The Vibrating String

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1 The Wave Equation for the Vibrating String

Let the displaced string be modeled by a parametric curve $(x(s), y(s)$ with parameter s, s being the distance along the string. That is, $(x(s), y(s)$ is the displaced position of the point that was initially at $(s, 0)$. The string tangent vector is

$$\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}.$$

The linear strain e is the change is the distance between two points divided by the initial distance between these two points. Thus

$$e = \left[\|(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j})ds\| - ds \right] /ds$$

$$= \|\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}\| - 1$$

$$= \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2} - 1.$$

Thus given Young's modulus for the string, we may obtain the tension at a point in the string and equate this to the acceleration to that portion of the string. By making some simplifying assumptions we arrive at the wave equation for the vibrating string. Let u be the displacement in the y direction. The wave equation for the string is

$$\frac{\partial^2 u}{\partial^2 t} - c^2 \frac{\partial^2 u}{\partial^2 x} = F(x,t),$$

for $0 < x < \ell, t > 0$, where $F(x,t)$ is a forcing function. The initial displacement is

$$u(x,0) = f(x).$$

The initial velocity is

$$\frac{\partial u}{\partial t}(x,0) = g(x).$$

The string is fixed at the ends

$$u(0,t) = 0, u(\ell,t) = 0.$$

This is an initial-boundary value problem, having both initial conditions and boundary conditions. If the forcing function $F(x,t) = 0$, and $g = 0$, then this is the unforced vibrating string problem, the vibration being caused by an initial displacement, such as the plucking of a guitar string.

2 Characteristics

3 Bibliography

[1]Weinberger Hans F., **A First Course in Partial Differential Equations**, Blaisdell, 1965