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\phi)}{\partial\phi}$$

**Proposition.** $\mathbf{a}_i = h_i\nabla u_i$.
Proof. $\nabla u_i$ is in the $u_i$ direction, say $\nabla u_i = \alpha\mathbf{a}_i$. We have $ds^2 = h_i^2 du_i^2$. Thus $du_i/ds = 1/h_i$. But the directional derivative is

$$du_i/ds = \nabla u_i\cdot\mathbf{a}_i = \alpha\mathbf{a}_i\cdot\mathbf{a}_i = \alpha$$

Thus $\mathbf{a}_i = h_i\nabla u_i$.

**Gradient**. The gradient in curvilinear orthogonal coordinates is

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\mathbf{a}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\mathbf{a}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\mathbf{a}_3$$

This follows by differentiating:

$$\nabla f = (\frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial x} + \frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial x} + \frac{\partial f}{\partial u_3}\frac{\partial u_3}{\partial x})\mathbf{i}$$

$$+(\frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial y} + \frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial y} + \frac{\partial f}{\partial u_3}\frac{\partial u_3}{\partial y})\mathbf{j}$$

$$+(\frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial z} + \frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial z} + \frac{\partial f}{\partial u_3}\frac{\partial u_3}{\partial z})\mathbf{k}$$

$$= \frac{\partial f}{\partial u_1}\nabla u_1 + \frac{\partial f}{\partial u_2}\nabla u_2 + \frac{\partial f}{\partial u_3}\nabla u_3$$

$$= \frac{1}{h_1}\frac{\partial f}{\partial u_1}\mathbf{a}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\mathbf{a}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\mathbf{a}_3.$$

**Divergence**. The divergence is

$$\nabla\cdot\mathbf{F} = \frac{1}{h_1h_2h_3}(\frac{\partial(h_2h_3F_1)}{\partial u_1} + \frac{\partial(h_1h_3F_2)}{\partial u_2} + \frac{\partial(h_1h_2F_3)}{\partial u_3}).$$

We prove this as follows. Let

$$\mathbf{F} = F_1\mathbf{a}_1 + F_2\mathbf{a}_2 + F_3\mathbf{a}_3.$$

Using facts such as

$$\mathbf{a}_1 = \mathbf{a}_2 \times \mathbf{a}_3 = h_2h_3\nabla u_2 \times \nabla u_3,$$

we find

$$\nabla \cdot \mathbf{F} = \nabla(F_xh_2h_3) \cdot (\nabla u_2 \times \nabla u_3) + F_xh_2h_3\nabla \cdot \nabla u_2 \times \nabla u_3 + ...$$

$$= (\frac{\partial(F_1h_2h_3)}{\partial u_1}\nabla u_1 + \frac{\partial(F_1h_2h_3)}{\partial u_2}\nabla u_2 + \frac{\partial(F_1h_2h_3)}{\partial u_3}\nabla u_3) \cdot \nabla u_2 \times \nabla u_3$$
$$+F_1h_2h_3\nabla \cdot \nabla u_2 \times \nabla u_3 + ...$$

$$= (\frac{\partial(F_1h_2h_3)}{\partial u_1}\nabla u_1 \cdot \nabla u_2 \times \nabla u_3 + 0 + 0$$

$$+F_1h_2h_3\nabla \cdot \nabla u_2 \times \nabla u_3 + ...$$
$$= \frac{1}{h_1h_2h_3}\frac{\partial(F_1h_2h_3)}{\partial u_1} + \frac{1}{h_1h_2h_3}\frac{\partial(F_2h_1h_3)}{\partial u_2} + \frac{1}{h_1h_2h_3}\frac{\partial(F_3h_1h_2)}{\partial u_3}.$$

The " + ... " stands for similar terms involving $F_2$ and $F_3$. We have used

$$\nabla u_1 \cdot \nabla u_2 \times \nabla u_3 = \frac{1}{h_1h_2h_3},$$

and

$$\nabla \cdot \nabla u_2 \times \nabla u_3 = 0.$$

The latter result follows from the identity involving divergence of a cross product and the fact that the curl of a divergence is zero.

**Laplacian**. The Laplacian is the divergence of the gradient. Thus

$$\nabla^2 f = \nabla \cdot \nabla f =$$

$$\frac{1}{h_1h_2h_3}(\frac{\partial((h_2h_3/h_1)\partial f/\partial u_1)}{\partial u_1} + \frac{\partial((h_1h_3/h_2)\partial f/\partial u_2)}{\partial u_2} + \frac{\partial((h_1h_2/h_3)\partial f/\partial u_3)}{\partial u_3}).$$

**Curl**. The Curl is

$$\nabla \times \mathbf{F} =$$

$$\frac{\mathbf{a}_1}{h_2h_3}(\frac{\partial(h_3F_3)}{\partial u_2} - \frac{\partial(h_2F_2)}{\partial u_3})+$$
$$\frac{\mathbf{a}_2}{h_1h_3}(\frac{\partial(h_1F_1)}{\partial u_3} - \frac{\partial(h_3F_3)}{\partial u_1})+$$
$$\frac{\mathbf{a}_3}{h_1h_2}(\frac{\partial(h_2F_2)}{\partial u_1} - \frac{\partial(h_1F_1)}{\partial u_2})$$

We shall prove this as follows. Let

$$\mathbf{f} = f_1h_1\nabla u_1 + f_2h_2\nabla u_2 + f_2h_2\nabla u_3.$$

Then because the curl of a gradient is zero, we have

$$\nabla \times \mathbf{f} = \nabla f_1h_1 \times \nabla u_1 + \nabla f_2h_2 \times \nabla u_2 + \nabla f_3h_3 \times \nabla u_3.$$

Now

$$\nabla f_1h_1 \times \nabla u_1 =$$
$$(\frac{\partial(f_1h_1)}{\partial u_1}\nabla u_1 + \frac{\partial(f_1h_1)}{\partial u_2}\nabla u_2 + \frac{\partial(f_1h_1)}{\partial u_3}\nabla u_3) \times \nabla u_1,$$

and similar expressions for the other terms. We have $\nabla u_1 \times \nabla u_1 = 0$ and

$$\nabla u_2 \times \nabla u_1 = -\frac{\mathbf{a}_1 \times \mathbf{a}_2}{h_1h_2} = -\frac{\mathbf{a}_3}{h_1h_2},$$

and so on. The result follows by making such substitutions.

# 17 A List of Vector Analysis Formulas

Physics books, such as books on Mechanics, and books on Electromagnetic Theory, usually have a chapter on Vector Analysis and a list of vector analysis identities.

# 18 Differential Forms

The theory of Differential Forms may be considered as a modern version of Vector Analysis. However, calculation may not appear as intuitively as it does in Vector Analysis. The book on General Relativity by John Wheeler et al, called "Gravitation," treats the use of Differential Forms in Physics. A simple iontroduction to differential forms is the book listed in the bibliography by Spivak.

# 19 Bibliography

[1]Hildebrand Francis B, **Advanced Calculus for Applications**, 1962, Prentice Hall.
[2]Powell John L, Crasemann Bernd, **Quantum Mechanics**, 1965, Addison-Wesley, (One of the texts used in my Quantum Mechanics course.)
[3]Spiegel Murray R., **Vector Analysis**, 1959, Schaum.
[4]Wills, **Vector Analysis**, Dover.
[5]Crowe, **The History of Vector Analysis**,
[6]Hay, **Vector and Tensor Analysis**, Dover
[7]Gibbs, J. Willard, **Vector Analysis**,
[8]Spivak, Michael, **Calculus on Manifolds**,
[9]Jackson, **Classical Electrodynamics**,
[10]Goldberg, **Tensor Analysis on Manifolds**,
[11] Wheeler, John, Thorne, Kip, et al **Gravitation**