Vibration

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Contents

1 Damped One-Dimensional Vibration

The differential equation for damped vibration is

$$m\ddot{x} + c\dot{x} + kx = F(t).$$

The mass is m, the viscous damping constant is c, the stiffness is k, and the applied force is $F(t)$. For free damped vibration the roots of the characteristic equation are

$$-\frac{c}{2m} \pm \sqrt{(c/2m)^2 - k/m}.$$

The natural undamped resonant angular frequency is

$$\omega_n = \sqrt{k/m}.$$

The critical damping constant is

$$c_c = 2m\omega_n.$$

The critical damping ratio is defined to be

$$\zeta = \frac{c}{c_c}.$$

We have

$$c = 2\zeta m\omega_n,$$

so that if we divide the damped vibration equation by the mass m, we can write it in the form

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F(t)}{m}.$$

Underdamped Case $\zeta < 1$

$$x = x_0 \exp(-\zeta\omega_n t) sin(\omega_d t + \phi),$$

where the angular frequency of damped oscillation is

$$\omega_d = \omega_n\sqrt{1-\zeta^2}.$$

The period is

$$\tau_d = \frac{2\pi}{\omega_d}.$$

The notation in this section follows *Thomson*: **Theory of Vibration With Applications**

2 The Quality Factor Q

The quality factor Q arises in electrical tuning circuits. It may be defined in terms of the half power points of the resonance peak. See the relevant section in the chapter of **Physics** by Jim Emery, called electrical circuits. There it is shown that Q is given by

$$\frac{\omega_n L}{R},$$

where ω_n is the resonant frequency, L is the inductance, and C is the capacitance. Also Q is equal to 2π times the average stored energy divided by the energy dissipated per cycle. This is completely analogous to the one dimensional vibration problem. The electrical problem is

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{V}.$$

Rather than rederive Q for the vibration problem we identify coefficients in the two problems. Then mass m corresponds to inductance L, viscous damping constant c corresponds to resistance R, and stiffness k corresponds to 1 over the capacitance C. Thus

$$\begin{aligned} Q &= \frac{\omega_n m}{c} \\ &= \frac{\omega_n m}{\zeta c_c} \\ &= \frac{1}{2\zeta}, \end{aligned}$$

where ζ is the critical damping ratio of the previous section.

3 Logarithmic Decrement

The term Logarithmic decrement is an old one. It occurs in Lord Rayleigh's **Theory of Sound**, which was published in 1877. Damping can be determined experimentally by measuring the rate of decay of free oscillation. The logarithmic decrement δ is defined as the natural logarithm of the ratio of displacement values measured at a time t and a time one period later $t + \tau_d$. Thus

$$\begin{aligned} \exp(\delta) = \frac{x_1}{x_2} &= \frac{\exp(-\zeta\omega_n t)}{\exp(-\zeta\omega_n(t+\tau_d))} \\ &= \exp(\zeta\omega_n\tau_d). \end{aligned}$$

So that

$$\delta = \ln(x_1/x_2) = \zeta\omega_n\tau_d.$$

Replacing τ_d by an expression involving ω_n and ζ, we get

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}.$$

4 Equivalent Viscous Damping

There are many damping mechanisms. Among these are damping proportional to velocity as in the motion of a body through a fluid, and structural or hysteresis damping in which the dissipation energy is proportional to the square of the amplitude. In Thompson, in the section on damping, it is shown that, for a harmonic solution of the one dimensional damping equation, the energy dissipated per cycle at frequency ω is given by

$$w_d = \pi c \omega X^2,$$

where c is the viscous damping constant and X is the amplitude of vibration. Also it is shown that a hysteresis curve for this damping is elliptical. Assuming structural damping given by aX^2, where a is a constant, we obtain an equivalent viscous damping constant given by

$$c_{equivalent} = \frac{a}{\pi\omega}.$$

But notice that this so called constant depends inversely on the frequency. Note also that for structural damping, because the stored energy is also proportional to the square of the amplitude, Q is constant and independent of each eigenfrequency. However, the literature seems somewhat ambiguous about the Q of materials being frequency independent. The book **Vibration Analysis** 2nd ed. by Vierck gives a more involved derivation of the viscous equivalent of hysteresis damping (pp82-84). The equivalent critical damping ratio is obtained from

$$2m\zeta_{equivalent}\omega_n = c_{equivalent} = \frac{a}{\pi\omega}.$$

Thus

$$\zeta_{equivalent} = \frac{a}{2m\pi\omega_n\omega},$$

where ω_n is the natural resonant frequency. Let us write this as

$$\zeta_{equivalent} = \frac{b}{\omega_n\omega},$$

where b is a constant. Let

$$\zeta_n = \frac{b}{\omega_n^2}$$

be the equivalent critical damping ratio at resonance. Then

$$b = \zeta_n\omega_n^2$$

and so

$$\zeta_{equivalent} = \frac{\zeta_n\omega_n}{\omega}.$$

5 A Method For Computing Damped Harmonic Motion Parameters

We wish to determine parameters from data for damped harmonic motion. Let

$$x = ae^{-bt}\sin(ct + d),$$

where a can be positive or negative, $b \geq 0$, $c > 0$,and $0 \leq d < \pi$. This is the solution of the differential equation

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + (b^2 + c^2)x = 0.$$

For the LRC circuit

$$b = R/2L \text{ and } c = \sqrt{1/LC - (R/2L)^2}.$$

Given an x-intercept t_1, a following x-intercept t_2 and t the maximum magnitude point between the intercepts, find a, b, c, d. Solution:

$ct_1 + d = k\pi$ and $ct_2 + d = (k + 1)\pi$, thus

$$c = \frac{\pi}{t_2 - t_1}$$

$$d = \pi - (t_1 c \bmod(\pi)).$$

Using t as the maximum point

$$0 = dx/dt = ae^{-bt}(c\cos(ct + d) - b\sin(ct + d))$$

so

$$c\cos(ct + d) - b\sin(ct + d) = 0$$

and then

$$\frac{c}{\sqrt{b^2 + c^2}}\cos(ct + d) - \frac{b}{\sqrt{b^2 + c^2}}\sin(ct + d) = 0.$$

Letting

$$\sin(\theta) = \frac{c}{\sqrt{b^2 + c^2}}$$

we have

$$\sin(\theta)\cos(ct + d) - cos(\theta)\sin(cx + d) = \sin(\theta - (ct + d)) = 0$$

Then

$$\theta = ct + d - n\pi.$$

Since

$$\sin(\theta) = \frac{c}{\sqrt{b^2 + c^2}}$$

we have

$$b^2 + c^2 = \frac{c^2}{\sin^2(\theta)}.$$

$$b = c\sqrt{\frac{1}{\sin^2(ct+d)} - 1}$$

The value of x at t gives a. We may write

$$ct + d - n\pi = \sin^{-1}(\frac{c}{\sqrt{b^2 + c^2}}).$$

Maximum magnitudes occur at the same phase point in each interval. As $b \to \infty$ the maximum phase point $ct+d$ goes to $n\pi$ from the right. As $b \to 0$ the maximum phase point goes to the midpoint of the intercepts from the left. Let c and d be fixed and without loss of generality let $n = 0$ and $a = 1$. Then

$$x_{max}(b) = (\frac{c}{\sqrt{b^2 + c^2}}\ e^{-(b/c)(\sin^{-1}(\frac{c}{\sqrt{b^2+c^2}}))}$$

Thus $x_{max}(b)$ is decreasing and $x_{max}(0) = 1$ and $x_{max}(\infty) = 0$.

The determination of b is sensitive to error in the maximum point. If we have a sequence of values we may use a least squares method. Suppose we have a sequence of $n+1$ intercepts s_i and n maximum points t_i. Compute

$$c = \frac{\pi}{n}\sum_{i=1}^{n} \frac{1}{s_{i+1} - s_i}$$

$$d = \pi - \frac{1}{n}\sum_{i=1}^{n}(t_i c \bmod(\pi))$$

We may determine a and b by a least squares fit of

$$ae^{-bt_i} = \frac{x_i}{\sin(ct_i + d)}.$$

We note that

$$|\sin(ct_i + d)| = \sin(ct_i + d - k\pi) = \sin(ct_i + d - (cs_i + d)) = \sin(c(t_i - s_i)).$$

6 Decoupling Equations

Consider the equation

$$M\ddot{X} + KX = F,$$

where M is a square n-dimensional symmetric mass matrix, K is a square n-dimensional symmetric stiffness matrix, and F is a n-dimensional force. Both M and K must be positive definite because they represent the kinetic and strain quadratic forms. If either of them were not positive definite, there would be a nodal displacement or a nodal displacement velocity giving a negative energy. We assume a solution of the form

$$X = X_0 \exp(i\omega t).$$

Consider the homogeneous problem with $F = 0$. We have

$$-M\omega^2 X_0 + KX_0 = 0.$$

Let us write X for X_0. Then we have

$$(K - \omega^2 M)X = 0$$

This is a generalized eigenvalue problem. This homogeneous problem has a nonzero solution if and only if the determinant is zero:

$$|K - \omega^2 M| = 0.$$

This determinant is called the characteristic function, which is an nth degree polynomial in the variable ω^2. The roots are called eigenvalues. In the case of symmetric matrices the eigenvalues are real. To show this, let the inner product be

$$(u, v) = \sum_{i=1}^{n} u_i \bar{v}_i,$$

where the subscripted values are the vector components, and where the bar indicates the complex conjugate. Then if λ is an eigenvalue, with eigenvector v, because M and K are self-adjoint (real symmetric), we have

$$\lambda(Mv, v) = (\lambda Mv, v) = (Kv, v) = (v, \bar{K}^T v)$$

$$= (v, Kv) = (v, \lambda Mv) = \bar{\lambda}(v, Mv) = \bar{\lambda}(\bar{M}^T v, v) = \bar{\lambda}(Mv, v).$$

Therefore

$$\lambda = \bar{\lambda},$$

and so λ is real.

In this vibration problem the eigenvalues are not only real, but positive, because the matrices are positive definite. Under certain conditions, (1) If they are distinct, or (2) If one of the matrices is positive definite, then the vibration problem can be decoupled. Suppose that the eigenvalues are distinct. That is, the eigenfrequencies $\omega_1, ..., \omega_n$ are distinct. Since the determinant of the matrix

$$K - \omega_i^2 M$$

is zero for each i, its column vectors are linearly dependent, so that there exists a nonzero vector X_i such that

$$(K - \omega_i^2 M) X_i = 0.$$

The vectors $X_1, ..., X_n$ are called eigenvectors or modal vectors. Let P be a matrix which has as columns the eigenvectors

$$P = [X_1 | X_2 | ... | X_n]$$

By the definition of the eigenvectors, we have

$$K[X_1|X_2|...|X_n] = [\omega_1^2 M X_1 | \omega_2^2 M X_2 | ... | \omega_n^2 M X_n]$$

$$= M[X_1|X_2|...|X_n]\Lambda = MP\Lambda,$$

where Λ is the diagonal matrix of eigenvalues. Then

$$P^T K P = P^T M P \Lambda.$$

We will show that the eigenvectors have certain orthogonality properties. We will show that if ω_i^2 is not equal to ω_j^2, then vector X_i is perpendicular to X_j. Let A be a matrix, then we write A^T for the transpose. Because K and M are symmetric

$$K^T = K$$

and

$$M^T = M$$

We have

$$X_i^T K X_j = X_i^T \omega_j^2 M X_j = \omega_j^2 X_i^T M X_j$$

On the other hand

$$X_i^T K X_j = (K^T X_i)^T X_j = (K X_i)^T X_j = (\omega_i^2 M X_i)^T X_j$$
$$= \omega_i^2 X_i^T M^T X_j = \omega_i^2 X_i^T M X_j.$$

So

$$\omega_j^2 X_i^T M X_j = \omega_i^2 X_i^T M X_j.$$

But ω_j and ω_i are not equal, so that we must have

$$X_i^T M X_j = 0,$$

and then

$$X_i^T K X_j = 0.$$

Now let

$$X_i^T M X_i = \mu_i^2.$$

Any multiple of an eigenvector is still an eigenvector, so we scale the eigenvectors so that each

$$\mu_i^2 = 1$$

Then

$$P^T M P = I.$$

Then we have

$$P^T K P = P^T M P \Lambda = I \Lambda = \Lambda.$$

Both matrices have been diagonalized.

7 Example: Coupled Oscillators

Consider two masses m_1 and m_2. They can each move in a horizontal direction. Let the masses be connected with springs, k_1, k_2, and k_3. Spring k_1 is connected on the left with a fixed point, and on the right with mass m_1. Spring k_2 is connected on the left with m_1, and on the right with mass m_2. Spring k_3 is connected on the left with m_2, and on the right with a fixed point. Let u_1 and u_2 be the displacement from equilibrium of masses m_1 and m_2 respectively. The kinetic energy of the system is

$$T = \frac{1}{2} m_1 \dot{u}_1^2 + \frac{1}{2} m_2 \dot{u}_2^2$$

and the potential energy is

$$V = \frac{1}{2}k_1 u_1^2 + \frac{1}{2}k_2(u_2 - u_1)^2 + \frac{1}{2}k_3 u_2^2.$$

The Lagrangian is

$$L = T - V.$$

The equations of motion are given by Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}_1} - \frac{\partial L}{\partial u_1} = 0$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}_2} - \frac{\partial L}{\partial u_2} = 0.$$

We have

$$\frac{\partial L}{\partial \dot{u}_1} = \frac{\partial T}{\partial \dot{u}_1} = m_1 \dot{u}_1$$

and

$$\frac{\partial L}{\partial \dot{u}_2} = \frac{\partial T}{\partial \dot{u}_2} = m_2 \dot{u}_2.$$

Also

$$-\frac{\partial L}{\partial u_1} = \frac{\partial V}{\partial u_1} = k_1 u_1 - k_2(u_2 - u_1)$$

and

$$-\frac{\partial L}{\partial u_2} = \frac{\partial V}{\partial u_2} = k_2(u_2 - u_1) + k_3 u_2.$$

Thus Lagranges's equations are

$$m_1 \ddot{u}_1 + k_1 u_1 - k_2(u_2 - u_1) = 0$$

and

$$m_2 \ddot{u}_2 + k_2(u_2 - u_1) + k_3 u_2 = 0.$$

In matrix form the equations become

$$M \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + K \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix},$$

and

$$K = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_3 + k_2) \end{bmatrix}.$$

Let

$$u_1 = U_1 e^{i\omega t}$$

and

$$u_2 = U_2 e^{i\omega t}.$$

Then

$$(K - \omega^2 M) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

This equation has a nonzero solution if and only if

$$\det(K - \omega^2 M) = 0.$$

Let $\lambda = \omega^2$. The determinant is

$$(k_1 + k_2 - \lambda m_1)(k_3 + k_2 - \lambda m_2) - k_2^2,$$

which we equate to zero to find the eigenvalues λ. Now let us specialize the problem and set the spring constants k_1 and k_3, to a common value k. And also set the two masses m_1 and m_3, to a common value m. Let the coupling spring be a weak spring.

That is, let spring constant k_2 be equal to a fraction of k. The equation for the eigenvalues is the quadratic equation

$$(k + k_2 - \lambda m)(k + k_2 - \lambda m) - k_2^2 = 0.$$

That is, it is

$$A\lambda^2 + B\lambda + C = 0,$$

with

$$A = m^2$$

$$B = -2(k + k_2)m$$

and

$$C = (k + k_2)^2 - k_2^2 = k^2 + 2kk_2.$$

Let $a = (k + k_2 - \lambda m)$ and $b = -k_2$. Then since the determinant is 0, we must have

$$a^2 = b^2$$

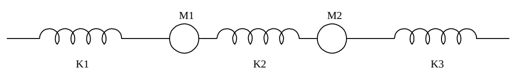

Figure 1: Coupled springs, With masses m_1 and m_1. The spring constants are k_1, k_2 and k_3. k_2 is the constant for the coupling spring and is much smaller than the other two constants.

So either

$$a = b,$$

or

$$a = -b.$$

The equations for the eigenvectors are

$$aU_1 + bU_2 = 0,$$

and

$$bU_1 + aU_2 = 0.$$

If $a = b$, then the eigenvector is

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

On the other hand, if $a = -b$, then the eigenvector is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenvalues are λ_1 and λ_2. Let a_i be the value of a for eigenvalue λ_i, and let b_i be the value of b for eigenvalue λ_i. Now let us compute with values assigned to the constants. Let $m = 1$, $k = 1$, and $k_2 = 1/10$. The computation is done with a Matlab script (Octave script). Call it **coupled.m**:

```
m=1.;
k=1.;
k_2=1/10;
A= m*m;
B= -2*(k+k_2)*m ;
C=  k^2 + 2 * k * k_2;
lambda_1=(-B + sqrt(B*B - 4*A*C))/(2*A)
omega_1=sqrt(lambda_1)
a_1 = (k + k_2 - lambda_1 * m)
b_1 = -k_2
lambda_2=(-B - sqrt(B*B - 4*A*C))/(2*A)
omega_2=sqrt(lambda_2)
a_2 = (k + k_2 - lambda_2 * m)
b_2 = -k_2
omega=(omega_1+omega_2)/2
t1=2*pi/omega
phi=(omega_1 -omega_2)/2
t2=2*pi/phi
```

The output is:

```
octave-3.0.0.exe:5> coupled
lambda_1 =  1.2000
omega_1 =  1.0954
a_1 = -0.10000
b_1 = -0.10000
lambda_2 =  1.00000
omega_2 =  1.00000
a_2 =  0.10000
b_2 = -0.10000
omega =  1.0477
t1 =  5.9970
phi =  0.047723
t2 =  131.66
```

```
octave-3.0.0.exe:6> exit
```

The first eigenvalue is

$$\lambda_1 = 1.2$$

a_1 and b_1 are equal, so the eigenvector is

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The corresponding frequencies are

$$\omega_1 = \pm\sqrt{\lambda_1} = \pm 1.0954$$

Hence two linearly independent solutions are

$$e^{i\omega_1 t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and

$$e^{-i\omega_1 t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

So any linear combination of these solutions is also a solution. That is, the space spanned by these two solutions is a solution. But both $cos(\omega_1 t)$ and $sin(\omega_1 t)$ can be written as a linear combination of $e^{i\omega_1 t}$ and $e^{-i\omega_1 t}$. Hence this solution space is the same as the space spanned by

$$\cos(\omega_1 t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and

$$\sin(\omega_1 t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The second eigenvalue is

$$\lambda_2 = 1.0$$

a_2 and b_2 are not equal, so the eigenvector is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The corresponding frequencies are

$$\omega_1 = \pm\sqrt{\lambda_2} = \pm 1.0$$

Hence two linearly independent solutions are

$$e^{i\omega_2 t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$e^{-i\omega_2 t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Again the solution space is spanned by

$$\cos(\omega_2 t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\sin(\omega_2 t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Suppose the initial conditions are

$$u_1(0) = A$$

$$\frac{du_1(0)}{dt} = 0$$

$$u_2(0) = 0$$

$$\frac{du_2(0)}{dt} = 0$$

Let the solution be

$$u = c_{11}\cos(\omega_1 t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_{12}\sin(\omega_1 t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_{21}\cos(\omega_2 t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_{22}\sin(\omega_2 t)\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where the constants are to be determined by the initial conditions.

From the first initial condition we have

$$c_{11} + c_{21} = A$$

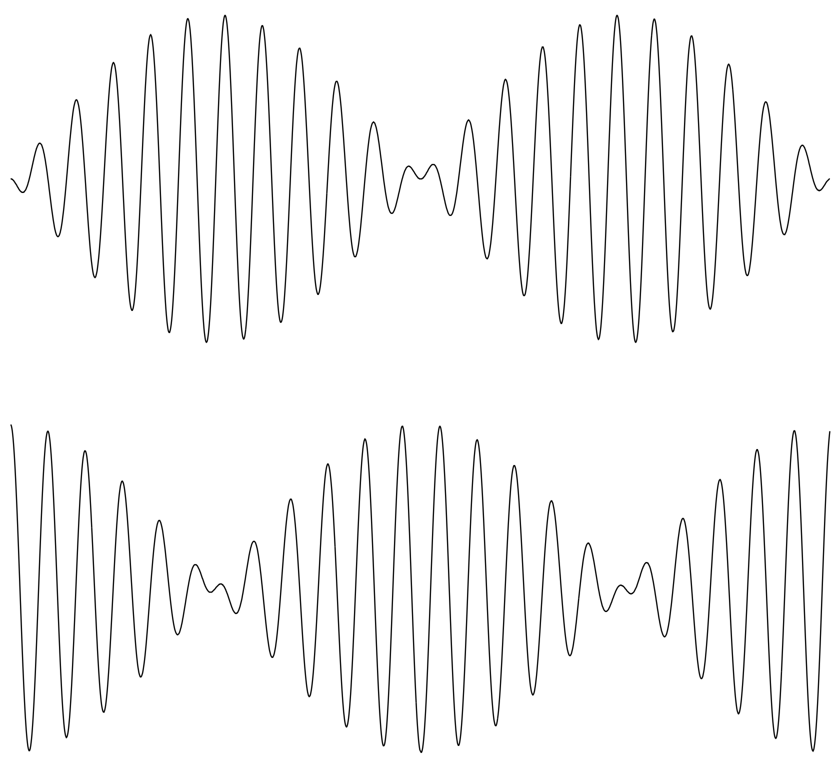

Figure 2: Coupled oscillations. The lower curve is the oscillation of mass m_1. The upper curve is the oscillation of mass m_2.

From the third initial condition we have

$$-c_{11} + c_{21} = 0$$

Thus

$$c_{21} = A/2$$

and

$$c_{11} = A/2.$$

Differentiating our solution, we have

$$\frac{du}{dt} = -c_{11}\omega_1 \sin(\omega_1 t) \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_{12}\omega_1 \cos(\omega_1 t) \begin{bmatrix} 1 \\ -1 \end{bmatrix} - c_{21}\omega_2 \sin(\omega_2 t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_{22}\omega_2 \cos(\omega_2 t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

From the second initial condition we have

$$c_{12}\omega_1 + c_{22}\omega_2 = 0.$$

From the forth initial condition we have

$$-c_{12}\omega_1 + c_{22}\omega_2 = 0.$$

Therefore $c_{22} = 0$ and $c_{12} = 0$. Therefore our solution is

$$u_1 = \frac{A}{2}\cos(\omega_1 t) + \frac{A}{2}\cos(\omega_2 t).$$

$$u_2 = -\frac{A}{2}\cos(\omega_1 t) + \frac{A}{2}\cos(\omega_2 t).$$

Now if $\omega = (\omega_1 + \omega_2)/2$ and $\phi = (\omega_1 - \omega_2)/2$, then

$$\cos(\omega_1 t) + \cos(\omega_2 t) = 2\cos(\omega t)\cos(\phi t)$$

So

$$u_1 = A\cos(\omega t)\cos(\phi t).$$

Similarly

$$u_2 = -A\sin(\omega t)\sin(\phi t).$$

For the numbers we have used here the oscillating frequency is

$$\omega = 1.0477,$$

and the corresponding period is

$$T_1 = 5.9970.$$

The modulating frequency is

$$\phi = 0.047723,$$

and the modulating period is

$$t_2 = 131.66.$$

The oscillators operate as follows. The first oscillator starts at maximum amplitude A, while the second oscillator is stopped at zero amplitude. The amplitude of the first oscillator begins to decrease, according to the modulating factor $\cos(\phi t)$. At $t = t_2/4$, the amplitude of the first oscillator has decreased to zero, whereas the amplitude of the second oscillator has increased to its maximum amplitude A, according to the modulating factor $\sin(\phi t)$. The energy of the first oscillator has been transferred to the second. This behavior repeats, as energy flows back and forth between the two oscillators.

8 Decoupling The Rayleigh Damping Equations

Consider the vibration problem with proportional damping

$$M\ddot{X} + (\alpha M + \beta K)\dot{X} + KX = 0.$$

We find the modal matrix P consisting of the eigenvectors as in the previous section. Let

$$X = PY.$$

Then

$$P^T[MP\ddot{Y} + (\alpha M + \beta K)P\dot{Y} + KPY] = 0.$$

Then

$$P^TMP\ddot{Y} + (\alpha P^TMP + \beta P^TKP)\dot{Y} + P^TKPY = 0.$$

$$I\ddot{Y} + (\alpha I + \beta\Lambda)\dot{Y} + \Lambda Y = 0.$$

Each of the matrices in this equation is a diagonal matrix. Written out, the system is

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \dots \\ \dots \\ \ddot{y}_n \end{bmatrix} + \begin{bmatrix} \alpha + \beta\omega_1^2 & 0 & \dots & 0 \\ 0 & \alpha + \beta\omega_2^2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \alpha + \beta\omega_n^2 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dots \\ \dots \\ \dot{y}_n \end{bmatrix}$$

$$+ \begin{bmatrix} \omega_1^2 & 0 & \dots & 0 \\ 0 & \omega_2^2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \omega_n^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ \dots \\ y_n \end{bmatrix} = 0$$

The equations have been decoupled, each of the n differential equations is independent of the others. The coordinates of vector Y are called normal coordinates. The ith coordinate equation is

$$\ddot{y}_i + (\alpha + \beta\omega_i^2)\dot{y}_i + \omega_i^2 y_i = 0.$$

We recognize this as a damped harmonic equation which in standard form is

$$\ddot{y}_i + 2\zeta_i\omega_i\dot{y}_i + \omega_i^2 y_i = 0.$$

Then we have

$$2\zeta_i\omega_i = \alpha + \beta\omega_i^2,$$

where ζ_i is the critical damping ratio for the ith mode of vibration. Thus

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2}.$$

If all modes have the same Q, then ζ_i is constant and

$$\zeta_i = \frac{1}{2Q}.$$

Given two or more eigenfrequencies we can use least squares to find values of the damping parameters α and β that are approximately correct for each eigenfrequency.

In the section **Equivalent Viscous Damping** we have shown that

$$\zeta_{equivalent} = \frac{\zeta_i \omega_i}{\omega},$$

for a nonresonant frequency ω. Thus over a small frequency range near the resonant frequency ω_i, $\zeta_{equivalent}$ may be taken to be ζ_i. Therefore we can use constant Rayleigh damping parameters α and β found through the least squares fitting described above. See the section on damping in the ANSYS theory manual.

9 A Damped and Forced Longitudinally Vibrating Bar

Let a long bar of cross sectional area A and length ℓ vibrate along its axis. Consider an element of the bar of length Δx. On the left side of the element there is a force

$$A\sigma(x)$$

and on the right a force

$$A\sigma(x + \Delta x),$$

where σ is the stress. Also let there be a viscous damping force opposing the displacement velocity given by

$$\gamma \frac{\partial u}{\partial t} \Delta x,$$

and an external force

$$f(x, t)\Delta x.$$

The mass of the element is

$$A\Delta x \rho.$$

The strain is

$$\epsilon = \frac{\partial u}{\partial x}$$

We have

$$\sigma = E\epsilon,$$

where E is Young's modulus. Equating the net force to the mass times the acceleration, we have

$$A(\sigma(x+\Delta x)-\sigma(x))-\gamma\frac{\partial u}{\partial t}\Delta x+f(x,t)\Delta x=A\Delta x\rho\frac{\partial^2 u}{\partial t^2}.$$

Replacing the stress by the strain

$$EA(\frac{\partial u(x+\Delta x)}{\partial x}-\frac{\partial u(x)}{\partial x})-\gamma\frac{\partial u}{\partial t}\Delta x+f(x,t)\Delta x=A\Delta x\rho\frac{\partial^2 u}{\partial t^2}.$$

Dividing by $EA\Delta x$ and letting Δx go to zero, we have

$$\frac{\partial^2 u}{\partial x^2}=\frac{\rho}{E}\frac{\partial^2 u}{\partial t^2}+\frac{\gamma}{EA}\frac{\partial u}{\partial t}-\frac{1}{EA}f(x,t).$$

We look at the free boundary condition case with the stress, and hence the strain zero at the bar ends. That is

$$\frac{\partial u}{\partial x}=0,$$

at $x=0$, and at $x=\ell$. We next calculate the vibration modes for the undamped, and unforced problem. We have

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where

$$c=\sqrt{\frac{E}{\rho}},$$

is the wave velocity. Using separation of variables

$$u(x,t)=U(x)T(t),$$

we have

$$c^2\frac{\ddot{U}}{U}=\frac{\ddot{T}}{T}=-L^2,$$

for a constant L. Then

$$\ddot{U}(x)+\frac{L^2}{c^2}U=0,$$

so that

$$U(x)=A\cos((L/c)x+B\sin((L/c)x).$$

Then

$$U'(x) = (L/c)[-A\sin((L/c)x + B\cos((L/c)x)].$$

And

$$0 = U'(0) = (L/c)B$$

so that $B = 0$. Then

$$0 = U'(\ell) = -(L/c)A\sin((L/c)\ell,$$

which implies that

$$L\ell/c = n\pi,$$

for $n = 0, 1, 2, 3, ...$ Hence for $n = 0, 1, 2, 3, ...$, each eigenfunction

$$U_n(x) = A_n\cos((n\pi/\ell)x),$$

multiplied by a time dependent function, is a solution to the problem.

Now we use these eigenfunctions to solve the damped and forced problem. We assume a solution of the form

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t)\cos((n\pi/\ell)x).$$

We substitute this into the damped wave equation, getting

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=0}^{\infty} -T_n(t)(n\pi/\ell)^2\cos((n\pi/\ell)x),$$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \sum_{n=0}^{\infty} \frac{1}{c^2}\ddot{T}_n(t)\cos((n\pi/\ell)x),$$

and

$$\frac{\gamma}{EA}\frac{\partial u}{\partial t} = \sum_{n=0}^{\infty} \frac{\gamma}{EA}\dot{T}_n(t)\cos((n\pi/\ell)x).$$

So we have

$$\sum_{n=0}^{\infty}[\frac{1}{c^2}\ddot{T}_n(t) + \frac{\gamma}{EA}\dot{T}_n(t) + (n\pi/\ell)^2 T_n]\cos((n\pi/\ell)x) = \frac{1}{EA}f.$$

Multiplying by c^2,

$$\sum_{n=0}^{\infty}[\ddot{T}_n(t) + \frac{c^2\gamma}{EA}\dot{T}_n(t) + (c^2 n\pi/\ell)^2 T_n]\cos((n\pi/\ell)x) = \frac{c^2}{EA}f.$$

Letting

$$\omega_n = \frac{n\pi}{\ell}c = \frac{n\pi}{\ell}\sqrt{E/\rho},$$

and

$$2\zeta_n\omega_n = \frac{c^2\gamma}{EA} = \frac{\gamma}{\rho A},$$

we can write the equation as

$$\sum_{n=0}^{\infty}[\ddot{T}_n(t) + 2\zeta_n\omega_n\dot{T}_n(t) + \omega_n^2 T_n]\cos((n\pi/\ell)x) = \frac{1}{\rho A}f.$$

We recognize the differential equations in t as damped harmonic equations, with ω_n the natural frequency, and ζ_n the critical damping ratio. Solving for ζ_n, we have

$$\zeta_n = \frac{\gamma}{\rho A 2\omega_n} = \frac{\gamma}{\rho A 2(n\pi/\ell)\sqrt{E/\rho}} =$$

$$\frac{\gamma\ell}{2n\pi A\sqrt{\rho E}}.$$

To determine experimentally the viscous damping constant γ, we can proceed as follows. We use a forcing functions whose frequencies are near a resonant frequency ω_n, and measure the response, from which we may calculate the critical damping constant ζ, which then gives γ.

Now suppose we have a lumped parameter mathematical model of the rod, namely a finite element model in which we use proportional or Rayleigh damping. The model is

$$M\ddot{u} + (\alpha M + \beta K)\dot{u} + Ku = F.$$

We can decouple this equation by using the modal matrix, that is the matrix consisting of eigenvectors of the free vibration. We then may compare the resulting one dimensional modal differential equations with the damped harmonic equations found above, thereby obtaining a relationship between the intrinsic viscous damping γ and the Rayleigh parameters α and β. As above we have

$$\frac{\gamma}{\rho A} = 2\omega_n \zeta_n = \alpha + \beta\omega_n^2.$$

This relates the Rayleigh damping parameters to the intrinsic viscous damping constant, which is a property of the material. Note that there is also hysteretic damping that does not depend on the displacement velocity.

10 The Thin Transverse Vibrating Beam

The equation for the vibrating beam may be obtained by reduction from the thin transverse vibrating plate equation (see Soedel). But here we shall use elementary concepts to obtain the equation.

Let r be the radius of curvature of the neutral axis of the thin beam. Let y be the distance from the neutral axis to a point in the beam cross section. When the beam is bent, a length ℓ changes to a length $\ell + \delta\ell$. The length at the neutral axis does not change. The length ℓ on the neutral axis, and the radius of curvature r, are lengths of sides of a triangle. The distance from the neutral axis y and the length change $\delta\ell$, when the deflection is small, are sides of a similar triangle. So using similar triangles, we find

$$\frac{\delta\ell}{y} = \frac{\ell}{r}.$$

Then the strain is

$$e = \frac{\delta\ell}{\ell} = \frac{y}{r}.$$

The stress is

$$\sigma = E\frac{y}{r}.$$

The net force is zero over the cross section, so integrating we get

$$0 = \int \sigma dA = \frac{E}{r}\int y dA.$$

The integral is the area centroid. The neutral axis passes through the centroid. The bending moment is given by

$$M = \int y\sigma dA = \frac{E}{r}\int y^2 dA = \frac{E}{r}I.$$

where I is the area moment of inertia. The curvature is the reciprocal of the radius of curvature and we have

$$\frac{1}{r} = \kappa = \frac{d^2y/dx^2}{(1 + (dy/dx)^2)^{3/2}}.$$

When the slope is small the curvature is equal approximately to the second derivative. Hence the beam equation is

$$\frac{d^2y}{dx^2} = \kappa = \frac{M}{EI}.$$

When the beam is subjected to point loads , the bending moment is linear between the point loads, hence on each segment of the beam we have an equation of the form

$$\frac{d^2y}{dx^2} = \frac{ax + b}{EI},$$

where a and b are constants depending on the point loads. This integrates to a cubic polynomial. Hence the solution deflection curve is a piecewise cubic polynomial with continuous first and second derivatives, and zero second derivatives at the ends. The deflection curve is a natural cubic spline. A natural cubic spline has zero second derivatives at the ends. The curve becomes straight at the ends.

Let Q be the transverse shear stress in the beam, and let p be the downward pressure per unit length. Then the conditions for equilibrium of a element of the beam of length dx is that the external vertical force vanish, so that the change in shear stress over length Δx is

$$\frac{\partial Q}{\partial x} \Delta x = p \Delta x.$$

The moment about an edge of the element vanishes so that the change in the moment over length Δx is

$$\frac{\partial M}{\partial x} \Delta x = Q \Delta x.$$

That is

$$\frac{\partial M}{\partial x} = Q.$$

Since the moment is proportional to the second derivative, the shear is proportional to the third derivative. Combining these equations, we get

$$\frac{\partial^2 M}{\partial x^2} = p.$$

Substituting for the moment M, we get

$$\frac{\partial^4 w}{\partial x^4} = \frac{p}{EI}.$$

Suppose the beam is freely vibrating. Then we replace the force on the element of length dx by a mass times acceleration term

$$\rho A dx \frac{\partial^2 w}{\partial t^2}.$$

Then the free vibration equation is

$$\frac{\partial^4 w}{\partial x^4} = \frac{\rho A}{EI}\frac{\partial^2 w}{\partial t^2}.$$

A particular solution to the equation depends on the boundary conditions at the beam ends. As a typical solution, we have

$$w = C \sin(kx) e^{i\omega t}.$$

The wave number is

$$k = \frac{2\pi}{\lambda},$$

where λ is the wavelength. We find that

$$k^4 = \omega^2 \frac{\rho A}{EI},$$

which defines the natural frequency ω as a function of a given wavelength. If two nodes of a standing wave are given, the wave length must fit an integral number of times between these two nodes.

11 A Damped Vibrating Cantilever Beam

The free vibration equation is

$$\frac{\partial^4 u_3}{\partial x_1^4} = -\frac{\rho A}{EI}\frac{\partial^2 u_3}{\partial t^2}.$$

Using separation of variables, we let

$$u_3(x,t) = f(x)g(t).$$

Substituting in the equation, we get

$$\frac{1}{f(x_1)}\frac{d^4 f(x_1)}{dx_1^4} = -\frac{\rho A}{EI}\frac{1}{g(t)}\frac{d^2 g(t)}{dt_1^2} = k^4.$$

We get two ordinary differential equations

$$\frac{d^4 f}{dx_1^4} - k^4 f = 0.$$

and

$$\frac{d^2 g}{dt^2} + \omega^2 g = 0,$$

where

$$\omega^2 = \frac{EI}{\rho A}k^4.$$

The general solutions are

$$f(x) = A\cos(kx) + B\sin(kx) + C\cosh(kx) + D\sinh(kx),$$

and

$$g(x) = E\cos(\omega t) + F\sin(\omega t).$$

Boundary conditions for the clamped cantilever beam: At $x = 0$,

$$u_3(0) = 0, \frac{\partial u_3}{\partial x_1} = 0.$$

At the end of the beam where $x_1 = \ell$, the moment is zero and the shear stress is zero, so that

$$\frac{\partial^2 u_3}{\partial x_1^2} = 0, \frac{\partial^3 u_3}{\partial x_1^3} = 0.$$

The first boundary condition gives

$$A + C = 0.$$

The second condition gives

$$Bk + Dk = 0$$

or if k is not zero

$$B + D = 0.$$

By the third condition

$$-A\cos(k\ell) - B\sin(k\ell) + C\cosh(k\ell) + D\sinh(k\ell) = 0$$

By the fourth condition

$$-A\sin(k\ell) + B\cos(k\ell) + C\sinh(k\ell) + D\cosh(k\ell) = 0$$

Substituting the first two equations into the last two, we get two homogeneous equations for A and B.

$$(\cos(k\ell) + \cosh(k\ell))A + (\sin(k\ell) + \sinh(k\ell))B = 0$$

$$(\sin(k\ell) - \sinh(k\ell))A - (\cos(k\ell) + \cosh(k\ell))B = 0$$

These equations can only have a solution if the determinant is zero. The determinant is

$$(\cos(k\ell) + \cosh(k\ell))^2 + \sin^2(k\ell) - \sinh^2(k\ell)$$

$$= 2\cos(k\ell)\cosh(k\ell) + \sin^2(k\ell) + \cos^2(k\ell) + \cosh^2(k\ell) - \sinh^2(k\ell)$$

$$= 2\cos(k\ell)\cosh(k\ell) + 2 = 0$$

That is

$$\cosh(k\ell)\cos(k\ell) + 1 = 0.$$

The roots of this equation are:

$$\{k_i\ell\} = \{1.87, 4.75, 7.85, ..., \}$$

Let us call these roots

$$\eta_1, \eta_2, \eta_3, ...$$

Then

$$k_i = \frac{\eta_i}{\ell}.$$

As i increases, the ith root will approach

$$\frac{2i-1}{2}\pi$$

very quickly. This is because $\cosh(x)$ will get very large as x increases. Then if x is a root of the equation

$$\cosh(x)\cos(x) = -1,$$

then it will have to be very close to a root of $\cos(x)$.

Once we have selected a root, we can assign a value to A, and then solve for B,C,and D to get the mode shape.

The eigenfrequencies are given by

$$\omega_i = k_i^2\sqrt{EI/\rho A} = (\eta_i/\ell)^2\sqrt{EI/\rho A}.$$

Let

$$f_i(x) = A_i\cos(k_i x) + B_i\sin(k_i x) + C_i\cosh(k_i x) + D_i\sinh(k_i x)$$

be the corresponding eigenfunctions.

For the damped problem we introduce a viscous damping constant γ. The damped forced equation is

$$\frac{EI}{\rho A}\frac{\partial^4 u_3}{\partial x_1^4} + \frac{\partial^2 u_3}{\partial t^2} + \frac{\lambda}{\rho A}\frac{\partial u_3}{\partial t} = \frac{q_3(x,t)}{\rho A}.$$

where q_3 is an external force per unit length. As a solution we take

$$\sum_{i=1}^{\infty} g_i(t) f_i(x).$$

Substituting this into the equation, we get

$$\sum_{i=1}^{\infty}(\ddot{g}_i(t) + \frac{\lambda}{\rho A}\dot{g} + \frac{EI}{\rho A}k^4 g_i)f_i(x) = \frac{q_3}{\rho A}.$$

Then

$$\sum_{i=1}^{\infty}(\ddot{g}_i(t) + 2\zeta_i\omega_i\dot{g} + \omega_i^2 g_i)f_i(x) = \frac{q_3}{\rho A}.$$

where

$$\zeta_i = \frac{\lambda}{2\omega_i\rho A} = \frac{\lambda\ell^2}{2\eta_i^2\sqrt{EI\rho A}}.$$

As in the case of the longitudinally vibrating rod, we get a set of damped one dimensional vibration equations. The critical damping constant ζ_i relates the intrinsic viscous damping constant to the Rayleigh parameters α and β.
Vibrating Beam Reference: Soedel, p65. See the chapter titled: **Arch, Beam and Rod**. The equation for free vibration of the beam is derived by reduction from Love's equation,

$$-EI\frac{\partial^4 u_3}{\partial x_1^4} + q_3' = \rho A\frac{\partial^2 u_3}{\partial t^2}.$$

Vibrating Beam Reference: Haberman, p234.
Vibrating Beam Reference: Burton **Vibration and Impact**, pp242-246.

12 The General Problem of Linear Vibration

The reference for this section is:
Bradbury T C, **Theoretical Mechanics**, John Wiley, 1968, p352.

The kinetic and potential energies are given as quadratic forms

$$T = \frac{1}{2}g_{ij}\dot{q}_i\dot{q}_j,$$

and

$$V = \frac{1}{2}k_{ij}q_iq_j.$$

The Lagrange equations of motion become

$$g_{ij}\ddot{q}_j + k_{ij}q_j = 0.$$

Write this as a matrix equation

$$G\ddot{Q} + KQ = 0,$$

where G and K are symmetric n by n matrices. This equation has a solution

$$Q = X \exp(\omega t).$$

Then

$$(K - \lambda G)X = 0,$$

where

$$\lambda = \omega^2.$$

This is a generalized eigenvalue problem. The quadratic forms are positive definite. The eigenvalues are real, and the eigenvectors (normal modes) are orthogonal. Let the rows of matrix S be constructed out of eigenvectors, which are scaled to be of unit length with metric G (S, or S^T is called the modal matrix in Thomson "Theory of Vibration."). That is,

$$X^T G X = \delta_{ij}.$$

Then

$$S G S^T = I.$$

Let Λ be a diagonal matrix of the eigenvalues. Then

$$K S^T = \Lambda G S.$$

Define normal coordinates Q' by

$$Q = Q' S^T.$$

Then the equations of motion become

$$G S^T \ddot{Q}' + K S^T Q' = 0.$$

Multiplying by S and simplifying, we get

$$\ddot{Q}' + \Lambda Q' = 0.$$

The problem could also be solved by diagonalizing the original energy quadratic forms. Any two positive definite quadratic forms can be simultaneously diagonalized. See for example, D. E. Littlewood **A University Algebra**, 2nd ed. 1958 Dover, p53.

13 The General Vibration Problem With Damping

Reference: Meirovitch **Analytical Methods of Vibration**. See Lagrange's equation with a dissipation function, p390. The dissipation function was named by Lord Rayleigh (**Theory of Sound**, p. 103):

$$F = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\dot{q}_i\dot{q}_j$$

For an example of the dissipation function, see p394.

14 Forced Vibration and Proportional Damping

Reference: Thomson, **Theory of Vibration**, p180.

$$M\ddot{X} + C\dot{X} + KX = F.$$

We use proportional damping so that

$$C = \alpha M + \beta K.$$

By diagonalizing and using the resulting normal coordinates Y, as was done in a previous section, the equation can be uncoupled and written as

$$\ddot{Y} + (\alpha I + \beta\Lambda)\dot{Y} + \Lambda Y = F'.$$

Then we get the one dimensional equations

$$\ddot{y}_i + (\alpha + \beta\omega_i^2)\dot{y}_i + \omega_i^2 y_i = Q_i(t)$$

Now using a sinusoidal forcing function at a set of frequencies ω near the resonant frequency ω_i, we can determine the coefficients α and β.

Each of the n equations represents a one dimensional damped vibration. Each such equation has a steady state solution for a sinusoidal forcing function $exp(i\omega t)$. Then computing the output for a set of frequencies around the resonant frequency ω_j for this equation, we can compute the damping ratio ζ and equate

$$2\zeta\omega_j = (\alpha + \beta\omega_j^2).$$

That is, if we have measured values of ζ at resonant frequencies, then we can compute values of α and β.

Reference: Zienkiewicz **The Finite Element Method** 1977, Chapter 20 *Field and Dynamic Problems*. The formula above is given on page 564: The critical damping ratio for the ith mode is

$$\zeta_i = \frac{1}{2\omega_i}(\alpha + \beta\omega_i^2),$$

which means that higher frequencies are overdamped and so decay, so that the approximation using a finite number of frequencies is justified.

15 Experimental Determination of Damping Parameters

- Reference: Inman, **Vibration, With Control, Measurement, and Stability** 1989, Chapter 8, *Modal Testing*.

- Reference: Clough and Penzien **Dynamics of Structures**, (This book is mentioned in Zienkiewicz as a source for experimental determination of damping parameters). See the section on earthquake Response.

- Inverse problem: Determine matrices M, C, K from measurements of responses.

- System Identification Theory: The process of determining the mathematical model from measured inputs and outputs. This is an active area of research in the field of control theory.

- Hardware: Electromagnetic Shaker, Stinger, Impulse Hammer, Impact Hammer.

16 Hysteresis Damping

When an elastic body is put under an alternating load, the stress-strain curve as the load is increasing differs from the stress-strain curve followed as the load decreases. The area enclosed by this hysteresis curve represents an energy loss. It turns out to be proportional to the square of the wave amplitude, but independent of the frequency, since it does not depend on the velocity as does viscous damping. This damping is called *hysteretic damping*, and *structural damping*.

References: Meirovitch, **Analytical Methods of Vibration**, pp400-404, and Soedel **Vibration of Shells and Plates**, 2nd ed., Chapter 14, *Hysteresis Damping*.

The measurement of the hysteresis loop requires very sensitive instruments. For example, according to the reference:
Harris and Crede, **Shock and Vibration Handbook** 2nd ed. 1976, Chap 36 page 3,

the maximum strain width of the hysteresis loop for chrome steel, while experiencing an alternating stress of 103 MPascals, is 2×10^{-6}. (Compute the energy lost per cycle)

17 Viscoelastic Damping

There is a possible viscous damping that is proportional to the strain velocity rather than to the displacement velocity. The resistance force is then proportional to the third derivative of the displacement. See Clough and Penzien, p301.

18 Modal Dynamic Analysis

The finite element program ABAQUS has a modal dynamic analysis described as follows: The eigenvalue problem is solved and the system is diagonalized as was shown in a previous section. So the equations are uncoupled. Each one dimensional homogeneous equation is solvable in closed form, and will be the underdamped, critically damped, or overdamped solution. At each time step a solution is found for this initial value problem, using as initial values, the final values of the previous step. The particular part of the solution is calculated by assuming a linear approximation to the forcing function at the given time point.

19 Modal Expansion

Given a linear operator L, suppose it has a set of orthonormal eigenfunctions ϕ_i.

$$L\phi_i = \lambda_i \phi_i.$$

Given an external force F, we may compute a Fourier representation

$$F = \sum c_i \phi_i$$

in the eigenspace (or a projection to a subspace), where the coefficients are inner products

$$c_i = (F, \phi_i).$$

Then let a solution of

$$LX = F,$$

be

$$X = \sum a_i \phi_i.$$

We have

$$LX = \sum \lambda_i a_i \phi_i,$$

so that

$$a_i = \frac{c_i}{\lambda_i}.$$

See also Werner Soedel **Vibrations of Shells and Plates**.

20 Damping References

Damping constants and ringdown formulas. Logarithmic decrement method. See p27 **Theory of Vibrations With Applications** William T. Thompson Ringdown:

$$e^{-\zeta\omega_n t}\sin(\omega_n t)$$

Determine ζ by calculating logarithms of ratios of successive periods of amplitudes. Matrix equation:

$$M\frac{d^2x}{dt^2} + C\frac{dx}{dt} + Kx = 0.$$

M is the mass matrix, C is the damping matrix, and K is the stiffness matrix.

Rayleigh damping (see Zienkkiewicz p. 532, p. 564, Desai and Abel). Also called proportional damping. Proportional damping is named Rayleigh damping because it is mentioned in one sentence in Rayleigh's **Theory of Sound**, V1, p130 Dover edition. See Mierovich Chapter 9 *Damped Systems*. Also there more general conditions on damping matrix which allows equations to be decoupled.

Determine experimentally (**Dynamics of Structures** R. W. Clough, J Penzien) Let

$$C = \alpha M + \beta K.$$

Let

$$x = x_0 e^{j\omega t}.$$

Substitute and diagonalize the resulting matrix. Find ζ experimentally, and then find α and β.

21 Steady State Solution

Consider the forced vibration problem.

$$M\ddot{X} + C\dot{X} + KX = F.$$

Let the force F be sinusoidal,

$$F = F_0 \exp(i\omega t).$$

Let the solution be

$$X = X_0 \exp(i\omega t).$$

Then we have

$$[(K - M\omega^2) + C\omega i]X_0 \exp(i\omega t) = F_0 \exp(i\omega t).$$

Then

$$Z_\omega X_0 = F_0,$$

where

$$Z_\omega = (K - M\omega^2) + C\omega i,$$

is the impedance matrix. Then

$$X_0 = Z_\omega^{-1} F_0.$$

If F is a periodic function, we may represent F as a Fourier series and apply the technique to each term.

22 The Vibration of a Cylindrical Plate

The equation for free vibration of a thin plate is

$$\nabla^4 u_3 + \frac{\rho h}{D}\frac{\partial^2 u_3}{\partial t^2} = 0.$$

The material density is ρ, the plate thickness is h, and the plate stiffness is D, where

$$D = \frac{Eh^3}{12(1-\nu^2)},$$

and E is Young's modulus, and ν is Poisson's ratio. The biharmonic operator is

$$\nabla^4 = \nabla^2\nabla^2 = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$$
$$= \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

The Laplace operator is, in polar coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

We shall assume harmonic time variation and that the solution has the form

$$u_3(r,\theta,t) = U_3(r,\theta)\exp(i\omega t).$$

Then we have

$$(\nabla^4 - \lambda^4)U_3(r,\theta)\exp(i\omega t) = 0,$$

where

$$\lambda^4 = \frac{\rho h \omega^2}{D}.$$

We can factor the operator and put the equation in the form

$$(\nabla^2 - \lambda^2)(\nabla^2 + \lambda^2)U_3 = 0,$$

The equation will be satisfied when either of the factors is zero.
Let us consider the factor

$$(\nabla^2 + \lambda^2)U_3 = 0.$$

We use the separation of variables technique and write

$$U_3(r,\theta) = f(r)g(\theta).$$

Then

$$\frac{r^2}{f}\frac{d^2 f}{dr^2} + \frac{r}{f}\frac{df}{dr} + \lambda^2 r^2 = -\frac{1}{g}\frac{d^2 g}{d\theta^2} = k^2,$$

for some constant k. Then we get two equations

$$\frac{d^2 g}{d\theta^2} + k^2 g = 0,$$

and

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \lambda^2 f = \frac{k^2 f}{r^2}.$$

We rewrite the second equation as

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + (\lambda^2 - \frac{k^2}{r^2})f = 0.$$

The first equation has solution

$$g(\theta) = C_1 \cos(k(\theta - \phi)).$$

where $k = 0, 1, 2, 3, 4, ...$. Let

$$\eta = \lambda r.$$

Then the second equation becomes

$$\frac{d^2 f}{d\eta^2} + \frac{1}{\eta}\frac{df}{d\eta} + (1 - \frac{k^2}{\eta^2})f = 0.$$

This is Bessel's equation. It has general solution

$$C_2 J_k(\lambda r) + C_3 Y_k(\lambda r).$$

Now let us consider the factor

$$(\nabla^2 - \lambda^2)U_3 = 0$$

We separate variables and write

$$U_3(r,\theta) = f(r)g(\theta).$$

Then

$$\frac{r^2}{f}\frac{d^2 f}{dr^2} + \frac{r}{f}\frac{df}{dr} + -\lambda^2 r^2 = -\frac{1}{g}\frac{d^2 g}{d\theta^2} = k^2,$$

for some constant k. Then we get two equations

$$\frac{d^2 g}{d\theta^2} + k^2 g = 0,$$

and

$$\frac{d^1 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \lambda^2 f = \frac{k^2 f}{r^2}.$$

We rewrite the second equation as

$$\frac{d^1 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - (\lambda^2 + \frac{k^2}{r^2})f = 0.$$

The first equation again has solution

$$g(\theta) = C_1 \cos(k(\theta - \phi)).$$

where $k = 0, 1, 2, 3, 4, ...$. Let

$$\eta = \lambda r.$$

Then the second equation becomes

$$\frac{d^1 f}{d\eta^2} + \frac{1}{\eta}\frac{df}{d\eta} - (1 + \frac{k^2}{\eta^2})f = 0.$$

This is the modified equation of Bessel, which has general solution

$$C_4 I_k(\lambda r) + C_5 K_k(\lambda r).$$

The kth mode of vibration of the plate is then

$$u_3(r, \theta, t) = C_1 \cos(k(\theta - \phi))[C_2 J_k(\lambda r) + C_3 Y_k(\lambda r) + C_4 I_k(\lambda r) + C_5 K_k(\lambda r)] \exp(i\omega t).$$

(see also elastic.tex).

23 The Boundary Conditions On A Cylindrical Plate

Let us consider a simple boundary value problem, the clamped cylindrical plate. Now $Y_k(\lambda r)$ and $K_k(\lambda r)$ go to infinity at the origin, so C_3 and C_5 are zero. Hence

$$u_3(r,\theta,t) = U_3(r,\theta)\exp(i\omega t) = C_1\cos(k(\theta-\phi))[C_2 J_k(\lambda r)+C_4 I_k(\lambda r)]\exp(i\omega t).$$

Let a be the boundary radius of the disk. Then we take as boundary conditions

$$f(a) = 0$$

$$\frac{\partial f(a)}{\partial r} = 0$$

We have two equations

$$C_2 J_k(\lambda a) + C_4 I_k(\lambda a) = 0,$$

$$C_2 \frac{\partial J_k(\lambda a)}{\partial r} + C_4 \frac{\partial I_k(\lambda a)}{\partial r} = 0.$$

Then the determinant must be zero

$$J_k(\lambda a)\frac{\partial I_k(\lambda a)}{\partial r} - I_k(\lambda a)\frac{\partial J_k(\lambda a)}{\partial r} = 0$$

Let λ_m be the mth root of this equation. We have

$$\lambda_m^4 = \frac{\rho k \omega^2}{D}.$$

This gives the frequencies of vibration, and the mode shapes.

For example values of the plate stiffness D, see maple program **pltstiff**.

For a set of boundary conditions, for the boundary value problem of the annular ring, see James Cumming's Masters theses, p87. Those boundary conditions are formulated in Raja P. **Vibrations of Annular Plates**, Journal of the Aeronautical Society of India, 1962. 14(2): pp 37-50.

24 History of Damping Research

Summary of: Robert Plunket **Damping Analysis: An Historical Perspective**, which is in the work edited by Kinra and Wolfenden.

- Theory for shells and plates: D'Alembert (1747), Euler, Bernoulli, Poisson, Germaine.
- Coulomb **Memoir on Torsion** (1784): Energy loss in cyclical strain, stress-strain hysteresis loop.
- Weber 1837: internal friction measured by torsional vibrations.
- Attempts were made at constructing molecular theories, giving elastic and damping constants, but not until the 1920's did Max Born compute, from molecular theory, values for the 21 elastic constants.
- In the early 20th century physicists found that damping was independent of amplitude, but they also found that the damping frequency dependence did not follow the prediction of simple viscous damping.
- Maxwell, Voigt, and Kelvin, tried to model the frequency dependence using combinations of springs and dashpots, but were unsuccessful.
- Linear viscosity is not a good model for damping for any material at low strain levels and normal temperatures.
- There are several different mechanisms of energy dissipation in polycrystalline metals, and the effect of each peaks at different critical frequencies.
- All known metals with high damping at moderate strain levels depend on phase transformations.
- Dislocation motion is the most prevalent mechanism for the low damping found in most metals.
- Zener C **Elasticity and Anelasticity of Metals**, University of Chicago Press, 1948, gives a comprehensive review of measurement techniques to 1948, which were of low frequency on simple structures. More modern methods: Truell R, Elbaum E, Chick B B **Ultrasonic Methods in Solid State Physics** 1969. Nowick A S, Berry B S **Anelastic Relaxation in Crystalline Solids** 1972.

- Most work was based on free decay, but some was based on forced vibration and resonance.
- Supporting methods in experiments include supports at nodal points. In another case the device was a tuning fork held at the handle.
- Most research in damping is aimed at finding materials with increased damping properties, which materials would have the ability to remove harmful vibrations.
- We are looking at the inverse problem: To locate metals with low damping properties..

25 Bibliography

[1]Bradbury T C, **Theoretical Mechanics**, John Wiley, 1968.
[2]Burton Ralph, **Vibration and Impact**, Dover, 1968.
[3]Clough R W, Penzien J, **Dynamics of Structures**, Rayleigh damping.
[4]Crede, **Vibration And Shock Isolation**, Moment Of Inertia By The Three Wire Method.
[5]Cummings Robert James, **Analysis of the Performance and Contact Behavior of the Piezoelectric Traveling-Wave Motor**, Masters Theses, University of Missouri-Rolla, MRTC (Manufacturing Research and Training Center), 1994.
[6]Harris Cyril, **Shock and Vibration Handbook**, 3rd ed 1988 Linda Hall, 4th ed 1996, ta355 .h35.
[7]Desai Chandrakant S, Abel John F, **Introduction To The Finite Element Method**, Van Nostrand 1972, Rayleigh Damping.
[8]Inman Daniel, **Engineering Vibration**, Matlab Disk 1994, Borders, Chapter: Experimental Modal Analysis, Elasticity.
[9]Inman Daniel J, **Vibration**, 1989, Proportional Damping (Rayleigh Damping) P60, Advanced, Mathematical, Graduate Level.
[10]Kikuchi And Oden, **Contact Problems In Elasticity: A Study Of Variational Inequalities And Finite Element Methods**, Linda Hall QA931.k47, 1988, Society For Industrial And Applied Mathematics.
[11]Kinra Vikram K,Wolfenden Alan Eds, **M3D: Mechanics And Mechanisms Of Materials**, ASTM Publications 1992, TA355.M13, (See Article On Material Damping History).

[12]Koehler D R, **Radial Vibration Modes In Thin Plates Of $BaTiO_3$ And $LiNbO_3$**, Memo, Sandia National Laboratory, October 16, 1989.
[13]Kolsky H, **Stress Waves In Solids**, Internal Friction and Damping, Dover 1963.
[14]Lazan B J, **Damping Of Materials And Members In Structural Mechanics**, Oxford 1968.
[15]Littlewood D E, **A University Algebra**, 2nd ed, Dover, 1958.
[16]Mclachlan, N. W, **Theory Of Vibrations**, pp129-144, Circular Membrane, Circular Plate, Bessel Functions.
[17]Meirovitch Leonard, **Elements Of Vibration Analysis**, Mcgraw-hill 1975 South Independence 531.32 M478.
[18]Meirovitch, Leonard, **Analytical Methods Of Vibration**.
[19]Morse, Philip M, **Vibration And Sound**, Bendix Ni, QC231. M7.
[20]Raja P. Narayana, **Vibration Of Annular Plates**, Journal Of The Aeronautical Society Of India, V14, No 2, May 1962, Boundary Conditions.
[21]Rayleigh Lord, **The Theory Of Sound**, V1,Vibrations Of Plates Ch. 10, P.352.
[22]Soedel Werner, **Vibrations Of Shells And Plates**, 1981, Second Edition, 1993, University of Illinois At Chicago.
[23]Stoker, J J, **Nonlinear Vibrations**.
[24]Stutts Dan, **Loss Factors**, FAX: Feb 23 1996, Various Materials Loss. Coef. Vs Young's Mod, Comparison Of Damping for Variety Metals, Magnetoeleastic Damping In Ferromagnetic Metals, Internal Friction VS Strain Amplitude, Experimental Setup With Pucot Drive Crystal And Specimin, Piezoelectricity.
[25]Stutts Dan, **Modal Analysis Of Free-Free Rod With Damping**, FAX April 9, 1996 (See Piezo96 Notebook), Piezoelectricity.
[26]Thomson William T, **Theory Of Vibration With Applications**, 1972, Equivalent Viscous Damping.
[27]Tiersten H F, **Linear Piezoelectric Plate Vibration**, Plenum Press 1969 pp 33-99 Fundamental Equations And Equivalent Circuit.
[28]Zienkiewicz O C, **The Finite Element Method**, 3rd Edition, 1977, Rayleigh Damping, Field And Dynamic Problems Ch. 20, Higher Frequencies Overdamped P564.
[29]Vierck Robert K, **Vibration Analysis** 2nd edition, Harper and Row, 1979.